Quotients of $\Omega^{(d+1)}$

$\Gamma \subseteq PGL(d+1, K)$ discrete cocompact subgrp.
$\Gamma \circlearrowright \Omega^{(d+1)}$ without fixed pts.

Aim: 1.) Realize $\Gamma \backslash \Omega^{(d+1)}$ as a rigid-analytic space
2.) Relate $H^*(\Omega^{(d+1)})$ + Group cohom. of Γ to $H^*(\Gamma\backslash\Omega^{(d+1)})$.

$X_\Gamma := \Gamma\backslash\Omega^{(d+1)}$, $pr: \Omega^{(d+1)} \longrightarrow X_\Gamma$

As a G-ringed space

Topology: • $\mathcal{U} \subseteq X_\Gamma$ adm. open, if $pr^{-1}(\mathcal{U})$ adm. open
• $\mathcal{U} \subseteq X_\Gamma$ adm. open, $\underline{\mathcal{U}}$ covering of $\mathcal{U}$ by adm. opens is admissible, if $pr^*\underline{\mathcal{U}}$ adm.
(Quotient topology)

Structure Sheaf: Γ-act. on $\Omega^{(d+1)}$ $\Rightarrow$ (right) Γ-act. on $pr_*\mathcal{O}_{\Omega^{(d+1)}}$

Put $\mathcal{O}_{X_\Gamma}(\mathcal{U}) := \mathcal{O}_{\Omega^{(d+1)}}(pr^{-1}(\mathcal{U}))^\Gamma$ $\quad \mathcal{U} \subseteq X_\Gamma$ adm. op.

$\Rightarrow pr: \Omega^{(d+1)} \longrightarrow X_\Gamma$ Γ-eq. morph. of G-ringed.

Now: X_Γ as rigid space

Main tool: map $\rho: \Omega^{(d+1)} \longrightarrow |\mathcal{BT}|$

Lemma (Goldman-Iwahori) There exists bij.:

$$|\mathcal{BT}| \xrightarrow[\text{bij.}]{\sim} \mathrm{Norm}(K^{d+1})/_\sim$$

$|\mathcal{BT}|_{\mathbb{Q}}$ corr. to rational norms ($\log_{q} v(K^{d+1}) \subseteq \mathbb{Q}$)

Vertices in $\mathcal{BT}$ corr. to integral norms ($\log_{q} v(K^{d+1}) \subseteq \mathbb{Z}$)

Proof: (Idea) • Let M be a $\mathcal{O}$-lattice in K^{d+1}. $M = \bigoplus_{i=0}^{d} \mathcal{O}e_i$
$v_M(\underbrace{w_0,\dots,w_d}_{=:w}) := \max |\lambda_i|$, λ_i are $(e_0,\dots,e_d)$-coord. of w

• $(\alpha_0,\dots,\alpha_n) \in |\sigma|$, $\sum_{i=0}^{n} \alpha_i = 1$, ~~$\sigma$~~ $\sigma = \{[M_0],\dots,[M_n]\}$

$$v := \max_{0 \le i \le n} \left\{ q^{\alpha_i + \dots + \alpha_n} v_{M_i} \right\}$$

Define $\rho : \Omega^{(d+1)} \longrightarrow |\mathcal{BT}|$ by

$$\underset{(\hat{K}\text{-rat pt.})}{(z_0 : \dots : z_d)} \longmapsto \left((w_0, \dots, w_d) \mapsto \left| \sum_{i=0}^{d} w_i z_i \right| \right)$$

$PGL(d+1, K)$-action on $|\mathcal{BT}|$:

$$g \cdot v(x) := v(g^{-1} x) \qquad v \in \mathrm{Norms}(K^{d+1})/\sim$$

$\Rightarrow$ ρ is $PGL(d+1, K)$-equivariant.

<u>Prop. (Drinfeld)</u>: there ex. a family $\mathcal{U}_\sigma^a$ of open affinoids in $\Omega^{(d+1)}$, $\sigma \in \mathcal{BT}$ simplex, $0 < a < 1$ rational, such that

1) $0 < a < 1$
$(\mathcal{U}_\sigma^a)_{\substack{\sigma \in \mathcal{BT} \\ \text{simplex}}}$ adm. covering of $\Omega^{(d+1)}$ such that the nerve $N((\mathcal{U}_\sigma^a)_{\sigma \in \mathcal{BT}})$ equals the barycentric subdivision $\mathcal{B}(\mathcal{BT})$.

2.) $g(\mathcal{U}_\sigma^a) = \mathcal{U}_{g\sigma}^a \qquad \forall g \in PGL(d+1, K)$

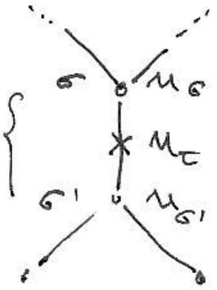
3.) $\mathcal{U}_\sigma^a \Subset_K \mathcal{U}_\sigma^b \qquad 0 < a < b < 1$

<u>Notations</u>:
- $N(\underline{\mathcal{U}})$, $\underline{\mathcal{U}}$ covering, is a simpl. compl.
  Vertices: $\mathcal{U} \in \underline{\mathcal{U}}$
  $\mathcal{U}_1, \dots, \mathcal{U}_r \in \underline{\mathcal{U}}$ simplex, if $\bigcap_{i=1}^{r} \mathcal{U}_i \neq \emptyset$.

- X simpl. compl.; $\mathcal{B}(X)$ a simpl. compl.
  Vertices $\mathcal{B}(X)$: simplices of X
  $\sigma_1, \dots, \sigma_r \in \mathcal{B}(X)$ simplex, if $\sigma_1 \le \dots \le \sigma_r$.
- $\mathcal{U} \Subset_K V = \mathrm{Sp}\, A$, if ex. affinoid gen. f_i of A over K s.t. $\mathcal{U} \subseteq \{ x \in V ;\ |f_i(x)| < 1 \ \forall i \}$.

<u>Meaning of 1.)</u>: <u>Ex.</u> $K = \mathbb{Q}_2$, $d=1$

$\Rightarrow |\mathcal{BT}| =$



$(1.) \Rightarrow \mathcal{U}_\sigma \cap \mathcal{U}_\tau \neq \emptyset$

$\mathcal{U}_\sigma \cap \mathcal{U}_{\sigma'} = \emptyset$

<u>Proof of Prop: (Sketch)</u>

ex. metric d on $|\mathcal{BT}|$. Let σ be simplex in $\mathcal{BT}$

$k := \operatorname{codim} \sigma$

$$V_\sigma^a := \Big\{ y \in |\mathcal{BT}|_{\mathbb{Q}} \,;\ d(x,y) \leq 1 - \frac{3-a}{4^{k+1}} \quad \forall x \in \sigma, \quad \sum_{x\in\sigma} d(x,y) \leq k + \frac{1+a}{4^{k+1}} \Big\}$$

has properties 1.) – 3.)

Drinfeld, Prop. 6.1: $X_a := \{ z \in \Omega^{(d+1)} \,;\ \sum_{i=0}^{d} d(x_i, \rho(z)) \leq a \}$

(x_i vertices) are affinoid open, $X_a \Subset_K X_b$ $0<a<b<1$.

$\widehat{\Omega}^{(d+1)}$ sep. $\qquad \rho^{-1}(V_\sigma^a) =: \mathcal{U}_\sigma^a$ do the job.

ρ surj. ($\to$ prop (1.))

ρ equivariant ($\to$ (2.))

□

<u>Admissible covering for X_Γ:</u>

Γ_σ stabilizer of $\sigma \in \mathcal{BT}$ simplex.

Goldman - Iwahori, 3.3 $\Longrightarrow \underset{\text{discrete}}{\Gamma_\sigma} \leq \underset{\text{compact}}{PGL(d+1, K)_\sigma}$

$\Rightarrow \Gamma_\sigma$ finite.

<u>Thm</u> (Schneider, Stuhler §5 Thm 1)

X_Γ is a proper smooth rigid variety over K.

<u>Proof</u>: $\sigma \in \mathcal{BT}$ simplex

$g\sigma$, σ are never adjacent in $\mathcal{B}(\mathcal{BT})$ for any $g \in \Gamma \setminus \Gamma_\sigma$.

Property 1.) $\Longrightarrow \mathcal{U}^a_{g\sigma} \cap \mathcal{U}^a_\sigma = \emptyset \quad \forall g \in \Gamma \setminus \Gamma_\sigma$.

Prop. 2.) $\Longrightarrow g\mathcal{U}^a_\sigma \cap g'\mathcal{U}^a_\sigma = \emptyset \Leftrightarrow g\Gamma_\sigma \neq g'\Gamma_\sigma$.

Put
$$\mathcal{U}^a_{\Gamma_\sigma} := \coprod_{g \in \Gamma/\Gamma_\sigma} g(\mathcal{U}^a_\sigma)$$

$$X^a_{\Gamma,\sigma} := pr(\mathcal{U}^a_{\Gamma_\sigma})$$

$\mathcal{BT}$ locally finite; Let τ be a simplex in $\mathcal{BT}$

$\Rightarrow \#\{g\sigma \in \mathcal{B}(\mathcal{BT});\ g \in \Gamma,\ g\sigma \text{ adj. to } \tau\} < \infty$

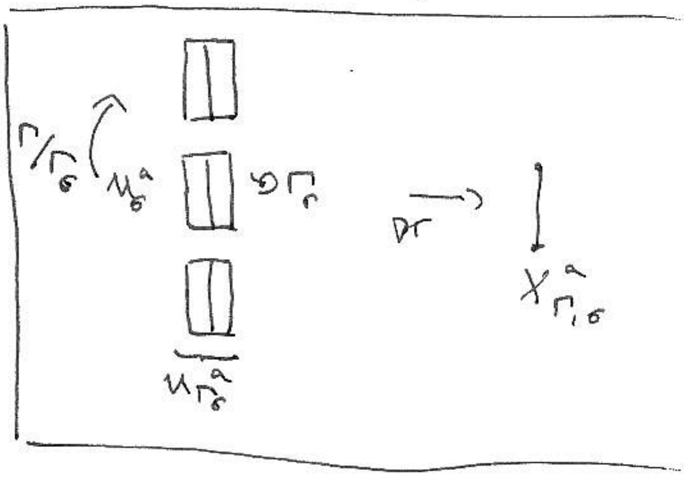
$\overset{\text{Propert. 1.)}}{\Longrightarrow}$ ex. $g_1, \dots, g_r \in \Gamma$ s.t.

$$\mathcal{U}^a_\tau \cap \mathcal{U}^a_{\Gamma_\sigma} = \bigcup_{i=1}^r \mathcal{U}^a_\tau \cap \mathcal{U}^a_{g_i\sigma}$$

$\Longrightarrow \mathcal{U}^a_{\Gamma_\sigma}$ admissible open, hence $X^a_{\Gamma,\sigma}$ adm. open.

$\Gamma_\sigma \circlearrowright \mathcal{U}^a_\sigma$; Drinfeld 6.3 $\Longrightarrow \Gamma_\sigma \setminus \mathcal{U}^a_\sigma = Sp\, \mathcal{O}_{\mathcal{U}^a_\sigma}(\mathcal{U}^a_\sigma)^{\Gamma}$.

$\Gamma_\sigma \setminus \mathcal{U}^a_\sigma = \Gamma \setminus \mathcal{U}^a_{\Gamma_\sigma} \underset{\text{by Def.}}{=} X^a_{\Gamma,\sigma}$



Γ cocompact

$\underset{\mathcal{BT} = Gl(d+1,K)/Gl(d+1,\mathcal{O})}{\Longrightarrow}$ ex. $\sigma_1, \dots, \sigma_r$ simpl. in $\mathcal{BT}$, s.t. $\mathcal{BT} = \coprod_{i=1}^r \Gamma\sigma_i$

$\Longrightarrow X_\Gamma = \bigcup_{i=1}^r X^a_{\Gamma,\sigma_i}$ adm. affinoid cover, s.t.

$$\underset{\Gamma_{\sigma_i} \cap \Gamma_{\sigma_j} \setminus \mathcal{U}^a_{\sigma_i} \cap \mathcal{U}^a_{\sigma_j}}{\underset{\parallel}{X^a_{\Gamma,\sigma_i} \cap X^a_{\Gamma,\sigma_j}}} \longrightarrow X^a_{\Gamma,\sigma_i} \times_K X^a_{\Gamma,\sigma_j} \quad \text{closed imm.}$$

affinoid

$\Longrightarrow X_\Gamma$ separated; $X^a_{\Gamma,\sigma} \Subset_K X^b_{\Gamma,\sigma} \Longrightarrow X_\Gamma$ proper $\square$

<u>Def.</u>: $f : Y \to X$ morphism of K-an. spaces.

- f <u>étale covering</u>, if f surj.) and $\forall x \in X$
  $\exists\ \mathcal{U}_x \subseteq X$ open aff. nbhd of x, s.t. there is a family $(V_{i,x})_{i \in I}$ of aff. open subspaces in Y ~~st.~~ with $f^{-1}(\mathcal{U}_x) = \coprod_{i \in I} V_{i,x}$, $f|_{V_{i,x}} : V_{i,x} \to \mathcal{U}_x$ isom.
- f <u>analytic covering</u>, if: same as above, but $(\mathcal{U}_x)_{x \in X}$, $(V_{i,x})$ adm. coverings

<u>Thm</u> (Schn, St. 5.1) cted.
pr is étale covering ($\Leftrightarrow X_\Gamma$ smooth)
Γ torsion free $\Rightarrow$ pr analytic covering

<u>Proof</u>: $x \in X^a_{\Gamma,\sigma_i} \subseteq X_\Gamma$, $pr_i := pr|_{\mathcal{U}^a_{\sigma_i}}$

Drin. 6.3 $\Rightarrow$ pr_i finite

vdPut $\Rightarrow$ pr_i ét. covering, iff $\forall y \in \mathcal{U}^a_{\sigma_i}$

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{X^a_{\Gamma,\sigma_i},pr_i(y)} & \longrightarrow & \widehat{\mathcal{O}}_{\mathcal{U}^a_{\sigma_i},y} \quad \text{Isom.} \\ \| & & \| \\ \widehat{\mathcal{O}_{X^a_{\Gamma,\sigma_i}}(X^a_{\Gamma,\sigma_i})_{pr_i(y)}} & \xrightarrow{\sim} & \widehat{\mathcal{O}_{\mathcal{U}^a_{\sigma_i}}(\mathcal{U}^a_{\sigma_i})_y} \end{array}$$

$\Uparrow$
Mumford, ab. Var.

$\Rightarrow$ pr étale covering.

Now let Γ be torsion free $\underset{\#\Gamma_\sigma < \infty}{\Longrightarrow}$ $\Gamma_\sigma = 1$.

$\Rightarrow X^a_{\Gamma,\sigma_i} = \mathcal{U}^a_{\sigma_i}$

and $\underbrace{pr^{-1}(X^a_{\Gamma,\sigma_i})}_{\overset{\text{by Def}}{=} \mathcal{U}^a_{\Gamma\sigma_i}} = \coprod_{g \in \Gamma} g\,\mathcal{U}^a_{\sigma_i}$ by definition $=$

$\Rightarrow$ pr analytic, as $(g\mathcal{U}^a_{\sigma_i}), (X^a_{\Gamma,\sigma_i})$ adm. $\square$

Cohomology of Quotient

Lemma: $\mathcal{V}$ = Cat. of smooth + sep. K-spaces + G-top (finer than analyt. top)

1.) $\mathcal{G}$ inj sheaf on $\mathcal{V}$ $\Rightarrow$ $\mathcal{G}(\Omega^{(d+1)})$ is an inj. Γ-module.

2.) If G-top on $\mathcal{V}$ finer than étale top. or if Γ torsion free
$\Rightarrow$ $\mathcal{G}(X_\Gamma) = \mathcal{G}(\Omega^{(d+1)})^\Gamma$ for any sheaf $\mathcal{G}$ on $\mathcal{V}$.

Proof: 2.) Γ torsion free $\xRightarrow[\text{above}]{\text{see}}$ pr analyt. covering.

$\Rightarrow$ $\mathcal{G}(X_\Gamma) \longrightarrow \mathcal{G}(\Omega^{(d+1)}) \overset{pr_1^*}{\underset{pr_2^*}{\rightrightarrows}} \mathcal{G}(\Omega^{(d+1)} \times_{X_\Gamma} \Omega^{(d+1)})$ exact. ⊛

(as $\mathcal{G}$ takes coprod. to prod).

pr is Galois-covering, i.e.

$$\coprod_{g\in\Gamma} \Omega^{(d+1)} \xrightarrow{\sim} \Omega^{(d+1)} \times_{X_\Gamma} \Omega^{(d+1)}$$

x in g-comp. $\longmapsto$ (gx, x) isomorphism.

$\overset{⊛}{\Longrightarrow}$ $\mathcal{G}(X_\Gamma) \to \mathcal{G}(\Omega^{(d+1)}) \overset{pr_1^*}{\underset{pr_2^*}{\rightrightarrows}} \prod_{g\in\Gamma} \mathcal{G}(\Omega^{(d+1)})$

$g \longmapsto g|_{\Omega^{(d+1)}}$

$f \overset{pr_1^*}{\longmapsto} (f\cdot 1, \dots, f\cdot g, \dots)$

$f \overset{pr_2^*}{\longmapsto} (f, f, \dots)$

hence: $f \in \ker(pr_1^*, pr_2^*) = \mathcal{G}(X_\Gamma) \Leftrightarrow f \in \mathcal{G}(\Omega^{(d+1)})^\Gamma$. ∎

The same argument holds, if top on $\mathcal{V}$ finer than ét. top.

1.) consider

$$\begin{tikzcd}
(\text{sheaves on } \mathcal{V}) \arrow[rr, "F'"] \arrow[dr, "\text{glob. sections}"'] & & (\Gamma\text{-modules}) \arrow[ll, bend right, "F"'] \arrow[dl, "(\ )^{\Gamma}"] \\
& (Ab) &
\end{tikzcd}$$

$$\mathcal{G} \longmapsto \mathcal{G}(\Omega^{(d+1)})$$

Construct F left-adj. + exact

$\Longrightarrow$ F resp. inj. objects.

Put, for any Γ-module M, any $Y \in Ob(\mathcal{V})$

$$\tilde{F}(M)(Y) := \bigoplus_{x \in \Omega^{(d+1)}(Y)} M \Big/ \text{subgrp gen. by } m - gm,\ g \in \Gamma, \text{ s.t. } m \text{ in } x\text{-cpt., } gm \text{ in } gx\text{-cpt.}$$

$F := \tilde{F}^{\#}$ ass. sheaf on $\mathcal{V}$.

$\square$

Cor.: Γ torsion~~free~~ or top on $\mathcal{V}$ finer than ét. top.

$\Rightarrow$ ex. spectral sequence

$$H^r(\Gamma, H^s(\Omega^{(d+1)})) \Longrightarrow H^{r+s}(X_\Gamma) \qquad (**)$$

Proof: Take spectr. seq ass. to „$(\ )^\Gamma \circ F'$".

$\square$

Claim: Spectr. seq. $(**)$ ex., if $H^* = H^*_{DR}$

Lemma: $f: X \to Y$ étale $\Rightarrow$ $f^*\Omega^1_{Y/k} = f^{-1}\Omega^1_{Y/k} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X = \Omega^1_{X/k}$

Proof: consider can. morph.

$$f^*\Omega^1_{Y/k} \longrightarrow \Omega^1_{X/K}.$$

$\Omega_{\cdot/k}$ coherent $\Rightarrow$ WLOG X, Y affinoid.

descent $\Rightarrow$ WLOG $k = \bar{k}$

Esn. talk $\Rightarrow$ f has finite fibres $\Rightarrow$ WLOG f injective

BGR 7.3.3/5 $\Rightarrow$ f is a local isom. at $x \in X$

iff $\hat{f}_x : \hat{\mathcal{O}}_{Y, f(x)} \longrightarrow \hat{\mathcal{O}}_{X,x}$ isom.

By SGA 1, Exp. 1, 4.4:

$\hat{f}_x$ isom. $\Leftrightarrow$ f_x étale (and x k-rat.) $\square$

Let $\mathcal{F}$ be a sheaf on X_{an}.

$W(\mathcal{F})(U \xrightarrow[\text{ét}]{f} X) := f^* \mathcal{F}(U)$ sheaf on étale site

By def: $H^*_{DR}(X) = \mathbb{H}^*(X_{an}, \Omega^\bullet)$

Lemma $\Rightarrow$ $W(\Omega^1_{X/k})(U \xrightarrow[\text{ét.}]{f} X) = \Omega^\cdot_{U/k}(U)$

Esn. talk (van der Put + de Jong) $\Rightarrow$ ét. cohom. of $W(\Omega^\cdot_X)$

$=$ an.-cohom. of $\Omega^\cdot_X$

Hochschild–Serre spectral seq for small ét. site $\Rightarrow$ claim