On period spaces for p-divisible groups

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In our talk we explained our results from [5] on the image of the Rapoport-Zink period morphism.

Fix a Barsotti-Tate group $\overline{X}_0$ over $\mathbb{F}_p^{\mathrm{alg}}$ of height h and dimension d. Let $W := W(\mathbb{F}_p^{\mathrm{alg}})$ be the ring of Witt vectors and let $K_0 := W[\frac{1}{p}]$. We consider Barsotti-Tate groups X over complete, rank one valued extensions $\mathcal{O}_K$ of W, $K := \operatorname{Frac}\mathcal{O}_K$, such that there exists an isogeny

$$\rho : X \otimes_{\mathcal{O}_K} \mathcal{O}_K/p\mathcal{O}_K \longrightarrow \overline{X}_0 \otimes_{\mathbb{F}_p^{\mathrm{alg}}} \mathcal{O}_K/p\mathcal{O}_K\,.$$

The theory of Grothendieck-Messing [7] associates to X an extension

$$0 \longrightarrow (\operatorname{Lie} X^\vee)_K^\vee \longrightarrow \mathbb{D}(X)_K \longrightarrow \operatorname{Lie} X_K \longrightarrow 0$$

where $\mathbb{D}(X)_K$ is the crystal of Grothendieck-Messing evaluated on K, and the isogeny ρ defines an isomorphism of crystals $\mathbb{D}(\rho)_K : \mathbb{D}(X)_K \xrightarrow{\sim} \mathbb{D}(\overline{X}_0)_K$. The K-subspace $\mathbb{D}(\rho)_K(\operatorname{Lie} X^\vee)_K^\vee$ defines a K-valued point in the Grassmannian $\mathcal{F} := \operatorname{Grass}(h-d, \mathbb{D}(\overline{X}_0)_{K_0})$ of $h-d$-dimensional subspaces of $\mathbb{D}(\overline{X}_0)_{K_0}$. In [4] Grothendieck posed the following

Problem. (A. Grothendieck, 1970)
Describe the subset of $\mathcal{F}$ formed by the points $\mathbb{D}(\rho)_K(\operatorname{Lie} X^\vee)_K^\vee$ for varying K, X, ρ.

A first approximation to this problem was given by Rapoport and Zink [8] who constructed a rigid analytic period domain $\mathcal{F}_{wa}^{\mathrm{rig}}$ for Barsotti-Tate groups, which consists of all weakly admissible filtrations on the isocrystal $\mathbb{D}(\overline{X}_0)_{K_0}$, and contains the subset $\mathcal{G}$ of Grothendieck's problem. In our talk we firstly showed that only in rare cases the Rapoport-Zink period domain equals $\mathcal{G}$, by exhibiting weakly admissible filtrations defined over infinite extensions K/K_0 which do not correspond to Barsotti-Tate groups X over $\mathcal{O}_K$. Secondly we described the solution of Grothendieck's problem as the open Berkovich subspace $\mathcal{F}_a$ of $\mathcal{F}$ consisting of those points for which the associated φ-module over the ("algebraic closure" of the) Robba ring is unit root. The space $\mathcal{F}_a$ is contained in the Berkovich subspace $\mathcal{F}_{wa}$ corresponding to the Rapoport-Zink period domain. The inclusion $\mathcal{F}_a \subset \mathcal{F}_{wa}$ induces an étale morphism of the associated rigid analytic spaces, which is a bijection on rigid analytic points by the theorems of Colmez-Fontaine [2], Breuil [1, Theorem 1.4] and Kisin [6]. The rational Tate module $T_pX_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of X gives rise to a local system of $\mathbb{Q}_p$-vector spaces on $\mathcal{F}_a$, whose associated space of $\mathbb{Z}_p$-lattices is the generic fiber of the Rapoport-Zink space, which parametrizes pairs (X, ρ) of Barsotti-Tate groups X over $\mathcal{O}_K$ and isogenies ρ as above. In [5] we constructed the space $\mathcal{F}_a$ and showed that it contains Grothendieck's set $\mathcal{G}$. Subsequently it was shown by Faltings [3] that $\mathcal{G} = \mathcal{F}_a$.

References

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