The Tits building and the Bruhat–Tits building of $\mathrm{GL}_{d+1}(K)$

Ralf Diepholz
Oberseminar talk 21.05.08

Parts 1 and 2 of this talk explain the basic concepts "simplicial set" and "simplicial complex" which underlie many objects we work with in [4]. Part 3 defines the Tits building and the Bruhat–Tits building of $\mathrm{GL}_{d+1}(K)$ as two special cases of the building of a (B, N)–pair. Sources are [2], [3] for part 1, [1], [2] for part 2, [1], [4] for part 3.

1. Simplicial objects

The category Ord. For $n \in \mathbb{N}$ put $\Delta_n = \{0, \ldots, n\}$. Let Ord be the category with the sets Δ_n $(n \in \mathbb{N})$ as objects and order–preserving maps as morphisms. A generating system for the morphisms is given by

$$d^i = d^i_n : \Delta_{n-1} \to \Delta_n,\ j \mapsto \begin{cases} j & ; j < i \\ j+1; & j \geq i \end{cases} \quad (n \geq 1, 0 \leq i \leq n) \ \ \textbf{(face maps)},$$

$$s^i = s^i_n : \Delta_{n+1} \to \Delta_n,\ j \mapsto \begin{cases} j & ; j \leq i \\ j-1; & j > i \end{cases} \quad (n \geq 0, 0 \leq i \leq n) \ \ \textbf{(degeneracy maps)},$$

satifying the following relations:

$$(*) \qquad \begin{aligned} d^j\, d^i &= d^i\, d^{j-1} \ \text{ if } i < j \\ s^i\, s^j &= s^{j-1}\, s^i \ \text{ if } i < j \\ s^j\, d^i &= \begin{cases} d^i\, s^{j-1} & ; \text{if } i < j \\ \mathrm{id} & ; \text{if } j \leq i \leq j+1 \\ d^{i-1}\, s^j & ; \text{if } i > j+1 \end{cases} \end{aligned}$$

The topological space $|\Delta_n| := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}_{\geq 0} ; \sum x_i = 1\}$ is called the **geometric n–simplex**. We define

$$d^i_* : |\Delta_{n-1}| \to |\Delta_n|, (x_0, \ldots, x_{n-1}) \mapsto (x_0, \ldots, \hat{x}_i, \ldots, x_{n-1}) \quad (n \geq 1, 0 \leq i \leq n),$$

$$s^i_* : |\Delta_{n+1}| \to |\Delta_n|, (x_0, \ldots, x_{n+1}) \mapsto (x_0, \ldots, x_i + x_{i+1}, \ldots, x_{n+1}) \quad (n \geq 0, 0 \leq i \leq n),$$

obtaining a covariant functor Ord $\to$ {topological spaces}.

Simplicial objects. Let $\mathcal{K}$ be a category. A **simplicial object in $\mathcal{K}$** is a contravariant functor Ord $\to \mathcal{K}$. We may think of a simplicial object as a family $M = (M_r)_{r \in \mathbb{N}}$ together with morphisms $d^{i*}_r : M_r \to M_{r-1}$, $s^{i*}_r : M_r \to M_{r+1}$ (the images of the above maps) satisfying the relations dual to $(*)$. (Remark: the older terminology (e.g. in [2], [3]) is "semi–simplicial object".)

Example. Let G be an abelian group, $G_1, \ldots, G_m \subset G$ subgroups. Define

$$M_r := \bigoplus_{i_0,\dots,i_r\in\{1,\dots,m\}} G_{i_0} \cap \dots \cap G_{i_r},$$
$$d^j : M_r \to M_{r-1},\ (g_{i_0 \dots i_r})_{i_0,\dots,i_r} \mapsto \left(\sum_{1\le i\le r} g_{i_0 \dots i_j\, i\, i_{j+1} \dots i_{r-1}}\right)_{i_0,\dots,i_{r-1}},$$
$$s^j : M_r \to M_{r+1},\ (g_{i_0 \dots i_r})_{i_0,\dots,i_r} \mapsto \left(g_{i_0 \dots \hat{i}_j \dots i_{r+1}}\right)_{i_0,\dots,i_{r+1}}.$$

These data constitute a simplicial abelian group (i.e. a simplicial object in the category of abelian groups). This example occurs in Section 2 of [4] where it is applied to the groups $G_i := H^0(X, U_i; I)$ ($U_1, \dots, U_m$ open subvarieties of a rigid variety X, I any injective sheaf). In [4] Section 3 this will be used on $X := \mathbb{P}$, $U_i := \mathbb{P} - H_i(|\pi|^n)$ ($H_1, \dots, H_m \in \mathcal{H}_n$).

Geometric Realization. Let $M = (M_r)$ be a simplicial topological space. Define

$$X = \dot{\bigcup_{r\in\mathbb{N}}} (M_r \times |\Delta_r|),$$

endowed with the sum topology (i.e. the final topology w.r.t. the inclusions $M_r \times |\Delta_r| \to X$). Consider the equivalence relation $\sim$ on X generated by

$$(d^i\, a,\, x) \sim (a,\, d^i_*\, x) \quad (r \ge 1,\ a \in M_r,\ x \in |\Delta_{r-1}|,\ 0 \le i \le r),$$
$$(s^i\, a,\, x) \sim (a,\, s^i_*\, x) \quad (r \ge 0,\ a \in M_r,\ x \in |\Delta_{r+1}|,\ 0 \le i \le r).$$

The quotient space $|M| := X/\sim$ is called the **geometric realization** of M.

The chain complex of a simplicial set with coefficients in an abelian group. Let $M = (M_r)$ be a simplicial set. For $r \in \mathbb{N}$ let $C_r(M)$ denote the free $\mathbb{Z}$−module with basis M_r. Let A be an abelian group. The groups

$$C_r(M; A) := C_r(M) \otimes A \quad (\text{"}r\text{−chains of } M \text{ with coefficients in } A\text{"}),$$

together with the differentials

$$\begin{array}{rccc} d_r: & C_r(M; A) & \longrightarrow & C_{r-1}(M; A) \\ & a & \longmapsto & \sum_{i=0}^{r} (-1)^i\, d^i_r(a) \quad (a \in M_r) \end{array} .$$

form a complex $C_*(M; A)$ of abelian groups; its homology is denoted by $H_*(M; A)$.

2. Simplicial complexes. Buildings

Simplicial complexes. Let $\mathcal{A}$ be an ordered set with the following properties:

(a) any two $A, B \in \mathcal{A}$ possess an infimum;

(b) for any $A \in \mathcal{A}$ there is an integer $n \ge -1$ such that the ordered set $\{B \in \mathcal{A};\ B \le A\}$ is isomorphic to the power set $\mathcal{P}(\Delta_n)$ (ordered by inclusion).

Then $\mathcal{A}$ is called a **simplicial complex**. The elements of $\mathcal{A}$ are called **faces**. The relation "$B \le A$" is spelled "B **is a face of** A". The (unique) number n in (b) is called the **dimension**, the number $n+1$ the **rank** of A. The rank−1−faces are called **vertices**. The (unique) rank−0−face is called the **empty face** of $\mathcal{A}$. A subset $\mathcal{B} \subset \mathcal{A}$ is called a **subcomplex** if $A \in \mathcal{B}$ implies $B \in \mathcal{B}$ for all faces $B \le A$.

Example. Let K be a set. Let $\mathcal{A}$ be a set of finite subsets of K such that

- if $A \in \mathcal{A}$ and $B \subset A$ then $B \in \mathcal{A}$;
- $\{x\} \in \mathcal{A}$ for all $x \in K$.

Then $(\mathcal{A}, \subseteq)$ is a simplicial complex, the faces of an element $A \in \mathcal{A}$ being its subsets, the vertices being the singletons $\{x\}$ $(x \in K)$.

In fact, every simplicial complex $(\mathcal{B}, \le)$ is isomorphic to a simplicial complex of this kind: Consider $K := \{\text{vertices of } \mathcal{B}\}$, $\mathcal{A} := \{\{V \in K;\ V \le B\};\ B \in \mathcal{B}\}$.

Simplicial complexes as simplicial sets. For $n \in \mathbb{N}$ let

$$\mathcal{A}_n := \{n+1\text{–tupels } (V_0, \ldots, V_n) \text{ of vertices of } \mathcal{A};\ \exists\, A \in \mathcal{A}\colon V_0, \ldots, V_n \le A\}.$$

Then $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is a simplicial set with face maps

$$d_n^i : \mathcal{A}_n \to \mathcal{A}_{n-1}, (V_0, \ldots, V_n) \mapsto \left(V_0, \ldots, \hat{V}_i, \ldots, V_n\right)$$

and degeneracy maps

$$s_n^i : \mathcal{A}_n \to \mathcal{A}_{n+1}, (V_0, \ldots, V_n) \mapsto (V_0, \ldots, V_i, V_i, \ldots, V_n),$$

the so−called **simplicial set associated to** K. Up to isomorphy, $\mathcal{A}$ may be recovered from $(\mathcal{A}_n)_{n\in\mathbb{N}}$: The set $\mathcal{A}' = \{\{V_0, \ldots, V_n\};\ n \in \mathbb{N}, (V_0, \ldots, V_n) \in \mathcal{A}_n\}$, ordered by inclusion, is isomorphic to $\mathcal{A}$.

In particular, we may speak of the chain complex (with coefficients in some abelian group) and of the geometric realization of a simplicial set. E.g., the geometric realization of the power set $\mathcal{P}(\Delta_n)$ is isomorphic to the geometric n−simplex $|\Delta_n|$.

Dangerous curve. An element of $\mathcal{A}_n$ is called an "n−simplex" or "n−dimensional simplex" of the simplicial complex $\mathcal{A}$. On the other hand many authors say "simplex" instead of "face", so an "n−dimensional simplex" in this latter sense is a certain subset of $\mathcal{A}$ and not a tupel in $\mathcal{A}$. Don't mix it up.

Chamber complexes. Let $\mathcal{A}$ be a simplicial complex with bounded dimensions such that every maximal face has the same dimension d. A face of dimension n is then said to have **codimension** $d - n$. A codimension−0−face is called a **chamber**. $\mathcal{A}$ is called a **chamber complex** if any two chambers C, C' are "joined by a gallery", i.e. there is a sequence of chambers $C = C_0, C_1, \ldots, C_n = C'$ such that C_i and C_{i-1} have a common codimension−1−face $(1 \le i \le n)$. $\mathcal{A}$ is called **thin** if in addition every codimensional−1−face is a face of exactly two chambers.

Buildings. A simplicial complex $\mathcal{B}$ is called a (**thick**) **building** if there exists a family $(\mathcal{A}_i)_{i\in I}$ of thin chamber−subcomplexes covering $\mathcal{B}$ such that:

(B1) for any two faces $A, B \in \mathcal{B}$ there is an $\mathcal{A}_i$ containing A and B;

(B2) for any two $\mathcal{A}_i, \mathcal{A}_j$ containing A and B there is an isomorphism $\mathcal{A}_i \to \mathcal{A}_j$ fixing A and B pointwise;

(B3) any codimension−1−face of $\mathcal{B}$ is a face of at least 3 chambers ("thickness").

Such a family $(\mathcal{A}_i)_{i\in I}$ is called a **system of apartments** of $\mathcal{B}$. From (B2) (applied to the empty face) it follows that any two apartments are isomorphic.

3. The building of a (*B*, *N*)−pair

Let G be a group. Let $B, N \subset G$ be subgroups generating G and such that $B \cap N$ is normal in N. Let S be a generating set of the group $N / B \cap N$. The pair (B, N) is called a **(*B*, *N*)−pair** (or **Tits system**) if

• $s\,B\,w \subset B\,s\,B \bigcup B\,s\,w\,B \;\; \forall\, s \in S,\, w \in N$,

• $s\,B\,s^{-1} \neq B \;\; \forall\, s \in S$.

A subgroup $P \subset G$ containing a conjugate of B is called **parabolic**.

Theorem. The set $\mathcal{T}(B, N)$ of parabolic subgroups of G, ordered by inclusion, is a building.

The building $\mathcal{T}(B, N)$, together with the conjugation action of G, is called the **building of (*B*, *N*)**. The parabolic subgroups of the form $w\,P\,w^{-1}$ $(w \in N,\ P \supset B)$ form a subcomplex $\mathcal{A} \subset \mathcal{B}$, the so−called **fundamental apartment**. A system of apartments of $\mathcal{T}(B, N)$ is then given by $(g\,\mathcal{A}\,g^{-1})_{g\in G}$.

Example A. The **Tits building of $G := \mathrm{GL}_d(K)$** (K any field) is defined to be the building of the following (B, N)−pair:

B = upper triangular matrices in G),

N = monomial matrices in G.

The fundamental apartment of $\mathcal{T}(B, D)$ is finite and has $d!$ chambers, namely the conjugates of the given Borel subgroup B under the *Weyl group* ${}^{v}W := N / B \cap N$ (isomorphic to the symmetric group S_d). Another construction of this building is obtained as follows: Let

$$\mathcal{T} := \text{set of all } K\text{−linear flags } 0 \subset W_0 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} W_r \underset{\neq}{\subset} K^d,$$

ordered by inclusion, and endowed with the natural G−action. We have the isomorphism of ordered G−sets

$$\begin{array}{ccc} \mathcal{T} & \to & \mathcal{T}(B, N) \\ \tau & \mapsto & \text{stabilizer } G_\tau \text{ of } \tau \end{array} .$$

From now on let K be a local field, o the valuation ring, π a prime element.

Structure of a simplicial profinite set on $\mathcal{T}$: Recall that a profinite set is a topological space which is the limit of a projective system of finite discrete sets (equivalently: which is compact and totally disconnected). The simplicial set associated to $\mathcal{T}$ is given by

$$\mathcal{T}_r := \text{set of all } K\text{–linear flags } 0 \underset{\neq}{\subseteq} W_0 \subseteq W_1 \subseteq \ldots \subseteq W_r \underset{\neq}{\subseteq} K^d \qquad (r \in \mathbb{N})$$

together with the natural face and degeneracy maps. We topologize $\mathcal{T}_r$ as follows: Endowing G with the p–adic topology the quotients G/G_τ ($\tau \in \mathcal{T}_r$) are profinite. This defines a topology on the orbits $G\tau \simeq G/G_\tau$, and we take the sum topology on $\mathcal{T}_r = \bigcup G\tau$ (finite disjoint union).

Example B. The **Bruhat–Tits building of $G := \mathrm{SL}_d(K)$** is the building of the following (B, N)–pair:

B = preimage of the upper triangular matrices in $\mathrm{SL}_d(o/\pi)$ under the map

$$\mathrm{SL}_d(o) \to \mathrm{SL}_d(o/\pi),$$

N = monomial matrices in G.

Remarks: (1) B is called the "Iwahori subgroup" of G; the parabolic subgroups in the above sense are called the "parahoric subgroups". The group $W := N/N \cap B$ is called the "affine Weyl group" of G and is a semi–direct product of the finite Weyl group ${}^vW$ (Example A) by a free abelian group of rank $d-1$.

(2) The conjugation action of G on $\mathcal{T}(B, N)$ extends to $\mathrm{GL}_d(K)$, and $\mathcal{T}(B, N)$, endowed with this extended action, is also called the "Bruhat–Tits building of $\mathrm{GL}_d(K)$".

Here we also have another construction of the building: For an o–lattice $L \subset K^d$ let $[L]$ denote its homothety class. Let $\mathcal{BT}$ be the set of finite sets $\{[L_0], \ldots, [L_r]\}$ of classes of o–lattices with the property

$$L_0 \underset{\neq}{\subset} L_1 \underset{\neq}{\subset} \ldots \underset{\neq}{\subset} \pi^{-1} L_0 \quad \text{(for suitable representatives } L_i\text{)},$$

ordered by inclusion, and endowed with the natural G–action. We have the isomorphism

$$\begin{array}{ccc} \mathcal{BT} & \longrightarrow & \mathcal{T}(B, N) \\ \tau = \{[L_0], \ldots, [L_r]\} & \longmapsto & \text{stabilizer of } \tau \text{ in } G \end{array} .$$

[1] Kenneth Brown, Buildings, Springer 1989

[2] Roger Godement, Topologie algébrique et théorie des faisceaux, Hermann 1958

[3] John Milnor, The realization of a semi–simplicial complex, Ann. of Math. 65, 1957

[4] Schneider/Stuhler 1991