Purpose of today's and tomorrow's talk: Compute $H^s(\Omega^{(d+1)})$ in terms of generalized harmonic cochains.

Main objective:

$$H^s(\Omega^{(d+1)}) \xrightarrow[\text{with subspace}]{\text{identify}} \left\{ \text{s-dim. cochains on } \mathcal{BT}/SL_{d+1}(K) \right\}.$$

More precisely: R.'s last corollary

At several points throughout this talk I will relate my proceeding to the above objective, in particular: to what will happen tomorrow.

__First:__ let us give some setup

let $G = GL_{d+1}(K)$, as before. $(K, |\cdot|), \mathcal{O}, \pi \in \mathcal{O}$ as before

Consider $p: GL_{d+1}(\mathcal{O}) \longrightarrow GL_{d+1}(\mathcal{O}/\pi)$,

Set $B := p^{-1}\left(\left\{\begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}\right\}\right)$,

i.e.

$$B = \left\{\begin{pmatrix} \mathcal{O}^\times & & \mathcal{O} \\ & \ddots & \\ \pi\mathcal{O} & & \mathcal{O}^\times \end{pmatrix}\right\}$$

Standard Iwahori subgroup of G

Again, let

$$P := P_\emptyset = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix}\right\}$$

G-stabilizer of the flag $\mathcal{T}_\emptyset: Ke_1 \subsetneq \dots \subsetneq K^{d+1}$ (cf. Jan's talk)

let χ = char. fct. of $\underbrace{BP/P}_{\text{cpt., open in } G/P} \subset G/P$

For later use:
- For $y \in G$ let χ_y = char. fct. of $ByB \subset B\backslash G/B$
- For $M \subset G$ arb. subset let χ_M = char. fct. of M

Recall: G/P cpt.: $G/P = \underbrace{\mathbb{G}/\mathbb{P}(K)}_{\substack{\text{proj. variety}/K \\ \text{i.e. closed subset of cpt. space}}}$

We study

$$H: \underbrace{C_c^\infty(\overbrace{G/B}^{\substack{\text{discrete} \\ (B \in G \text{ open})}}, \mathbb{Z})}_{\ni f \text{, s.t. } f \text{ has finite support}} \longrightarrow C^\infty(G/P, \mathbb{Z}), \quad \varphi \longmapsto \varphi * \chi, \quad := \sum_{g \in G/B} \underbrace{\varphi(g)}_{\in \mathbb{Z}} \cdot \underbrace{g(\chi)}_{\substack{\text{cf. action} \\ \text{of } G \text{ on} \\ C^\infty(G/P, \mathbb{Z})}}$$

$\Longrightarrow H$ is a homomorphism of G-modules

Note: The G-action is given by translation, i.e.

$$gf: x \longmapsto \underbrace{f(g^{-1}x)}_{=: gf}$$

(f, x appropriate),
in part.: $\mathbb{Z}[G]$-module structure.

In this talk:

- $\#$ is surj.,
- ⇝ computation of $\ker(\#)$

Set

$$T^{++} := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_{d+1} \end{pmatrix} \in G,\ 1 \geq |t_1| \geq \dots \geq |t_{d+1}| \right\} \subset Gl_{d+1}(\mathcal{O})$$

⚠ T^{++} is only a semigroup $\subset Gl_{d+1}(\mathcal{O})$
(plain, looking at $|.|$)

Furthermore, set

$$t := \begin{pmatrix} 1 & & & \\ & \pi & & \\ & & \ddots & \\ & & & \pi^d \end{pmatrix}, \qquad y_j = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \pi & & \\ & & & & \ddots & \\ & & & & & \pi \end{pmatrix} \Big]\, j \qquad (j = 0, \dots, d)$$

Reminder:

P_I = stabilizer of the flag τ_I (cf. Jan's talk), $I \subset \Delta = \{1, \dots, d\}$,
$\Longrightarrow P_I$ parabolic subgroup of G, $\supset \underbrace{P}_{= P_\emptyset}$

Set

U_I = unipotent radical of P_I
= largest normal and unipotent subgroup of P_I,
(cf. [Borel, LAG], 11.21)

$$U := U_\emptyset = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\}, \qquad U^- := w_\Delta U w_\Delta$$

where

w_Δ = unique elt. of max. length in W_Δ = {Permutation matrices}
$\hat{=} \begin{pmatrix} 1 & 2 & \dots & d+1 \\ d+1 & d & \dots & 1 \end{pmatrix}$

⚠ $w_\Delta = w_\Delta^{-1}$, $\longrightarrow U^- = \left\{ \begin{pmatrix} 1 & & \\ & \ddots & \\ * & & 1 \end{pmatrix} \right\}$

Facts:

- Iwahori decomposition $B = \underbrace{(B \cap U^-)}_{= \left\{ \begin{pmatrix} 1 & \\ \pi\mathcal{O} & 1 \end{pmatrix} \right\}} \underbrace{(B \cap P)}_{= \left\{ \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ & \mathcal{O}^\times \end{pmatrix} \right\}}$
- $(+) \begin{cases} (1.)\ y (B \cap U^-) y^{-1} \subset B \cap U^- \\ (2.)\ y^{-1} (B \cap P) y \subset B \cap P \end{cases} \quad (y \in T^{++})$

Why study H? There is a bijection (incl. pres.)

$$\text{Subsets of } \Delta \xrightarrow{1:1} \text{subgroups of } Gl_{d+1}(\mathcal{O}), \supset B$$

$$I \longmapsto B_I := BW_IB,$$

the sets B_I investigated by Rajneesh.

There is a diagram

$$\begin{tikzcd} C_c^\infty(G/B, \mathbb{Z}) \arrow[r, two heads, "H", "G\text{-epim}"'] & C^\infty(G/P, \mathbb{Z}) \\ G_c^\infty(G/B_I, \mathbb{Z}) \arrow[u, hook] \arrow[r, two heads, "H_I", "G\text{-epim}"'] & C^\infty(G/P_I, \mathbb{Z}) \arrow[u, hook] \end{tikzcd}$$

$\hookrightarrow \hat{=}$ G-submodules

and $\ker(H_I) = \langle \chi_{By_jB_I} - \chi_{B_I}, I \rangle$

It will turn out: $H^s(\Omega^{(d+1)}) \underset{G\text{-equivar.}}{\simeq} \mathrm{Hom}_{\mathbb{Z}}\left(C_c^\infty(G/B_I, \mathbb{Z})_{/R_I}, A\right)$

R_I: certain relations in terms of char. fcts. of cosets

$I := \{1, \dots, d-s\}$

$\boxed{\text{R.h.s. for } I = \emptyset \ \hat{=} \text{ space of harmonic cochains in the sense of [Gar]}}$

Now:

Surjectivity of H $\quad 1^{st}$ step: $C^\infty(G/P, \mathbb{Z}) = \langle \chi \rangle_G$

Propos. 4.7 let $b, b' \in Gl_{d+1}(\mathcal{O})$, $y \in T^{++}$, then

$$byBP \cap b'yBP \neq \emptyset \quad \text{implies} \quad byBP = b'yBP,$$

and $byB = b'yB$. In particular, $bB = b'B$.

Proof.

Mult with some $c \in Gl_{d+1}(\mathcal{O}) \implies$ WLOG $b' = 1$.

Iwahori decomp $\implies$

$$byBP = by(B \cap U^-)(B \cap P)P$$
$$= by(B \cap U^-)y^{-1}\underbrace{y(B \cap P)P}_{=P}$$

Show both inclusions elementwise

$\triangle$ $y^{-1} \in G$, not nec. $y^{-1} \in T^{++}$ (in general: NO)

Likewise $yBP = y(B \cap U^-)y^{-1}P$,

$$\implies \emptyset \underset{\text{provided}}{\subsetneq} byBP \cap yBP = b\underbrace{y(B \cap U^-)y^{-1}}P \cap \underbrace{y(B \cap U^-)y^{-1}}P$$
$$\subset b(B \cap U^-)P \cap (B \cap U^-)P$$
$$\subset bBP \cap BP,$$

$\implies \exists c \in bBP \cap BP$, i.e. $c = bb_0p = b_1p'$ for some $b_0, b_1 \in B$, $p, p' \in P$,

i.e. $c = bb_0 = b_1p \quad (p' \mapsto p^{-1}p')$,

$$\implies p \in \underbrace{Gl_{d+1}(\mathcal{O})}_{p = b_1^{-1}bb_0} \cap P \underset{\text{(easy)}}{\subset} B \cap P, \implies b = b_1\underset{B}{\hat{p}}b_0^{-1} \in B,$$

$\underset{\text{Iwahori}}{\Longrightarrow}$ $b \in \underbrace{(B\cap U^-)}_{\ni b''}\underbrace{(B\cap P)}_{\ni b'''}$, $\Longrightarrow$ $b = b''b'''$,

$\Longrightarrow (b'')^{-1}b \in B\cap P$, $\overset{(*)}{\Longrightarrow}$ $b^{-1}b'' \in B\cap P$,

$\Longrightarrow b''yBP = \underbrace{bb^{-1}}_{=1}b''yBP \subset b(B\cap P)yBP = byy^{-1}\underbrace{(B\cap P)y}BP$
$\subset B\cap P$

$\subset by\underbrace{(B\cap P)}_{\subset B, \subset P}BP \subset byBP$, and vice versa (cf. $\underset{\text{"}\Leftarrow\text{"}}{(*)}$),

$\Longrightarrow b''yBP = byBP$

Now

$\emptyset \neq \underbrace{byBP}_{=b''yBP} \cap yBP = b''yBP \cap yBP \underset{\text{as above}}{=} b''y(B\cap U^-)y^{-1}P \cap y(B\cap U^-)y^{-1}P$,

$\subset B\cap U^-$ $\subset B\cap U^-$

$\Longrightarrow b''y(B\cap U^-)y^{-1} \cap \underbrace{y(B\cap U^-)y^{-1}}_{\text{nec. } \ni b''} \neq \emptyset$,

$\Longrightarrow b'' \in y(B\cap U^-)y^{-1}$,

$\underset{\text{via Iwahori decomp above}}{\Longrightarrow}$ $b \in y(B\cap U^-)y^{-1}(B\cap P)$, $\Longrightarrow$ $byBP = yBP$

$\|$ $\|$

$by(B\cap U^-)y^{-1}P = y(B\cap U^-)y^{-1}P$

concludes the proof $\square$

Considering $y = t^n$ $(n \geq 0)$, we state

Propos. 4.8

Let $C \subset G/P$ be compact, open, $\Longrightarrow$ for any $n \geq 0$ large enough, C has a representation as finite disjoint union of sets of type

$$bt^nBP/P \qquad (b \in Gl_{d+1}(\mathcal{O}))$$

Proof (outline) $\left.\begin{array}{l}C \text{ is compact (finite union requ.)}\\ \text{Prop. 4.7 on intersections (disj. union)}\end{array}\right\} \Longrightarrow$ s.t.s. $\left\{bt^nBP/P,\ \begin{array}{l}b \in Gl_{d+1}(\mathcal{O})\\ n \geq 0\end{array}\right\}$ basis of open sets in G/P

Set $\mu_n = t^n(B\cap U^-)t^{-n}$,

$\Longrightarrow \mu_n \supset \mu_{n+1}$. Set $\mathcal{L} := \{\text{cosets of } \mu_{n+1} \text{ in } \mu_n\}$,

$\Longrightarrow \underbrace{1}_{=b}t^nBP \underset{\text{above}}{=} \mu_nP = \bigcup_{x\in\mathcal{L}} x\mu_{n+1}P \underset{\text{above}}{=} \bigcup_{x\in\mathcal{L}} xt^{n+1}BP$

View the latter as a "representation",

we have $G = Gl_{d+1}(\mathcal{O})P$ (Iwasawa decomp.),

$\Longrightarrow$ S.t.s $\{t^nBP/P, n\geq 0\}$ is fundamental system of nbh. for $1 \in G/P$ ($b=1$, use translation of nbh. and Iwasawa)

Now $t^nBP = \underbrace{t^n(B\cap U^-)t^{-n}}_{\subset B^{(n)}}P \subset B^{(n)}P$ where $B^{(n)} = \{b \in B, b \equiv 1 \bmod \pi^n\}$ is fund system of nbh. for $1 \in G$ $\square$

<u>Corollary 4.9</u> $C^\infty(G/P,\mathbb{Z}) = \langle X_1 \rangle_G$, in particular H is surj.

Proof. $G \circlearrowright C^\infty(G/P,\mathbb{Z})$ by translation : $(g,\varphi) \mapsto \underbrace{(\bar{h} \mapsto \varphi(g^{-1}\bar{h}))}_{=: g\varphi}$

$f \in C^\infty(G/P,\mathbb{Z}) \Rightarrow \exists$ covering $G/P = \coprod_i \underbrace{U_i}_{\text{open}}$ st. $f|_{U_i} = a_i X_{U_i}$

where namely $U_i = f^{-1}(\{a_i\})$, $a_i \in f(G/P)$

each U_i is closed, $\rightarrow$ even cpt. (G/P cpt.), $\Rightarrow$ apply 4.8 to U_i and decompose

$X_{U_i} = \sum (\dots) X$, $f = \sum f|_{U_i}$.

<u>H surj.</u>: $\underset{C_c^\infty(G/B,\mathbb{Z})}{\overset{\cap}{X_1}} \mapsto X_1 * X_1 = X_1$, H maps generator to generator. □

Now that we know that H is an epimorphism : determine ker(H).

0) 4.7, 4.8, 4.9 will, for later use, be generalized according to $\begin{cases} P \mapsto P_I \\ B \mapsto B_I \end{cases}$.

<u>Remarks.</u> 1) The Hecke ring of B is $C_c^\infty(\underbrace{B\backslash G/B}_{=\{\text{double } B\text{-cosets of } G\}},\mathbb{Z})$

Ring structure (ass., unit) via convolution:

$$\varphi * \psi = \sum_{g \in G/B} \varphi(g)\, g\psi$$

as above.

2) Also via <u>convolution</u>: <u>right</u>-action $C_c^\infty(G/B,\mathbb{Z}) \circlearrowleft C_c^\infty(B\backslash G/B,\mathbb{Z})$

<u>Definition.</u> Let

$\mathcal{A}$ = subring of $C_c^\infty(B\backslash G/B,\mathbb{Z})$ generated by X_y $(y \in T^{++})$

<u>Recall.</u> $y_i = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \pi & \\ & & & \ddots \\ & & & & \pi \end{pmatrix} \Big] i$

<u>Lemma 4.10</u> $\mathcal{A} = \mathbb{Z}[X_{y_0},\dots,X_{y_d}]$; $\forall y,y' \in T^{++}$ $X_y * X_{y'} = X_{yy'}$

(„$y \mapsto X_y$ respects finite products")

<u>Proof outline</u>

Def. $\Rightarrow X_y * X_{y'} = \sum_{g \in ByB/B} 1\, gX_{y'} = \sum_{g \in ByB/B} X_{gBy'B}$

One shows (arguments: cf. 4.7)

$X_y * X_{y'} = \sum_{x \in \mathcal{L}'} X_{xyBy'B}$ $\quad \mathcal{L}' = \{$cosets of $y(B \cap U^-)y^{-1}$ in $B \cap U^-\}$

$ByyB = \underbrace{\coprod_{x \in \mathcal{L}'} xyBy'B}_{\text{union of sets which are disj., cf 4.7}}$ $\quad$ (we are interested in $X_{yy'}$!)

Combine $\Rightarrow X_y * X_y = X_{yy'}$

$\Longrightarrow$ $\exists$ epimorphism of rings

$$\varepsilon : \mathbb{Z}[X_0, \dots, X_d] \longrightarrow \mathcal{A}, \qquad X_j \longmapsto X_{y_j} \quad (j = 0, \dots, d)$$

Remains: ε inj.

Cartan decompos. $\Longrightarrow$ $\forall_{y \in T^{++}}$ $\exists! \, n_0, \dots, n_d \geq 0$: $ByB = B y_0^{n_0} \cdots y_d^{n_d} B$

(„unique fact. of repres.") $\square$

Convention: View $\mathbb{Z}$ as $\mathcal{A}$-module via

$$\mathcal{A} \longrightarrow \mathbb{Z}, \qquad X_{y_0}, \dots, X_{y_d} \longmapsto 1.$$

Propos. $\#$ induces a G-isomorphism

$$C_c^\infty(G/B, \mathbb{Z}) \underset{\mathcal{A}}{\otimes} \mathbb{Z} \longrightarrow C^\infty(G/P, \mathbb{Z})$$

Proof. $C := C_c^\infty(G/B, \mathbb{Z})$.

S.t.s.: $C \underset{\mathcal{A}}{\otimes} \mathbb{Z}$ is a quotient of C

Then namely:

$$\begin{tikzcd} C \arrow[r, "\#"] \arrow[d] & C^\infty(G/P, \mathbb{Z}) \\ C \underset{\mathcal{A}}{\otimes} \mathbb{Z} \arrow[ur, "\overline{\#}"'] & \end{tikzcd}$$

We know $C = \langle X_1 \rangle_G$. Define $C \xrightarrow{\alpha} C \underset{\mathcal{A}}{\otimes} \mathbb{Z}$, $X_1 \longmapsto X_1 \otimes \underbrace{1}_{\text{gen. of } \mathcal{A}\text{-module } \mathbb{Z}}$

S.t.s. $\forall_{y \in T^{++}}$ $\#(\underbrace{X_1 * (X_y - X_1)}_{\hat{=} \text{ generator of } \ker(\alpha)}) = 0$

Indeed: $\#(X_1 * (X_y - X_1)) = \#(X_y - X_1) = 0$

Namely $ByBP = \underbrace{\coprod_{g \in ByB/B} gBP}_{\text{union of disj. sets of cosets, i.e. disj union}}$,

$$\Longrightarrow \#(X_y - X_1) = \Big(\underbrace{\sum_{g \in ByB/B} X_{gBP}}_{= X_{ByBP}} \Big) - X_{BP} = \underbrace{X_{ByBP} - X_{BP}}_{ByBP = By(B \cap U^-) y^{-1} P = BP} = 0$$

$\#$ surj. $\Longrightarrow$ $\overline{\#}$ surj.

Remains: $\overline{\#}$ injective

Lemma $\forall g \in G \ \exists y \in T^{++}$ s.t. $gByB \subset Gl_{d+1}(O)\,T^{++}B$

Proof $T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \in G \right\}$

$N(T) =$ normalizer, $\Longrightarrow N(T) = WT$

We have $G = Gl_{d+1}(O)\,TB$ (Cartan decomp.), WLOG $g \in T$

$\Longrightarrow$ (theory of generalized Tits systems) $\exists$ finitely many $g_\alpha \in N(T)$

$\forall h \in N(T) \quad gBhB \subset \bigcup_\alpha B g_\alpha h B$

$N(T) = WT \Longrightarrow \forall_\alpha \ g_\alpha = w_\alpha t_\alpha$

Choose $y \in T^{++}$ s.t. $\forall_\alpha \ t_\alpha y \in T^{++}$, $\Longrightarrow gByB \subset \bigcup_\alpha B w_\alpha t_\alpha y B \subset Gl_{d+1}(O)\,T^{++}B$ $\square$

Claim: $C_c^\infty(Gl_{d+1}(O)\,T^{++}B/B) \hookrightarrow \overset{C}{\overset{\|}{C_c^\infty(G/B, \mathbb{Z})}} \longrightarrow \overset{C}{\overset{\|}{C_c^\infty(G/B, \mathbb{Z})}} \otimes_{\mathcal{A}} \mathbb{Z}$
is surj.

Reason: $\chi_{gB} + \chi_{gB} * (\chi_y - \chi_1) = \chi_{gB} * \chi_y = \chi_{gByB}$

above Lemma $\Longrightarrow$ claim $\square$

Propos. $\forall y \in T^{++}$ $\quad H|_{C_c^\infty(Gl_{d+1}(O)\,yB/B, \mathbb{Z})}$ is inj.

Proof Plain, using 4.7 $\square$

Want to apply this, in order to obtain the injectivity of Φ.

Claim $\bigcup_{n \geq 0} C_c^\infty(Gl_{d+1}(O)\,t^n B/B, \mathbb{Z}) \hookrightarrow C \longrightarrow C \otimes_{\mathcal{A}} \mathbb{Z}$

is surj.

Reason Given $y \in T^{++}$, $n \geq 0$ big enough, $\Longrightarrow \exists y' \in T^{++}$ $yy' = t^n$

Using Iwahori $\Longrightarrow yBy'B \subset Byy'B = Bt^nB$,

$$\Longrightarrow \chi_{yB} + \chi_{yB} * (\chi_{y'} - \chi_1) = \underbrace{\chi_{yBy'B}}_{\in C_c^\infty(Bt^nB/B, \mathbb{Z})} \quad \square$$

But now

$$\begin{tikzcd} \bigcup_{n \geq 0} C_c^\infty(Gl_{d+1}(O)\,t^n B/B, \mathbb{Z}) \arrow[r, hook] \arrow[drr, "\text{inj. of } H"'] & C \arrow[r] & C \otimes_{\mathcal{A}} \mathbb{Z} \arrow[d, "\Phi \ \text{inj. on the image of compos.}"] \\ & & C^\infty(G/P, \mathbb{Z}) \end{tikzcd}$$

$\square$

References:

[ScSt]: Schneider, P.; Stuhler, U.: The cohomology of p-adic symmetric spaces; Inv. Math. 105, 1991, pp. 47–122

[Gar]: Garland, H.: p-adic Curvature and the Cohomology of discrete subgroups of p-adic groups, Ann. Math., 2nd Series, Vol. 97, No. 3 (1973), pp. 375–423