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Semi-stability and base change

By

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Abstract. Let X and Y be regular strictly semi-stable varieties over a discrete valuation ring R and let $R \subseteqq \widetilde{R}$ be a finite ramified extension of discrete valuation rings. We explicitly give desingularization procedures for the base change $X \times_R \widetilde{R}$ and for the product $X \times_R Y$.

0. Introduction. The concept of semi-stable curves was extremely fruitful for the development of arithmetic algebraic geometry. It plays a central role in analyzing the degeneration of smooth curves and so is an important tool in passing from curves over the field of fractions K of a discrete valuation ring R to curves over the residue field k of R. Beyond the one dimensional case the notion of semi-stability can be generalized to higher dimensions; cf. [1]. There again semi-stable varieties possess the mildest sort of singularities one may hope for in the degeneration of smooth varieties. But whereas every smooth curve over K has a semi-stable model over R after allowing a finite extension of discrete valuation rings, the analogous statement in higher dimension is known only when the residue characteristic (k) is zero; cf. [3]. In general this is still a conjecture. On the other hand A. J. de Jong has shown in [1, Theorem 6.5] that any variety X over a complete discrete valuation ring admits an alteration $X' \longrightarrow X$, i.e. a proper surjective generically finite morphism, by a regular strictly semi-stable variety X' after a finite extension of discrete valuation rings. This is less than the desired general existence of semi-stable models but nevertheless it is valuable for plenty of applications as de Jong has demonstrated.

To further investigate this useful concept of semi-stability is the aim of this article. We explicitly give desingularization procedures for the product of two regular strictly semi-stable varieties over R and for the base change of a regular strictly semi-stable variety by a ramified extension of discrete valuation rings. They allow to recover regularity, whenever given regular strictly semi-stable varieties are subject to product or base change operations. For instance they can be used in the investigation of the rigid analytic Picard functor; cf. [2]. There one faces the problem of extending line bundles from the generic fiber $X_K \times_K Y_K$ of a product $X \times_R Y$ of two regular strictly semi-stable formal schemes to $X \times_R Y$. For this purpose one needs a regular model of $X_K \times_K Y_K$. Our desingularization procedures yield such a model.

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1. Definition and Examples. Let R be a discrete valuation ring with field of fractions K, uniformizer π and residue field k and let $S = \operatorname{Spec} R$. Let further $\widehat{R}$ be the π-adic completion of R.

Let us recall the definition of a semi-stable S-variety (cf. A. J. de Jong [1, 2.16]).

Definition 1.1. An *S-variety* is an irreducible, reduced and separated scheme X, flat and of finite type over S. We denote by $X_K := X \times_S \operatorname{Spec} K$ the generic fiber of X and by $X_0 := X \times_S \operatorname{Spec} k$ the special fiber. Let X_0^σ for $\sigma \in N$ be the irreducible components of the special fiber X_0 of X. For every $\emptyset \neq M \subseteqq N$ we define

$$X_0^M := \bigcap_{\sigma \in M} X_0^\sigma$$

as the scheme-theoretic intersection.

The S-variety X is called *regular strictly semi-stable* if

(a) X_K is smooth over K,
(b) X_0 is geometrically reduced,
(c) X_0^σ is a Cartier divisor on X for all $\sigma \in N$ and
(d) X_0^M is smooth over k for all $M \subseteqq N$ and equidimensional of dimension $\dim X - \#M$.

X is called *regular semi-stable over S* if étale locally X is regular strictly semi-stable.

Remark 1.1.1. Any regular, (strictly) semi-stable S-variety is in fact *regular*. This is a consequence of condition (c); cf. Proposition 1.3. Its special fiber is a divisor with normal crossings.

Remark 1.1.2. If X is proper over S, condition (a) is a consequence of the other three conditions, since then every closed point of X_K specializes to a point of X_0; cf. Proposition 1.3.

Example 1.1.3. Let X be a *semi-stable curve over S*, i.e. X is flat and proper over S and all geometric fibers of X are reduced connected curves having at most ordinary double points as singularities. Assume further that the generic fiber X_K is smooth over K. If we repeatedly blow up all the points in which X is not regular we obtain a desingularization $\widetilde{X} \longrightarrow X$ of X. Since X is a semi-stable curve, all irreducible components of its special fiber X_0 and all its singular points are defined over a separable field extension of k. Therefore the special fiber $\widetilde{X}_0$ of $\widetilde{X}$ is geometrically reduced and has at most ordinary double points as singularities. Since $\widetilde{X}$ is regular, all irreducible components of $\widetilde{X}_0$ are Cartier divisors. So we see that $\widetilde{X}$ is a regular semi-stable S-variety. If all irreducible components of the special fiber $\widetilde{X}_0$ are smooth then $\widetilde{X}$ is even regular strictly semi-stable. On the other hand if there are components of $\widetilde{X}_0$ which intersect themselves, one can make X regular strictly semi-stable only after an extension of discrete valuation rings $R \subseteqq R'$ of ramification index 2. Then blowing up all the non-smooth points of the special fiber of $\widetilde{X} \otimes_R R'$ yields a regular strictly semi-stable variety over R'. The base extension is necessary to avoid the appearance of non-reduced components.

Example 1.1.4. In [4, Theorem 4.6] K. Künnemann has shown that every abelian variety A_K over K which has semi-abelian reduction and is polarized by an ample line bundle admits a regular strictly semi-stable model over R after a finite extension of discrete valuation rings.

E x a m p l e 1.1.5. In the general case A. J. de Jong has shown in [1, Theorem 6.5] that every S-variety X over the spectrum S of a complete discrete valuation ring admits an alteration $X' \longrightarrow X$, i.e. a proper surjective generically finite morphism, by a regular strictly semi-stable S-variety X' after a finite extension of complete discrete valuation rings.

R e m a r k 1.1.6. If we further assume that R is a complete discrete valuation ring, we can do rigid geometry over K. In this context we can analogously define the notion of a *regular, (strictly) semi-stable formal model* of a rigid analytic space (cf. [2]) by imposing the same conditions (b), (c) and (d) on the special fiber of the formal model. In this case however the smoothness of the rigid space is always a consequence of the other three conditions, since every (closed) point of the rigid space specializes.

E x a m p l e 1.1.7. Let R be a complete discrete valuation ring. G. A. Mustafin [5] has given an example for a rigid analytic space over K that has a regular strictly semi-stable formal model. It is the *rigid analytic Hopf variety* which is defined analogously to its counterpart in complex algebraic geometry. Namely let X_K be the quotient of the pointed affine space $\mathbb{A}^n_K - \{0\}$ by the action of the group $\Gamma := g^{\mathbb{Z}} \subseteqq \mathrm{GL}_n(K)$ with

$$g = \begin{pmatrix} \pi^e u_1 & & 0 \\ & \ddots & \\ 0 & & \pi^e u_n \end{pmatrix}, \quad u_i \in R^{\times}, e \geqq 2\,.$$

According to [5, Theorem a] X_K is a proper smooth connected rigid analytic space over K with regular strictly semi-stable formal model X over R. The special fiber X_0 of X has the following shape. Let Y_0 be the blowing up of $\mathbb{P}^n_k$ in the origin. Then X_0 is a closed chain of e copies of Y_0 where two consecutive copies are glued by identifying the $\mathbb{P}^{n-1}_k$ at infinity in one copy with the exceptional divisor of the blowing-up in the next copy. X_K is not algebraizable.

We next want to give a characterization of regular strictly semi-stable S-varieties. We start with the following local description.

Lemma 1.2. *Let X be a regular strictly semi-stable S-variety and let $x \in X_0$ be a point lying on the irreducible components $X_0^{i_1}, \dots, X_0^{i_s}$ and not on the other components of X_0. Then there exists an open affine neighborhood* Spec A *of x such that the completion of A with respect to the ideal I corresponding to the closed subscheme $X_0^{\{i_1,\dots,i_s\}}$ is of the form*

$$\widehat{A^I} \cong C\,[\![\xi_{i_1}, \dots, \xi_{i_s}]\!]/(\xi_{i_1}\cdots\xi_{i_s} - \pi)$$

for a formally smooth $\widehat{R}$-algebra C complete and separated with respect to the ideal πC, with $A/I = C/\pi C$. The ξ_σ correspond to generators of the ideal associated to the Cartier divisor X_0^σ for $\sigma \in \{i_1, \dots, i_s\}$.

P r o o f. (cf. [1, 2.16]) Let Spec A be an open neighborhood of x such that the Cartier divisor X_0^σ is principal on Spec A, generated by $\xi_\sigma \in A$ for all $\sigma \in \{i_1, \dots, i_s\}$ and such that $C_0 := A/I$ is an integral domain where $I := (\xi_{i_1}, \dots, \xi_{i_s})$. By (d) the k-algebra C_0 is smooth and can therefore be lifted to a smooth R-algebra of finite type. Let C be the completion of that R-algebra with respect to the π-adic topology. It is formally smooth over $\widehat{R}$ with $C/\pi C = C_0$ and therefore an integral domain. Due to its smoothness we can lift the identity map

$$C[\![\xi_{i_1}, \dots, \xi_{i_s}]\!]/(\xi_{i_1}, \dots, \xi_{i_s}, \pi) = C_0 = A/I$$

to a surjective morphism

$$C[\![\xi_{i_1}, \dots, \xi_{i_s}]\!] \longrightarrow \widehat{A^I}.$$

There is an equation $\xi_{i_1} \cdots \xi_{i_s} = u \cdot \pi$ on A. After replacing ξ_{i_1} by $u \cdot \xi_{i_1}$, we may assume $u = 1$ and hence we obtain a morphism

$$C\,[\![\xi_{i_1}, \dots, \xi_{i_s}]\!]/(\xi_{i_1} \cdots \xi_{i_s} - \pi) \;\twoheadrightarrow\; \widehat{A^I}.$$

This epimorphism is an isomorphism since its source is an integral domain of the same dimension $s + \dim C_0$ as its target; use condition (d). □

This description leads to

Proposition 1.3. *An S-variety X is regular strictly semi-stable if and only if*

(a) *the generic fiber X_K is smooth over K and*
(b) *every closed point $x \in X_0$ of the special fiber admits an open neighborhood which for some $s \in \mathbb{N}$ is smooth over the scheme*

$$\operatorname{Spec} R[\xi_{i_1}, \dots, \xi_{i_s}]/(\xi_{i_1} \cdots \xi_{i_s} - \pi)\,.$$

If X is proper over S condition (a) *is a consequence of condition* (b) .

Proof. The sufficiency of conditions (a) and (b) is obvious since in the given neighborhood of x the irreducible components of X_0 are exactly the $V(\xi_\sigma)$ for $\sigma \in \{i_1, \dots, i_s\}$.

We now demonstrate the necessity of (a) and (b). Take a point $x \in X_0$. Say it lies on the irreducible components $X_{i_1}, \dots, X_{i_s}$ of X_0 and not on any other component. Let Spec A be the open neighborhood of x and ξ_σ for $\sigma \in \{i_1, \dots, i_s\}$ the generator of the principal Cartier divisor X_σ on Spec A from Lemma 1.2. Since the identity $\xi_{i_1} \cdots \xi_{i_s} = \pi$ holds in A we obtain a morphism

$$f : \operatorname{Spec} A \longrightarrow \operatorname{Spec} R[\xi_{i_1}, \dots, \xi_{i_s}]/(\xi_{i_1} \cdots \xi_{i_s} - \pi)\,.$$

It is flat in x because of Lemma 1.2. The fiber containing x is $X^{\{i_1,\dots,i_s\}} \longrightarrow \operatorname{Spec} k$ and thus is smooth. So f is smooth in x and therefore there exists an open neighborhood of x on which f is smooth. □

2. Desingularization of products of semi-stable *S*-varieties. We now want to describe a desingularization procedure for the product of two regular strictly semi-stable S-varieties.

Proposition 2.1. *Let X and Y be two regular strictly semi-stable S-varieties. Let X_0^σ for $\sigma \in M$, respectively Y_0^τ for $\tau \in N$ be the irreducible components of their special fibers. Then successively blowing up all the closed subschemes $X_0^\sigma \times_k Y_0^\tau$ of $X \times_R Y$ leads to a desingularization $X \times_R Y =: V^0 \longleftarrow V^1 \longleftarrow \dots \longleftarrow V^r$, where V^r is a regular strictly semi-stable S-variety with $X_K \times_K Y_K = V_K^r$.*

Proof. We pick a closed point of $X_0 \times_k Y_0$ and investigate the mentioned blowing-ups in a neighborhood of this point. By Proposition 1.3 the point has an open neighborhood U which is smooth over

$$T := \operatorname{Spec} R[\xi_1, \dots, \xi_m, \zeta_1, \dots, \zeta_n]/(\xi_1 \cdots \xi_m - \zeta_1 \cdots \zeta_n\,, \quad \xi_1 \cdots \xi_m - \pi)$$

for some n and m. So in this neighborhood each subscheme $X_0^\sigma \times_k Y_0^\tau$ is given as $\mathrm{V}(\xi_i, \zeta_j)$ for suitable i and j. This means that the blowing-ups restricted to U can all be defined as blowing-ups of T in the irreducible components $\mathrm{V}(\xi_i, \zeta_j)$ of the special fiber. Due to the flatness of U over T, the corresponding blowing-up of $X \times_R Y$ restricted to U is obtained by base change.

We now claim that at each step every closed point of the special fiber of V^ρ for $0 \leqq \rho \leqq r$ has an open neighborhood which is smooth over an S-variety

$$T := \operatorname{Spec} R[\xi_1, \dots, \xi_m, \ \zeta_1, \dots, \zeta_n]/(\xi_\mu \cdots \xi_m - \zeta_\nu \cdots \zeta_n, \ \xi_1 \cdots \xi_m \zeta_1 \cdots \zeta_{\nu-1} - \pi)$$

for some $1 \leqq \mu \leqq m$ and $1 \leqq \nu \leqq n$. This is true at the beginning for $\mu = \nu = 1$.

By the above at each step the blowing-up of V^ρ is given by blowing up an irreducible component of the special fiber of T. These components are

$$\mathrm{V}(\xi_i) \ \text{ for } i < \mu\,, \quad \mathrm{V}(\zeta_j) \ \text{ for } j < \nu \quad \text{and} \quad \mathrm{V}(\xi_i, \zeta_j) \ \text{ for } i \geqq \mu \text{ and } j \geqq \nu\,.$$

The first two types are Cartier divisors, therefore the blowing-up in those is an isomorphism. So we can without loss of generality assume that $\mathrm{V}(\xi_\mu, \zeta_\nu)$ is blown up. Thereby we obtain two charts:

- $\xi_\mu = \zeta_\nu \cdot \xi'_\mu$: The relations are equivalent to

$$\begin{aligned} &\xi'_\mu \cdots \xi_m - \zeta_{\nu+1} \cdots \zeta_n &&= 0 \quad \text{and} \\ &\xi_1 \cdots \xi'_\mu \cdots \xi_m \cdot \zeta_1 \cdots \zeta_\nu - \pi &&= 0\,. \end{aligned}$$

- $\zeta_\nu = \xi_\mu \cdot \zeta'_\nu$: The relations are equivalent to

$$\begin{aligned} &\xi_{\mu+1} \cdots \xi_m - \zeta'_\nu \cdots \zeta_n &&= 0 \quad \text{and} \\ &\xi_1 \cdots \xi_m \cdot \zeta_1 \cdots \zeta_{\nu-1} - \pi &&= 0\,. \end{aligned}$$

So we see, that after the blowing-up each point of the special fiber of $V^{\rho+1}$ indeed has a neighborhood which is smooth over an S-variety T of the specified type. Thereby either μ or ν was increased by one. This proves the claim.

At each step the number of irreducible components of T_0 which are not Cartier divisors on T is lowered. The remaining ones will be blown up in the sequel. If all components are Cartier divisors, then either $\mu = m$ or $\nu = n$ and T is regular. The open subscheme of V^r above T is then regular strictly semi-stable. □

We now want to describe the case of a ramified base extension. So let $R \subsetneqq \widetilde{R}$ be a finite extension of discrete valuation rings and let $\widetilde{S} = \operatorname{Spec} \widetilde{R}$. We denote by $\widetilde{K}, \widetilde{k}$ and $\widetilde{\pi}$ respectively the field of fractions, the residue field and a uniformizer of $\widetilde{R}$. Let e be the ramification index, i.e. we have $\pi = \widetilde{u}\, \widetilde{\pi}^e$ for a unit $\widetilde{u} \in \widetilde{R}$.

Proposition 2.2. *Let X be a regular strictly semi-stable S-variety with irreducible components X_0^σ for $\sigma \in M$ of its special fiber. Let $R \subsetneqq \widetilde{R}$ be a ramified ring extension as above. We assume that all the $X_0^\sigma \times_k \widetilde{k}$ are still irreducible. We perform the following blowing-ups of $X \times_R \widetilde{R}$:*

a) *We blow up all the irreducible components $X_0^\sigma \times_k \widetilde{k}$ of $X_0 \times_k \widetilde{k}$, each one maybe several times until it becomes a Cartier divisor.*
b) *In this process new irreducible components of the special fiber will appear. We also blow up all these, maybe several times, until they too become Cartier divisors.*

This leads to a desingularization $X \times_R \widetilde{R} =: V^0 \longleftarrow V^1 \longleftarrow \dots \longleftarrow V^r$, *where* V^r *is a regular strictly semi-stable* $\widetilde{S}$*-variety with* $X_K \times_K \widetilde{K} = V^r_{\widetilde{K}}$.

R e m a r k 2.2.1. The hypothesis on the irreducibility of the $X_0^\sigma \times_k \widetilde{k}$ is not necessary. If we drop it, only the description of the blowing-ups in b) becomes slightly more complicated (cf. the proof).On the other hand if R is henselian, there exists an intermediate extension $R \subseteqq R' \subseteqq \widetilde{R}$, which is étale over R and whose residue field k' is the separable closure of k in $\widetilde{k}$. Let X be a regular strictly semi-stable S-variety. Then $X' := X \times_R R'$ is still regular strictly semi-stable over Spec R'. Moreover all the irreducible components of $X'_0 := X' \times_{R'} k'$ remain irreducible in $X'_0 \times_{k'} \widetilde{k}$.

P r o o f. We pick a closed point of $X_0 \times_k \widetilde{k}$ and investigate the mentioned blowing-ups in a neighborhood of that point. By Proposition 1.3 the point has an open neighborhood U which is smooth over

$$T := \operatorname{Spec} \widetilde{R}[\xi_1, \dots, \xi_m]/(\xi_1 \cdots \xi_m - \widetilde{\pi}^e)$$

for some m. So in this neighborhood each subscheme $X_0^\sigma \times_k \widetilde{k}$ is given as $\mathrm{V}(\xi_i, \widetilde{\pi})$ for a suitable i. This means that all the blowing-ups in a) can be defined as blowing-ups of T in the irreducible components $\mathrm{V}(\xi_i, \widetilde{\pi})$ of the special fiber. Due to the flatness of U over T, the corresponding blowing-up of $X \times_R \widetilde{R}$ restricted to U is obtained by base change.

We now claim that at each step every closed point of the special fiber of V^ρ for $0 \leqq \rho \leqq r$ has an open neighborhood which is smooth over an S-variety

$$T := \operatorname{Spec} \widetilde{R}\,[\xi_0, \dots, \xi_m]\Big/\left(\xi_\mu \cdots \xi_m - \prod_{i=0}^{\mu-1} \xi_i^{e_i}\ ,\ \ \xi_0 \cdots \xi_{\mu-1} - \widetilde{\pi}\right)$$

for integers $1 \leqq \mu \leqq m$ and $0 \leqq e_i \leqq e$. This is true at the beginning for $\mu = 1$, $e_0 = e$ and $\xi_0 = \widetilde{\pi}$.

By the above at each step the blowing-up of V^ρ is given by blowing up an irreducible component of the special fiber of T. These components are

$$\mathrm{V}(\xi_i) \ \text{ for } 0 \leqq i < \mu \ \text{ if } e_i = 0 \ \text{ and } \ \mathrm{V}(\xi_i, \xi_j) \quad \text{for } 0 \leqq i < \mu \leqq j \quad \text{if } e_i > 0\,.$$

Components of the first type are Cartier divisors, therefore the blowing-up in those is an isomorphism. So we can without loss of generality assume $e_0 \geqq 1$ and that $\mathrm{V}(\xi_0, \xi_\mu)$ is blown up. Thereby we obtain two charts:

- $\xi_0 = \xi_\mu \cdot \xi'_0$: The relations are equivalent to

$$\begin{aligned} \xi_{\mu+1} \cdots \xi_m - {\xi'_0}^{e_0} \cdot \prod_{i=1}^{\mu-1} \xi_i^{e_i} \cdot \xi_\mu^{e_0-1} &= 0 \quad \text{and} \\ \xi'_0 \cdot \xi_1 \cdots \xi_\mu - \widetilde{\pi} &= 0\,. \end{aligned}$$

- $\xi_\mu = \xi_0 \cdot \xi'_\mu$: The relations are equivalent to

$$\begin{aligned} \xi'_\mu \cdot \xi_{\mu+1} \cdots \xi_m - \xi_0^{e_0-1} \cdot \prod_{i=1}^{\mu-1} \xi_i^{e_i} &= 0 \quad \text{and} \\ \xi_0 \cdots \xi_{\mu-1} - \widetilde{\pi} &= 0\,. \end{aligned}$$

We see, that after the blowing-up indeed each point of the special fiber of $V^{\rho+1}$ has a neighborhood which is smooth over an S-variety T of the specified type. The blown up component

$V(\xi_0, \xi_\mu)$ has as strict transform in the second chart $V(\xi_0)$ if $e_0 = 1$ respectively $V(\xi_0, \xi'_\mu)$ if $e_0 > 1$. It is a Cartier divisor only if $e_0 = 1$. Otherwise it has to be blown up further.

The following new components appear. In case $e_0 = 1$ there is only one component, $V(\xi_0, \xi_\mu)$. In case $e_0 > 1$ there are $V(\xi_0, \xi_\mu, \xi_j)$ for each $\mu < j \leqq m$. They are the new irreducible components mentioned under b) and they too have to be blown up in the sequel. These blowing-ups can also be defined in T. If we drop the irreducibility assumption (cf. Remark 2.2.1), then they may not be irreducible, but they still correspond to irreducible components of T_0. This proves the claim.

The scheme T is regular if and only if each irreducible component of its special fiber is a Cartier divisor. Blowing up an irreducible component which is not a Cartier divisor lowers the number $e(m-\mu) + \sum_{i=0}^{\mu-1} e_i$. This shows that the procedure will stop after finitely many blowing-ups at a regular strictly semi-stable $\widetilde{S}$-variety V^r. □

R e m a r k 2.2.2. The procedures of Propositions 2.1 and 2.2 are compatible with passing to the henselization R^h of R, i.e. if we consider a desingularization of $V^0 \times_R R^h$ according to Propositions 2.1 or 2.2 then this desingularization can be obtained as a desingularization of V^0 with subsequent base change to R^h. The same is true for passing to the completion of R or for a base extension to a discrete valuation ring with ramification index 1 and purely inseparable residue field extension.

R e m a r k 2.2.3. In the context of formal and rigid geometry (cf. Remark 1.1.6) the above procedure can also be applied to obtain desingularizations of a product of two regular strictly semi-stable formal schemes and of the base change of a regular strictly semi-stable formal scheme by a ramified ring extension. This is used for instance in [2] in the investigation of the rigid analytic Picard functor.

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References

[1] A. J. DE JONG, Smoothness, semi-stability and alterations. Publ. Math. IHES **83**, 51–93 (1996).
[2] U. HARTL and W. LÜTKEBOHMERT, On rigid-analytic Picard varieties. J. Reine Angew. Math. **528** (2000).
[3] G. KEMPF, F. KNUDSEN, D. MUMFORD and B. SAINT-DONAT, Toroidal embeddings I. LNM **339**, Berlin-Heidelberg-New York 1973.
[4] K. KÜNNEMANN, Projective regular models for abelian varieties, semistable reduction, and the height pairing. Duke Math. J. **95**, 161–212 (1998).
[5] G. A. MUSTAFIN, *p*-adic Hopf varieties. Functional Anal. Appl. **11**, 234–235 (1977).

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