Spectral Sequences

Let $\mathcal{A}$ be an abelian category and $r_0 \in \mathbb{N}$. A spectral sequence in $\mathcal{A}$ starting on page r_0 consists of the following data:

α) For every $r \in \mathbb{N}_{\geq r_0}$ a family $(E_r^{pq})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ of objects in $\mathcal{A}$.

β) For every $r \in \mathbb{N}_{\geq r_0}$ a family $(d_r^{pq} : E_r^{pq} \to E_r^{p+r,\, q-r+1})_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$ of morphisms in $\mathcal{A}$, such that

$$d_r^{pq} \circ d_r^{p-r,\, q+r-1} = 0 \qquad \forall\, (p,q) \in \mathbb{Z}\times\mathbb{Z},\ r \in \mathbb{N}_{\geq r_0}$$

γ) For every $r \in \mathbb{N}_{\geq r_0}$ a family

$$\left(\alpha_r^{pq} : \ker(d_r^{pq}) / \operatorname{im}(d_r^{p-r,\, q+r-1}) \xrightarrow{\sim} E_{r+1}^{pq}\right)_{(p,q)\in\mathbb{Z}\times\mathbb{Z}}$$

of isomorphisms in $\mathcal{A}$.

$\rightsquigarrow$ morphisms between spectral sequences

Notation: $E = (E_r^{pq})_{r \geq r_0}$

Definition A spectral sequence E in $\mathcal{A}$ starting on page r_0 is said to be bounded if for every $n \in \mathbb{Z}$ the set

$$\{(p,q) \mid p+q=n,\ E_{r_0}^{pq} \neq 0\}$$

is finite.

Let $E = (E_r^{pq})_{r \geq r_0}$ be a bounded spectral sequence in $\mathcal{A}$.

For every $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ there exists $r_{(p,q)} \in \mathbb{N}_{\geq r_0}$, such that

$$E_r^{pq} \xrightarrow{\sim} E_{r+1}^{pq} \qquad \forall\, r \geq r_{(p,q)}$$

We define

$$E_\infty^{pq} := \varinjlim_{r \geq r_{(p,q)}} E_r^{pq}$$

Definition Let $E = (E_r^{pq})_{r \geq r_0}$ be a bounded spectral sequence and $E^n \in \mathrm{Ob}(\mathcal{A})$ for every $n \in \mathbb{Z}$. We say that E converges to $(E^n)_{n \in \mathbb{Z}}$ if for each $n \in \mathbb{Z}$ we are given a finite exhaustive decreasing filtration $(F^p(E^n))_{p \in \mathbb{Z}}$ (i.e. $F^p(E^n) = E^n$ for p small and $F^p(E^n) = 0$ for p big) and a family $\left(\beta^{pq} : E_\infty^{pq} \xrightarrow{\sim} gr_p(E^{p+q})\right)_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}$ of isomorphisms in $\mathcal{A}$.

Notation: $\quad E_{r_0}^{pq} \Rightarrow E^n$

Example Let $E_{r_0}^{pq} \Rightarrow E^n$ be a convergent spectral sequence.

i) Let $r_0 = 1$ and $E_1^{pq} = 0$ whenever $q \neq 0$. Then $E_\infty^{pq} = \ker d_1^{pq} / \operatorname{im} d_1^{p-1,q}$

and $\underbrace{gr_p(E^{p+q})}_{= F^p E^{p+q} / F^{p+1} E^{p+q}} = E_\infty^{pq} = 0$ for $q < 0$ or $q > 0$.

$$\Rightarrow \quad F^n(E^n) = E^n \quad \text{and} \quad F^{n+1}(E^n) = 0$$

We get $E^n = E_\infty^{n0} = \ker d_1^{n0} / \operatorname{im} d_1^{n-1,0}$

ii) Let $r_0 = 2$ and $E_2^{pq} = 0$ whenever $p < 0$ or $p > 1$.

$$\Rightarrow \quad F^0(E^n) = E^n \quad \text{and} \quad F^2(E^n) = 0.$$

We have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & F^1(E^n) \arrow[r] \arrow[d, equal] & E^n \arrow[r] & E^n / F^1(E^n) \arrow[r] \arrow[d, equal] & 0 \\
& F^1(E^n)/F^2(E^n) \arrow[d, equal] & & F^0(E^n)/F^1(E^n) \arrow[d, equal] & \\
& gr_1(E^n) \arrow[d, equal] & & gr_0(E_n) \arrow[d, equal] & \\
& E_\infty^{1,n-1} \arrow[d, equal] & & E_\infty^{0n} \arrow[d, equal] & \\
& E_2^{1,n-1} & & E_2^{0n} &
\end{tikzcd}$$

Spectral sequence of a filtration

Let $K^\bullet$ be a complex in $\mathcal{A}$ and suppose we have a decreasing filtration $(F^p(K^\bullet))_{p\in\mathbb{Z}}$ of $K^\bullet$, such that for every $n\in\mathbb{Z}$ the filtration on K^n is finite and exhaustive.

Then there is a canonical spectral sequence $E=(E_r^{pq})_{r\geq 0}$ starting at page zero with $E_0^{pq} = F^p K^{p+q}/F^{p+1}K^{p+q}$ converging to $E^n = H^n(K^\bullet)$. In other words

$$E_0^{pq} \Rightarrow H^{p+q}(K^\bullet)$$

Spectral sequence of a double complex

A <u>double complex</u> $L^{\bullet\bullet} = (L^{ij}, d_I^{ij}, d_{II}^{ij})$ <u>in</u> $\mathcal{A}$ is a collection of objects L^{ij} of $\mathcal{A}$ and morphisms $d_I^{ij}: L^{ij} \to L^{i+1,j}$, $d_{II}^{ij}: L^{ij} \to L^{i,j+1}$ for $(i,j)\in\mathbb{Z}\times\mathbb{Z}$ satisfying

$$d_I^2 = 0, \quad d_{II}^2 = 0, \quad d_I d_{II} + d_{II} d_I = 0.$$

For every $(i,j)\in\mathbb{Z}\times\mathbb{Z}$ define

$$H_I^{ij}(L^{\bullet\bullet}) := \ker d_{II}^{ij} / \operatorname{im} d_{II}^{i,j-1}$$

$$H_{II}^{ij}(L^{\bullet\bullet}) := \ker d_I^{ij} / \operatorname{im} d_I^{i-1,j}$$

and these become complexes $H_{I}^{\bullet j}(L^{\bullet\bullet})$ and $H_{II}^{i\bullet}(L^{\bullet\bullet})$.

At each position we get cohomology objects $H^{i}(H_{I}^{\bullet j}(L^{\bullet\bullet}))$ and $H^{j}(H_{II}^{i\bullet}(L^{\bullet\bullet}))$.

Next we define two filtrations for the total complex $\mathrm{Tot}(L^{\bullet\bullet})$ of $L^{\bullet\bullet}$. Recall that

$$\mathrm{Tot}(L^{\bullet\bullet})^{n} := \bigoplus_{i+j=n} L^{ij} \quad \text{with}$$

$$d := d_{I} + d_{II}$$

The first filtration is

$$F_{I}^{p}(\mathrm{Tot}(L^{\bullet\bullet}))^{n} := \bigoplus_{i \geq p} L^{i,n-i}$$

and the second

$$F_{II}^{p}(\mathrm{Tot}(L^{\bullet\bullet}))^{n} := \bigoplus_{i \geq p} L^{n-i,i}$$

Like this we obtain two spectral sequences ${}^{I}E$ and ${}^{II}E$. We have the following

<u>Proposition</u> Let $L^{\bullet\bullet}$ be a double complex in $\mathcal{A}$. We have two canonical spectral sequences ${}^{I}E$ and ${}^{II}E$ with

$${}^{I}E_{0}^{pq} = L^{pq}, \quad {}^{I}E_{1}^{pq} = H_{I}^{pq}(L^{\bullet\bullet}), \quad {}^{I}E_{2}^{pq} = H^{p}(H_{I}^{\bullet,q}(L^{\bullet\bullet}))$$

$${}^{II}E_{0}^{pq} = L^{qp}, \quad {}^{II}E_{1}^{pq} = H_{II}^{qp}(L^{\bullet\bullet}), \quad {}^{II}E_{2}^{pq} = H^{p}(H_{II}^{q,\bullet}(L^{\bullet\bullet}))$$

both "converging" to $H^{p+q}(\mathrm{Tot}(L^{\bullet\bullet}))$.

Hypercohomology spectral sequence

<u>Proposition</u> Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories where $\mathcal{A}$ has enough injectives and let $K^\bullet$ be a complex in $\mathcal{A}$ which is bounded below. Then we get two spectral sequences ${}^{I}E$ and ${}^{II}E$ with

$${}^{I}E_2^{pq} = H^p(R^q F(K^\bullet)) \Rightarrow {}^{h}R^{p+q} F(K^\bullet)$$

and

$${}^{II}E_2^{pq} = R^p F(H^q(K^\bullet)) \Rightarrow {}^{h}R^{p+q} F(K^\bullet)$$

<u>Remark</u> For the construction we use a Cartan-Eilenberg resolution of $K^\bullet$ and apply the constructions for the two spectral sequences of a double complex. Like this we get

$${}^{I}E_1^{pq} = H_{I}^{pq}(F\mathcal{J})$$

where $\mathcal{J}$ is a Cartan-Eilenberg resolution of $K^\bullet$.

$\mathcal{V}$ = category of smooth separated (rigid) analytic varieties over K

We fix a Grothendieck-topology on $\mathcal{V}$ which is finer than the analytic topology.

$D^{\geq 0}(\mathcal{V})$ = derived category of complexes of sheaves on $\mathcal{V}$ in nonnegative degree

Let $\mathcal{F}$ be an object in $D^{\geq 0}(\mathcal{V})$.

For $X \in Ob(\mathcal{V})$ we put

$$H^*(X) := H^*(X, \mathcal{F}) \quad .$$

If $\mathcal{U}$ is an (admissible) open subvariety we call

$$H^*(X, \mathcal{U}) := H^*(X, \mathcal{U}; \mathcal{F})$$

the relative cohomology, which is the derived functor of the functor "sections on X which vanish on $\mathcal{U}$".

Let X be a variety in $\mathcal{V}$ and let $\mathcal{U} \subseteq X$ be an open subvariety which possesses an admissible covering $(\mathcal{U}_n)_{n \in \mathbb{N}}$ with $\ldots \subseteq \mathcal{U}_n \subseteq \mathcal{U}_{n+1} \subseteq \ldots$ of open subvarieties.

Proposition There is a natural exact sequence

$$0 \to \varprojlim{}^{(1)} H^{(*-1)}(X, \mathcal{U}_n) \to H^*(X, \mathcal{U}) \to \varprojlim H^*(X, \mathcal{U}_n) \to 0$$

Proof First we will show the following claim:

(*) For any injective sheaf I on $\mathcal{V}$ we have

a) $H^0(X, \mathcal{U}; I) = \varprojlim H^0(X, \mathcal{U}_n; I)$

b) $\varprojlim{}^{(r)} H^0(X, \mathcal{U}_n; I) = 0$ for every $r \geq 1$

proof of (*): We have an exact sequence of projective systems (with respect to n)

$$0 \to H^0(X, \mathcal{U}_n; I) \to H^0(X, I) \to H^0(\mathcal{U}_n, I) \to 0$$

because: It is clear that for every $n \in \mathbb{N}$ the sequence is exact at $H^0(X, \mathcal{U}_n; I)$ and at $H^0(X, I)$. By [SGA4, V 4.7] the mapping $H^0(X, I) \to H^0(\mathcal{U}_n, I)$ is surjective because I is injective.

After applying $\varprojlim$ we get the following sequence

$$0 \to \varprojlim H^0(X, \mathcal{U}_n; I) \to \underbrace{\varprojlim H^0(X, I)}_{= H^0(X, I)} \to \underbrace{\varprojlim H^0(\mathcal{U}_n, I)}_{= H^0(\mathcal{U}, I)} \to 0$$

which is again exact for the same reason as before. We get $\varprojlim H^0(X, \mathcal{U}_n; I) = H^0(X, \mathcal{U}; I)$ and because $(H^0(X, I))_{n \in \mathbb{N}}$ and $(H^0(\mathcal{U}_n, I))_{n \in \mathbb{N}}$ are acyclic [J, Th. 1.8] we see that (b) is correct. Therefore (*) has been shown.

Now let $\mathcal{F} \xrightarrow{\sim} I^\cdot$ be an injective resolution. From the complex $H^0(X, \mathcal{U}_n; I^\cdot)$ we get two hypercohomology spectral sequences

$$ {}^{I}E_1^{rs} = \varprojlim{}^{(s)} H^0(X, \mathcal{U}_n; I^r) \Rightarrow {}^{h}R^{r+s} \varprojlim (H^0(X, \mathcal{U}_n; I^\cdot)) $$

$$ {}^{II}E_2^{rs} = \varprojlim{}^{(r)} H^s(H^0(X, \mathcal{U}_n; I^\cdot)) \Rightarrow {}^{h}R^{r+s} \varprojlim (H^0(X, \mathcal{U}_n; I^\cdot)) $$

The first one degenaretes (because of (*).(b)) and we get

$$ H^s(H^0(X, \mathcal{U}; I^\cdot)) \underset{(*).(a)}{=} H^s(\varprojlim H^0(X, \mathcal{U}_n; I^\cdot)) \underset{\text{Example.(i)}}{=} {}^{h}R^s \varprojlim (H^0(X, \mathcal{U}_n, I^\cdot)) $$

Therefore Example. (ii) gives the desired exact sequence (because $\varprojlim{}^{(r)} = 0$ for $r \geq 2$ [J, Th. 2.2])

$$ 0 \to \varprojlim{}^{(1)} H^{(*-1)}(X, \mathcal{U}_n) \to H^*(X, \mathcal{U}) \to \varprojlim H^*(X, \mathcal{U}_n) \to 0 $$

__Corollary__ If in the projective system $(H^{s-1}(X,\mathcal{U}_n))_{n\in\mathbb{N}}$ all A-modules are finitely generated, then we have $H^s(X,\mathcal{U}) = \varprojlim H^s(X,\mathcal{U}_n)$.

__Proof__ [J, Corollary 7.2] $\Rightarrow$ $\varprojlim^{(i)} H^{s-1}(X,\mathcal{U}_n) = 0$ for all $i > 0$.

Now let $\mathcal{U}_1, \dots, \mathcal{U}_m \subseteq X$ be a finite family of open subvarieties of the variety X in $\mathcal{V}$. We set $\mathcal{U} := \mathcal{U}_1 \cap \dots \cap \mathcal{U}_m$.

We want to construct a "strongly" convergent spectral sequence

$$(+) \qquad E_1^{rs} = \bigoplus_{1 \le i_0 < i_1 < \dots < i_{-r} \le m} H^s(X, \mathcal{U}_{i_0} \cap \dots \cap \mathcal{U}_{i_{-r}}) \Rightarrow H^{r+s}(X,\mathcal{U})$$

For the construction we need the following definition:

Let $G_1, \dots, G_m$ be a finite family of subgroups of some abelian group G. For every $r \in \mathbb{N}$ and $l \in \{0,\dots,r\}$ we have the following morphism

$$G_{r,l} : \bigoplus_{i_0,\dots,i_r} G_{i_0} \cap \dots \cap G_{i_r} \to \bigoplus_{i_0,\dots,i_{r-1}} G_{i_0} \cap \dots \cap G_{i_{r-1}}$$

$$(g_{i_0 \dots i_r})_{i_0 \dots i_r} \mapsto \left(\sum_{j=1}^{m} g_{i_0 \dots i_{l-1} j i_l \dots i_{r-1}} \right)_{i_0 \dots i_{r-1}}$$

Or in other words

$$g_{i_0 \dots i_r} \mapsto (f_{j_0 \dots j_{r-1}})_{j_0 \dots j_{r-1}}$$

with $f_{j_0 \dots j_{r-1}} = g_{i_0 \dots i_r}$ if $i_0 \dots \hat{i_l} \dots i_r = j_0 \dots j_{r-1}$ and $f_{j_0 \dots j_{r-1}} = 0$ else.

Now let $C(G_1, \dots, G_m)$ be the complex

$$\bigoplus_{i_0} G_{i_0} \xleftarrow{\sigma_1} \bigoplus_{i_0, i_1} G_{i_0} \cap G_{i_1} \xleftarrow{\sigma_2} \bigoplus_{i_0, i_1, i_2} G_{i_0} \cap G_{i_1} \cap G_{i_2} \leftarrow \dots$$

with

$$\sigma_r := \sum_{l=0}^{r} (-1)^l \sigma_{r,l}$$

<u>Proposition</u> Suppose that

$$\left(\sum_{i \in V} G_i \right) \cap \left(\bigcap_{j \in W} G_j \right) = \sum_{i \in V} \left(G_i \cap \left(\bigcap_{j \in W} G_j \right) \right)$$

for all subsets $V, W \subseteq \{1, \dots, m\}$. Then $C(G_1, \dots, G_m)$ is an acyclic resolution of $\sum_{i=1}^{m} G_i$.

<u>Remark</u> The given proof shows that $C(G_1, \dots, G_m)$ is homotopy equivalent to the subcomplex $C^+(G_1, \dots, G_m)$:

$$\bigoplus_{i_0} G_i \leftarrow \bigoplus_{i_0 < i_1} G_{i_0} \cap G_{i_1} \leftarrow \bigoplus_{i_0 < i_1 < i_2} G_{i_0} \cap G_{i_1} \cap G_{i_2} \leftarrow \dots$$

Lemma For any injective sheaf I on $\mathcal{V}$ we have (where all groups are considered as subgroups of $I(X)$):

i) $\sum_{i=1}^{m} H^0(X, \mathcal{U}_i; I) = H^0(X, \mathcal{U}; I)$

ii) $\left(\sum_{i \in V} H^0(X, \mathcal{U}_i; I)\right) \cap \left(\bigcap_{j \in W} H^0(X, \mathcal{U}_j; I)\right) = \sum_{i \in V} \left(H^0(X, \mathcal{U}_i; I) \cap \left(\bigcap_{j \in W} H^0(X, \mathcal{U}_j; I)\right)\right)$ for all subsets $V, W \subseteq \{1, \dots, m\}$

Proof i) If $m=2$ and $s \in H^0(X, \mathcal{U}; I)$ we have $s \in I(X)$ and $s|\mathcal{U} = s|\mathcal{U}_1 \cap \mathcal{U}_2 = 0$. By the sheaf property we find a section $s_1' \in I(\mathcal{U}_1 \cup \mathcal{U}_2)$ such that $s_1'|\mathcal{U}_1 = 0$ and $s_1'|\mathcal{U}_2 = s|\mathcal{U}_2$. By the injectivity of I the section s_1' extends to a global section $s_1 \in I(X)$. We set $s_2 := s - s_1$. Then $s_2|\mathcal{U}_2 = 0$ and $s = s_1 + s_2$. The general case follows by induction.

ii) We have $\bigcap_{j \in W} H^0(X, \mathcal{U}_j; I) = H^0(X, \bigcup_{j \in W} \mathcal{U}_j; I)$. Therefore it is enough to consider the case $W = \{1\}$.

$$\left(\sum_{i \in V} H^0(X, \mathcal{U}_i; I)\right) \cap H^0(X, \mathcal{U}_1; I) \underset{(i)}{=} H^0(X, \bigcap_{i \in V} \mathcal{U}_i; I) \cap$$

$$H^0(X, \mathcal{U}_1; I) = H^0(X, (\bigcap_{i \in V} \mathcal{U}_i) \cup \mathcal{U}_1; I) = H^0(X, \bigcap_{i \in V} (\mathcal{U}_i \cap \mathcal{U}_1); I)$$

$$\underset{(i)}{=} \sum_{i \in V} H^0(X, \mathcal{U}_i \cap \mathcal{U}_1; I) = \sum_{i \in V} \left(H^0(X, \mathcal{U}_i; I) \cap H^0(X, \mathcal{U}_1; I) \right) .$$

Now let $\mathcal{F} \simeq I^\cdot$ be an injective resolution. Combining the last Proposition and the Lemma we get an augmented double complex

$$\begin{array}{ccccc}
0 \to & H^0(X, \mathcal{U}; I^0) & \longrightarrow & H^0(X, \mathcal{U}; I^1) & \to \quad \dots \\
\uparrow & \uparrow & & \uparrow & \\
0 & C(H^0(X, \mathcal{U}_1; I^0), \dots, H^0(X, \mathcal{U}_m; I^0)) & \longrightarrow & C(H^0(X, \mathcal{U}_1; I^1), \dots, H^0(X, \mathcal{U}_m; I^1)) & \to \dots
\end{array}$$

in which the columns are acyclic, which gives us the fact that the homology of the total complex is equal to the homology of the complex $H^0(X, \mathcal{U}; I^\cdot)$ which is $H^*(X, \mathcal{U})$.

The homology of the r-th row (counted from above) is $\bigoplus_{i_0, \dots, i_r} H^*(X, \mathcal{U}_{i_0} \cup \dots \cup \mathcal{U}_{i_r})$. Writing down the second spectral sequence of this double complex gives us $(+)$.

Remark on convergence: This sequence is "strongly" convergent since the double complex above is homotopy equivalent to the sub complex $C^+(H^0(X, \mathcal{U}_1; I^\cdot), \dots, H^0(X, \mathcal{U}_r; I^\cdot))$ whose r-th row is zero for $r \geq m$.

[J] Jensen, C.U.: Les foncteurs Dérivés de $\varprojlim$ et leurs Applications en Théorie des Modules. Lect. Notes Math., vol. 254