Extensions of generalized Steinberg representations

"torsion free or ∇ finite étale coh., or ∇ = de Rham cohomology"

Recall: Under certain assumptions (on Γ, our cohomology theory ...) have SS

$$H^r(\Gamma, H^s(\Omega^{d+1})) \Rightarrow H^{r+s}(X_\Gamma)$$

- $\Gamma \subseteq PGL_{d+1}(K)$ cocompact discrete, free action on Ω^{d+1}
- $X_\Gamma := \Gamma \backslash \Omega^{d+1}$ ("proper smooth rigid variety over K")

Assume the SS is at our disposal.

In this talk: Analyze the initial terms $H^r(\Gamma, H^s(\Omega^{d+1}))$ with the help of (smooth) rep. theory

Introduce for $I \subseteq \Delta := \{1, \dots, d\}$ "set of generators of the W: $\Delta \xrightarrow{\sim} W$, $i \mapsto (i\,i{+}1)$"

"had a bijection $\{\text{subsets of } \Delta\} \xrightarrow{\sim} \{\text{parabolics containing } P_\emptyset\}$, inclusion preserving, $I \mapsto P_I$".

and M any abelian group

$$V_I(M) := C^\infty(G/P_I, M) \Big/ \sum_{i \in \Delta \setminus I} C^\infty(G/P_{I \cup \{i\}}, M)$$

$$= V_I(\mathbb{Z}) \otimes_{\mathbb{Z}} M.$$

"G/P_I is compact $\Rightarrow$ each function on $C^\infty(G/P_I, M)$ is a finite sum of characteristic functions."

"$V_I(M)$ is a smooth G-representation: Generalized M-valued Steinberg representation".

– Our main result says that $H^s(\Omega^{d+1}) = 0$ for $s > d$ and

$$H^s(\Omega^{d+1}) \cong \mathrm{Hom}_{\mathbb{Z}}(V_{\{1,\dots,d-s\}}(\mathbb{Z}), A)$$

$$= \mathrm{Hom}_A(V_{\{1,\dots,d-s\}}(A), A) \quad \text{for } 0 \le s \le d. \quad (\S 3 \text{ Thm } 1 + \S 4 \text{ Cor. } 17)$$

adjointness of $\otimes$ and "res".

~~(... equal A, for $s=1$ ...)~~

"$C_c^\infty(G/B_I, \mathbb{Z})/R_I \overset{H}{\cong} V_I(\mathbb{Z})$." operator H, "$I = \{1, \dots, d-s\}$" (see talk on generalized harmonic cochains)

Moreover: $H^r(\Gamma, H^s(\Omega^{d+1})) = H^r(\Gamma, \mathrm{Hom}_A(V_{\{1,\dots,d-s\}}(A), A))$

$$\underset{(*)}{=} \mathrm{Ext}^r_{A[\Gamma]}(V_{\{1,\dots,d-s\}}(A), A) \quad \text{for } 0 \le s \le d$$

(*): from right adjointness of $\mathrm{Hom}_A(V_I(A), \cdot)$ to the exact functor $V_I(A) \otimes_A (\cdot)$ on $A\Gamma$-mod. free A-module by §4 Cor. 5. $\to$ $V_I(\mathbb{Z})$ has a finite filtration with quotients of the form $C_c^\infty(S, \mathbb{Z})$, S locally profinite, metrizable $\Rightarrow$ Borel-Serre $\mathbb{Z}$-free.

$\Rightarrow \mathrm{Hom}_A(V_I(A), \cdot)$ preserves injectives.

Rem: In case A is a field use that we have standard injective⊛ resolutions $A \to C^\bullet(\Gamma, A)$ (is $A\Gamma$-mod) & $\mathrm{Hom}_A(V_I(A), A) \to C^\bullet(\Gamma, \mathrm{Hom}_A(V_I(A), A))$ since the A-modules A & $\mathrm{Hom}_A(V_I(A), A)$ are A-free(!?). (⊛ use Frob-reciprocity!)

~~Then ... use $\mathrm{Hom}_A(V_I(A), C^\bullet(\Gamma, A)) \xrightarrow{\sim} C^\bullet(\Gamma, \mathrm{Hom}_A(V_I(A), A))$, take Γ-invariants and homology h...~~

~~"explicitly: $A \to C^\bullet(\Gamma, A)$ standard resolution, $C^q(\Gamma, A) = \{\Gamma^q \to A\} \circlearrowleft \Gamma$~~

~~Since $V_I(A)$ is A-free $\to \mathrm{Hom}_A(V_I(A), A)$ A-free whence $C^\bullet(\Gamma, \mathrm{Hom}_A(V_I(A), A)) := C^\bullet(\Gamma, A) \otimes_A \mathrm{Hom}_A(V_I(A), A)$~~

~~injective resolution of $\mathrm{Hom}_A(V_I(A), A)$. Now use the $A\Gamma$-isomorphism~~

~~$\mathrm{Hom}_A(V_I(A), C^\bullet(\Gamma, A)) \xrightarrow{\sim} C^\bullet(\Gamma, \mathrm{Hom}_A(V_I(A), A))$~~

~~$\varphi \mapsto \{\gamma \mapsto \{v \mapsto \varphi(v)(\gamma)\}\}$.~~

~~Taking Γ-invariants the LHS computes $\mathrm{Ext}^\bullet_{A\Gamma}(V_I(A), A)$ and the RHS computes $H^\bullet(\Gamma, \mathrm{Hom}_A(V_I(A), A))$."~~

~~(compare A.3 [13])~~

~~⊛ because $\mathrm{Hom}_{A\Gamma}(W, C^\bullet(\Gamma, M)) = \mathrm{Hom}_A(W, M)$ by Frobenius reciprocity.~~

$\Rightarrow$ Study the groups $\mathrm{Ext}^r_{A\Gamma}(V_I(A), A)$!

Proposition 41:

i. $V_I(\mathbb{Z})$ has a proj. res by finite free $\mathbb{Z}\Gamma$-modules

ii. $\mathrm{Ext}^r_{\mathbb{Z}\Gamma}(V_I(\mathbb{Z}),\mathbb{Z})$ is finitely generated for $r \geq 0$

iii. There is a natural exact sequence for any $r \geq 0$

$$0 \longrightarrow \mathrm{Ext}^r_{\mathbb{Z}\Gamma}(V_I(\mathbb{Z}),\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow \mathrm{Ext}^r_{A\Gamma}(V_I(A),A) \longrightarrow \mathrm{Tor}^{\mathbb{Z}}_1(\mathrm{Ext}^{r+1}_{\mathbb{Z}\Gamma}(V_I(\mathbb{Z}),\mathbb{Z}),A) \to 0.$$

Pr:

i. Uses all of § 6 (cf. Prop. 16).

ii. follows from i. since $\mathbb{Z}$ is noetherian.

iii. $F_\bullet \to V_I(\mathbb{Z})$ proj. resolution by finite free $\mathbb{Z}\Gamma$-modules

$\Rightarrow$ $F_\bullet \otimes A \to V_I(A)$ is proj. res by finite free $A\Gamma$-modules ("$F_\bullet \otimes A$ computes $\mathrm{Tor}^{\mathbb{Z}}_*(V_I(\mathbb{Z}),A)$ which is zero $\forall * > 0$.")

$V_I(\mathbb{Z})$ is $\mathbb{Z}$-free

Fact (Universal coefficient theorem / Künneth formula): R commutative unital.

Let $I^\bullet$ chain complex of flat R-modules s.t. each submodule $d(I^n_\bullet) \subseteq I^{n+1}$ is also flat. Then for any $n \geq 0$ and M R-module $\exists$ exact sequence

$$0 \longrightarrow H^n(I^\bullet) \otimes_R M \longrightarrow H^n(I^\bullet \otimes_R M) \longrightarrow \mathrm{Tor}^R_1(H^{n+1}(I^\bullet), M) \longrightarrow 0$$

Set $R := \mathbb{Z}$, $I^\bullet := \mathrm{Hom}_{\mathbb{Z}\Gamma}(F_\bullet,\mathbb{Z})$, $M := A$ ("note that $I^\bullet$ consists of free finite $\mathbb{Z}$-modules $\Rightarrow$ coboundaries as well")

and observe that

$$I^\bullet \otimes_{\mathbb{Z}} A = \mathrm{Hom}_{\mathbb{Z}\Gamma}(F_\bullet,\mathbb{Z}) \otimes_{\mathbb{Z}} A \xrightarrow{\sim} \mathrm{Hom}_{A\Gamma}(F_\bullet \otimes A, A)$$ as complexes of A-modules

$$\varphi \otimes a \longmapsto \{ v \otimes a' \mapsto \varphi(v)a'a \}.$$

$\square$

In the following compute the groups $\mathrm{Ext}^r_{A\Gamma}(V_I(A),A)$ under assumption:

- Γ is a discrete cocompact subgroup of $PGL_{d+1}(K)$ ("PGL_{d+1} is semisimple")
- A is a field of char $= 0$.

Since in this case, Prop. 41(iii) shows that middle term is finite dim. A-vectorspace W with $\dim_A W$ only depending on the torsion parts of $\mathrm{Ext}^r_{\mathbb{Z}\Gamma}(V_I(\mathbb{Z}),\mathbb{Z})$ $\Rightarrow$ may assume $A = \mathbb{C}$ to compute this dimension.

$\mathrm{Rep}(\bar{G})$ = abelian category of smooth $\bar{G} := PGL_{d+1}(K)$-reprsentations/$\mathbb{C}$, enough injectives/projectives ← $C^\infty_c(G)$ is a projective object (Blanc)

"$V \in \mathrm{Rep}\, G$ then $V \to \mathrm{Ind}^G_1 V$"

Now consider $\mathrm{Ind}_\Gamma := C^\infty(\bar{G}/\Gamma, \mathbb{C})$, admissible* smooth $\bar{G}$-repr.

(* fixed-vectors under any compact open subgroup lie in a finite dimensional $\mathbb{C}$-vectorspace". This is here the case since $PGL_{d+1}(K)/\Gamma$ is compact $\Rightarrow$ any compact open subgroup has finite index")

$\Rightarrow$ (admissibility, Γ discrete) Shapiro's lemma: ([A3] A.8)

$$\mathrm{Ext}^*_{\mathbb{C}\Gamma}(V_I(\mathbb{C}),\mathbb{C}) \cong \mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), \mathrm{Ind}^{\bar{G}}_\Gamma \mathbb{C}) \overset{\mathrm{def}}{=} \mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), \mathrm{Ind}_\Gamma)$$

$\Rightarrow$ "Analyze the RHS, in particular the representation Ind_Γ"

as in ordinary group cohomology: $P_\bullet \to V_I$ proj. resolution

Frobenius reciprocity $\mathrm{Hom}_\Gamma(P_\bullet, \mathbb{C}) \simeq \mathrm{Hom}_G(P_\bullet, \mathrm{Ind}^G_\Gamma \mathbb{C})$

plus observation that $P_\bullet$ remains projective over H.

"sesquilinear ($b(av,w) = a\,b(v,w)$ & $b(v,aw) = \bar{a}\,b(v,w)$ form $+\mathbb{Z}$-linear)
and hermitian ($b(v,w) = \overline{b(w,v)}$)

$\Rightarrow$ Analyze Ind_p :

Def + Fact : • $V \in \mathrm{Rep}\,\bar{G}$ is called unitary if $\exists$ non-degenerate hermitian form $b: V \times V \to \mathbb{C}$ which is
[AA] $\bar{G}$-invariant (i.e. $b(gv,w) = b(v,gw) = b(v,w)$). ("as in theory of real or complex Lie groups")

• Any $V \in \bar{G}$ which is $\underset{\wedge}{\text{unitary}}$ is completely reducible ("semisimple") "decomposes into a direct sum of irreducible admissible unitary representations" (with finite multiplicity)
and admissible

Here: $b(f,h) := \int_{\bar{G}/p} f\bar{h}\, d\mu$, $f,h \in \mathrm{Ind}_p$ $\Rightarrow$ Ind_p unitary and completely reducible (1)
(use that G/p compact here unimodular $\Rightarrow b$ is G-invariant)

Furthermore, let $B :=$ upper Iwahori in G $\underset{\substack{B \text{ compact} \\ \& \text{ here} \\ V^B \text{ is finite dimensional}}}{\Longrightarrow}$ only finitely many irreducible constituents V_j have $V_j^B \neq 0$. (2)

$\underset{(1)+(2)}{\Rightarrow}$ $\mathrm{Ind}_p \cong V_0 \oplus V_1 \oplus \dots \oplus V_m$ with V_j admissible unitary and
• $V_0^B = 0$ ("put all those constituents together that have no non-zero "Iwahori-fixed vector" $=: V_0$)
• V_j irreducible with $V_j^B \neq 0$ for $1 \leq j \leq m$.

$$\Rightarrow \quad \mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), \mathrm{Ind}_p) \cong \bigoplus_j \underbrace{\mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), V_j)}_{\text{"study these extension groups"}}.$$

Proposition 5: $V, V' \in \mathrm{Rep}\,\bar{G}$ such that $V^B = 0$ and V' generated (over $\bar{G}$) by $(V')^B$.
Then $\mathrm{Ext}^*_{\bar{G}}(V', V) = 0$.

Pr: "Interpret Ext^* as Yoneda-Ext groups"
Consider a Yoneda extension of V' by V in $\mathrm{Ext}^r(V',V)$:

$$\begin{array}{ccccccccccccc} E: & 0 & \to & V & \to & E_1 & \to & E_2 & \to \dots \to & E_r & \to & V' & \to 0 \\ & & & \cup & & \cup & & \cup & & \cup & & \| & \\ E^B: & 0 & \to & 0 & \to & \hat{E}_1 & \to & \hat{E}_2 & \to \dots \to & \hat{E}_r & \to & V' & \to 0 \end{array}$$

. In any E_j have subrepresentation gen. by $E_j^B =: \hat{E}_j$.
"complex" in $\mathrm{Rep}\, G$".

if E^B exact then $E =$ image of $[E^B] \in \mathrm{Ext}^r(V', 0) = 0$ via $\mathrm{Ext}^r(V', c)$ where $c: 0 \xrightarrow{\subseteq} V$, i.e. $[E] = 0$.

Standard fact that E exact implies $0 \to V^B \to E_1^B \to \dots \to E_r^B \to (V')^B \to 0$ exact (B compact)

Fact: "$V \in \mathrm{Rep}\,\bar{G}$ is generated by V^B and $V' \subseteq V$ subrepresentation then V' is generated by $(V')^B$.
(standard if B replaced by a principal congruence subgroup $1 + \pi^i M_{n+1}(\mathcal{O}_K)$, $i \geq 1$,
same proof works with minor changes, [2] 4.4)
$\to V$ cuspidal "same proof"
$\to V$ not cuspidal "Induction using suitable Jacquet module of V: V_N"

$\Rightarrow$ $\left.\begin{array}{l} \ker d_* \subseteq E_* \\ \mathrm{Im}\, d_* \subseteq E_* \end{array}\right\}$ generated by $(\ker d_*)^B = (\mathrm{Im}\, d_*)^B$ $\Rightarrow$ E^B exact.
d_* is $\bar{G}$-equivariant

and that $V \to V_N$ defines $V^B \xrightarrow{\sim} (V_N)^{{}^\circ M}$
where $P = MN$
and ${}^\circ M = B \cap M$

□.

Remark: $V_I(\mathbb{C})$ generated by $V_I(\mathbb{C})^B$ ("$V_I(\mathbb{C})$ quotient of an unramified principal series representation" of $GL_{d+1}(K)$)
$\underset{\text{Prop. 5}}{\Rightarrow}$ $\mathrm{Ext}^*(V_I(\mathbb{C}), V_0) = 0$.
later

Proposition 6: $V \in \mathrm{Rep}\,\bar{G}$ irreducible admissible with $V^B \neq 0$ and $V \not\cong V_J(\mathbb{C})$ $\forall J \subseteq \Delta$.
Then $\mathrm{Ext}^*(V_I(\mathbb{C}), V) = 0$.

$T \subseteq G$ diagonal torus, $\delta: P \to \mathbb{C}^*$ modulus character (recall $P = P_\emptyset$ minimal parabolic) "Lower Borel")

Alg Groups (Borel ⇒) Thm 22.6. $\bar{P} = \bar{T}\bar{N}$ Levi-decomposition in $\bar{G}$ $\quad$ ($\delta(tn) = vol_N(t^{-1}N_0 t)/vol_N(N_0)$ with $P = TN$, $N_0 \subseteq N$ compact open subgroup

Recall (unramified principal series of G w.r.t. P) ([5].X.3.2.)

χ unramified character of T ("trivial on ${}^0T$ = maximal compact subgroup = image of $\begin{pmatrix} {}^0L^* & & \\ & \ddots & \\ & & {}^0L^* \end{pmatrix}$ in T")

$PS(\chi) := \mathrm{Ind}_P^G \chi$ $\quad$ (unnormalized) unramified principal series representation

$= \{ f \in C^\infty(G, P) : f(gp) = \chi(p)^{-1} f(g) \ \forall g \in G, p \in P \} \circlearrowleft G$

Ex: $C^\infty(G/P_I, \mathbb{C})$, $I \subseteq \Delta$

Properties of $PS(\chi)$:

(1) admissible smooth of finite length ("finite composition series / Jordan-Hölder series")

(2) if $P = TN$ then $(PS(\chi)_N)^{ss} \cong \bigoplus_{w \in W} (w.\chi\delta^{-1/2})\delta^{1/2}$ $\quad$ $V \mapsto V_N$, $\mathrm{Rep}(G) \to \mathrm{Rep}(T)$ Jacquet functor.

("by exactness of $V \mapsto V_N$ $PS(\chi)_N$ is of finite length.")

"~~[illegible]~~"

(3) $V \in \mathrm{Rep}\,G$ irreducible admissible is a JHF of some $PS(\chi)$ if and only if $V^B \neq 0$.

(4) $V = V_I(\mathbb{C})$, $I \subseteq \Delta$ $\iff$ V JHF of $PS(\chi)$, $\chi \in \{\delta_w : w \in W\}$ where $\delta_w = (w.\delta^{-1/2})\delta^{1/2}$.

Remark: (3) ⇒ $PS(\chi)$ generated by $PS(\chi)^B$

Since $V_I(\mathbb{C})$ = quotient of $C^\infty(G/P_I, \mathbb{C})$ ⇒ $V_I(\mathbb{C})$ generated by $V_I(\mathbb{C})^B$ $\underset{\text{Prop 5}}{\Rightarrow}$ $\mathrm{Ext}^*(V_I(\mathbb{C}), V_0) = 0$

Proof Prop. 6: (3)+(4) ⇒ V JHF of some $PS(\chi)$ with $\chi \notin \{\delta_w : w \in W\}$. [illegible]

~~[illegible]~~

Proof that: $\mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), PS(\chi)) = 0$: $\circledast$

~~(general case i.e. V subquotient similar but with more theory on $PS(\chi)$ & any constituent of $PS(\chi)$ is a submodule of some $PS(\chi')$.)~~

An induction argument on "$*$" gives: If V' is a subquotient of $PS(\chi)$ then $\mathrm{Ext}^*(V_I(\mathbb{C}), V') = 0$. □

(goes roughly like this: $* = 0$: $V_I(\mathbb{C}) \hookrightarrow V'$ ⇒ $V_I(\mathbb{C})$ JHF of $PS(\chi)$. ⇒ ↯ to (4)

Induction step: Assume $V' \subseteq PS(\chi)$ submodule. Then consider

("general theory of $PS(\chi)$: Any irreducible subquotient V of $PS(\chi)$ is a submodule of some $PS(\chi')$")

$0 \to V' \to PS(\chi) \to W \to 0$ with $W := PS(\chi)/V'$

$\circledast$ ⇒ $\mathrm{Ext}^{*-1}_{\bar{G}}(V_I(\mathbb{C}), W) \to \mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), V') \to \mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), PS(\chi))$ □

Remains to prove $\mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), PS(\chi)) = 0$:

$\mathrm{Ext}^*_{\bar{G}}(V_I(\mathbb{C}), PS(\chi)) \underset{\text{Shapiro}}{\cong} \mathrm{Ext}^*_{\bar{P}}(V_I(\mathbb{C}), \chi) \underset{\substack{\text{HS-SS} \\ \text{[A3] A.9.}}}{\cong} \mathrm{Ext}^*_{\bar{P}/\bar{N}}(V_I(\mathbb{C})_{\bar{N}}, \chi) \underset{(2)}{\cong} \bigoplus_{w \in W^I} \mathrm{Ext}^*_{\bar{T}}(\delta_w, \chi)$

$W^I = \{w \in W : w$ of minimal length in $wW_I\}$

Now $\delta_w \neq \chi$ $\forall w \in W^I$, $\bar{T}$ abelian.

Use the fact (X.1.9. in [5]): $\mathcal{U}, V \in \mathrm{Rep}(G)$ and exists $g \in Z(G)$ s.t. $g = id$ on $\mathcal{U}$ and $g \neq id$ on V ⇒ $\mathrm{Ext}^*(\mathcal{U}, V) = 0$. □

Proposition 7: If $V_J(\mathbb{C})$ is [illegible] of Ind_P, some $J \subseteq \Delta$ then $J \in \{\emptyset, \Delta\}$.

$V_\emptyset(\mathbb{C})$ Steinberg — trivial representation.

Pr: Suffices to prove: If $V_J(\mathbb{C})$ is unitary then $J \in \{\emptyset, \Delta\}$.
("non-unitary theorem" of Casselman [5] XI.4.5, note that PGL_{d+1} is (almost) simple) "$\Rightarrow$ plays a role like vanishing theorem of Garland for $H^*(\Gamma, \mathbb{C})$"

" ~~[illegible]~~ using a theorem of Howe: Converse also holds: if $J \in \{\emptyset, \Delta\}$ then $V_J(\mathbb{C})$ is unitary.
Clear for $V_\Delta(\mathbb{C}) = \mathbb{C}$. $V_\emptyset(\mathbb{C})$ is square-integrable mod centre $\Rightarrow$ unitary
Casselman, lecture notes Prop. 2.5.4.

"Here we are using that we are dealing with PGL_{d+1}-representations and that PGL_{d+1} is ~~[illegible]~~ almost simple (i.e. having a simple Lie algebra) $\cong sl_{d+1}$"

Applying Prop. 5–7. to $\mathrm{Ind}\,\Gamma = \bigoplus_{j=0}^{m} V_j \Rightarrow$ multiplicity $s\Gamma^0$

$$\mathrm{Ext}^*(V_I(\mathbb{C}), \mathrm{Ind}_P) = \begin{cases} \mathrm{Ext}^*(V_I(\mathbb{C}), \underset{\sim}{V_\Delta} + V_\emptyset^s) \\ ~~\mathrm{Ext}^*(V_I(\mathbb{C}), V_\Delta) \text{ else}~~ \end{cases}$$

(it's known that V_Δ has multiplicity 1 in $\mathrm{Ind}\,\Gamma$!?) $= \mathbb{C}^s$

$$= \mathrm{Ext}^*(V_I(\mathbb{C}), \mathbb{C}) \oplus \left[\mathrm{Hom}_G(V_\emptyset(\mathbb{C}), \mathrm{Ind}\,\Gamma) \otimes_{\mathbb{C}} \mathrm{Ext}^*(V_I(\mathbb{C}), V_\emptyset(\mathbb{C}))\right]$$

follows for $* = 0$ from fully faithfulness of the $(\;)^\sim$ on admissible repr.

(observe that $\mathrm{Hom}_G(V_\emptyset(\mathbb{C}), \mathrm{Ind}\,\Gamma) = \mathbb{C}^{s\,s\Gamma^0}$ if $V_\emptyset(\mathbb{C}) \subseteq \mathrm{Ind}\,\Gamma$ since $\mathrm{Ind}\,\Gamma = V_\emptyset(\mathbb{C})^s \oplus V_\Delta^*(\mathbb{C})$ then
and $\mathrm{Hom}_G(V_\emptyset(\mathbb{C}), V_\Delta(\mathbb{C})) = 0$ since $V_\emptyset(\mathbb{C}) \not\simeq V_\Delta(\mathbb{C})$ e.g. GL_2: $V_\emptyset(\mathbb{C}) = C^\infty(\mathbb{P}^1)/\mathbb{C}$, $V_\Delta(\mathbb{C}) = \mathbb{C}$.)

$\Rightarrow$ "Study here the first summand, the rest is left to Christian ~~[illegible]~~ tomorrow":

Proposition 8:
$$\mathrm{Ext}^r(V_I(\mathbb{C}), \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if } r = \#\Delta\setminus I \\ 0 & \text{else} \end{cases}$$

Pr: Put $\bar{I} := \{d+1-i : i \in I\}$.
Claim: $V_I(\mathbb{C})^\sim \xrightarrow{\simeq} V_{\bar{I}}(\mathbb{C})$.
"Pr: ([13], page 915): $t \in \mathbb{R}$, $\bar{\pi}_I^t := \mathrm{Ind}_{\bar{P}_I}^{\bar{G}} \delta_I^{-t}$, $\delta_I: \bar{P}_I \to \mathbb{C}^*$ modulus
(so that $\bar{\pi}_I^0 = C^\infty(\bar{G}/\bar{P}_I, \mathbb{C}) = C^\infty(G/P_I, \mathbb{C})$)
$\Rightarrow (\bar{\pi}_I^t)^\sim \underset{\langle,\rangle}{\simeq} \bar{\pi}_I^{-t-1}$ via pairing $\langle f_1, f_2 \rangle := \int_{\bar{G}/\bar{P}_I} f_1 f_2\, dh$ $(f_1 \in \bar{\pi}_I^t, f_2 \in \bar{\pi}_I^{-t-1} \Rightarrow f_1 f_2 ~~[illegible]~~ \in \bar{\pi}_I^{-1})$

$\bar{\pi}_I^0 \xrightarrow{\simeq} \bar{\pi}_{\bar{I}}^{-1}$ s.t. $V_I(\mathbb{C}) \xrightarrow{\simeq}$ Image. Now use: $\exists$ isom: $T_I^0: \bar{\pi}_I^0 \xrightarrow{\simeq} \bar{\pi}_{\bar{I}}^{-1}$ ([13] Prop. 1.5.6))

$\Rightarrow V_I(\mathbb{C})^\sim \xrightarrow{\simeq} (\text{Image})^\sim \hookrightarrow (\bar{\pi}_{\bar{I}}^{-1})^\sim \underset{\langle,\rangle}{\xrightarrow{\simeq}} \bar{\pi}_{\bar{I}}^0 \longrightarrow\!\!\!\!\to V_{\bar{I}}(\mathbb{C})$ is an iso.

Fact: $\mathcal{U}, V \in \mathrm{Rep}(\bar{G})$, V admissible $\Rightarrow \mathrm{Ext}_G^*(\mathcal{U}, V) \xrightarrow{\simeq} \mathrm{Ext}_G^*(\tilde{V}, \tilde{\mathcal{U}})$ (dualize resolutions) ([13] A.11)

Here: $\mathcal{U} := V_I(\mathbb{C})$, $V := \mathbb{C} \Rightarrow \mathrm{Ext}_G^*(V_I(\mathbb{C}), \mathbb{C}) \xrightarrow{\simeq} \mathrm{Ext}_G^*(\mathbb{C}, V_{\bar{I}}(\mathbb{C}))$

$= H_{sm}(\bar{G}, V_{\bar{I}}(\mathbb{C}))$ ~~[illegible]~~ $=$ Thm of Casselman [13], A.13. + $\bar{G}$ almost simple.

$$\mathrm{RHS} = \begin{cases} \mathbb{C} & \text{for } * = \text{parabolic rank of } P_{\bar{I}} = \#\Delta\setminus I \\ 0 & \text{else.} \end{cases}$$ □

"parabolic rank" = rank of the radical of $\bar{P}_{\bar{I}}$ = rank of the radical of $P_{\bar{I}} = \#\Delta\setminus I$. see also Backside!

On the parabolic rank:

Let G be connected reductive and P a parabolic.

The parabolic rank of P is defined as $rk(R(P))$.

Now $R(P) = Z_M \cdot R_u(P)$ where Z_M = center of M and $P = MN$ is a Levi-decomposition. (Comp. Borel/Tits: Groupes réductifs, I §2 (2.15))

$\Rightarrow$ $rk(R(P)) = \dim Z_M$ = dimension of maximal torus in Z_M = dimension of the maximal torus Z_M (Z_M is a torus!)

Example: if $G = GL_{d+1}$, $P = P_I$ (→ $P_\emptyset$ = lower Borel)

where $\Delta \setminus I = \{i_0 < i_1 < \dots < i_r\}$

(compare §4 of SS)

$$P_I = \begin{array}{c} \\ i_0 \\ i_1 \\ \\ \end{array}\begin{pmatrix} \boxed{*} & * & * \\ & \boxed{*} & * \\ & & \ddots \end{pmatrix}$$

$$\Rightarrow Z_M = \begin{array}{c} i_0 \\ i_1 \\ \\ \end{array}\begin{pmatrix} \boxed{a_0 \ddots a_0} & & \\ & \boxed{a_1 \ddots a_1} & \\ & & \ddots \end{pmatrix}$$ with a_i, $i = 0, \dots, r$ ranging through K^*

$\Rightarrow \dim Z_M = r+1$.

<u>Remark:</u> $\# \Delta \setminus \bar{I} = r+1$ $\Rightarrow$ parabolic ranks of P_I & $P_{\bar{I}}$ are the same.