Construction of the upper half-plane

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1 Prerequisites

Throughout this talk we fix a local field K in the sense of [Nk], i.e. a field which is complete with respect to a non-trivial non-archimedean discrete valuation $|\cdot|$ and such that its associated residue field is finite. We assume $|\cdot|$ to be normalized. Furthermore we let $p > 0$ be the characteristic of its residue field, o the valuation of K, π a uniformizing element and $\hat{\bar{K}}$ the completion of a fixed algebraic closure $\bar{K}$ of K. We also fix a natural number d and denote by $\mathbf{P}_K^d$ the d-dimensional projective space as a K-analytic variety.

2 Main Proposition

If we denote by $\mathscr{H}$ the set of all K-rational hyperplanes in $\mathbf{P}_K^d$ (which is by definition the kernel of a linear form defined over K, i.e. maps K^{d+1} into K), then our object of interest is

$$\Omega^{(d+1)} := \mathbf{P}_K^d - \bigcup_{H \in \mathscr{H}} H.$$

Proposition 1 (Aim of the talk)**.**

(i) We first construct a canonical admissible covering of open affinoid subvarieties of P_K^d to show that $\Omega^{(d+1)}$ is an admissible open subset and consequently an open analytic subvariety of $\mathbf{P}_K^d$.

(ii) Then we show that this covering (by a slight modification) is already nice enough to attest that $\Omega^{(d+1)}$ is even a Stein space (recall that a Stein space is a rigid analytic variety which is covered by a strictly increasing filtration of certain well-behaved open affinoid subvarieties).

As we soon want to speak of the norm of the coordinates of a point in $\mathbf{P}_K^d$, we introduce the following conventions:

1. If not indicated otherwise, we assume the representing vector of $z = [z_0 : \cdots : z_d] \in P^d_K$ to be of unit length (unimodular), i.e. $\max\{|z_i| \, | i = 0, \ldots, d\} = 1$.
2. For any hyperplane $H \in \mathscr{H}$, we let $l_H \in L_0^*$ with $L_0 := o^{d+1}$ always be a unimodular vector such that $H(\hat{\bar{K}})$ is its kernel in $P^d_K(\hat{\bar{K}})$, i.e.

$$H(\hat{\bar{K}}) = \{z \in P^d_K(\hat{\bar{K}}) : l_H(z) = 0\}.$$

The admissible covering of $\Omega^{(d+1)}$ will consist of the complement of ϵ-neighborhoods of the hyperplanes $H \in \mathscr{H}$ with ϵ tending to 0. Each of those is a finite intersection of open d-dimensional polydiscs and therefore an admissible open subdomain.

Definition 1. *If $\epsilon > 0$ is a rational number, the set*

$$H(\epsilon) := \{z \in P^d(\hat{\bar{K}}) : |l_H(z)| \leq \epsilon\}$$

is called the ϵ-neighborhood *of the hyperplane $H \in \mathscr{H}$.*

This definition is independent of the choices of l_H and z if we restrict ourselves to the above conventions.

From now on, we speak of any subset of $P^d(\hat{\bar{K}})$ as a subset of the underlying point set of P^d_K by replacing $\hat{\bar{K}}$ through the orbits of the absolute Galois group of K in $\bar{K}$. Note that there is no ambiguity in the definition of the norm of a Galois orbit of $\bar{K}$ as all Galois automorphisms are isometries.

Definition 2. *Two hyperplanes $H, H' \in \mathscr{H}$ are called* congruent $\mod (\pi^n)$ *if their appropriate representing vectors l_H and $l_{H'}$ in L_0^* are congruent modulo $\pi^n L_0^*$.*

If $\mathscr{H}_n$ denotes the set of equivalence classes of hyperplanes $H \in \mathscr{H} \mod (\pi^n)$, then we have

$$\mathscr{H}_n = (L_0^*/\pi^n L_0^*)/\text{scalars in } o^* = \mathbf{P}(L_0^*/\pi^n L_0^*)$$

and $\mathscr{H} = \varprojlim \mathscr{H}_n$ (as L_0^* is π-adically complete and $\mathscr{H} = \mathbf{P}(L_0^*)$). In this manner $\mathscr{H}$ inherits a natural profinite topology.

When do have two hyperplanes the same ϵ-neighborhood? The next lemma gives an answer to this question.

Lemma 1. *Two hyperplanes are congruent $\mod (\pi^n)$ if they have the same ϵ-neighborhoods for $\epsilon = |\pi^n|$.*

Proof. Let H and H' be those hyperplanes.
If their representing linear forms differ only by a multiple of π^n, then we find that $|l_H(z)| \leq \max\{|l_H(z) - l_{H'}(z)|, |l_{H'}(z)|\} \leq |\pi^n|$ if $z \in H'(|\pi^n|)$ and vice versa.

The other way around, we first note generally that since l_H and $l_{H'}$ are unimodular, they induce *surjective* linear maps

$$\bar{l}_H, \bar{l}_{H'} : L_0/\pi^n L_0 \to o/\pi^n o.$$

If we assume that $H(|\pi^n|) = H'(|\pi^n|)$, then $\bar{l}_H$ and $\bar{l}_{H'}$ have the same kernel. As a linear map which vanishes on a certain submodule is the same as a linear map on the quotient module, which in this case is free of rank one, we have $\bar{l}_{H'} = \bar{\alpha} \cdot \bar{l}_H$ for some *unit* $\bar{\alpha} \in o/\pi^n o$ (by surjectivity). If therefore $\alpha \in o$ represents $\bar{\alpha}$, then $l_{H'} - \alpha l_H \in \pi^n L_0^*$, i.e. H and H' are congruent $\mod (\pi^n)$ (as αl_H represents H as well). ■

Note that somewhat counter-intuitively, the ϵ-neighborhoods of hyperplanes are more likely to coincide as they become thicker. I.e., if their ϵ_0-neighborhoods coincide for some ϵ_0, they also coincide for all $\epsilon > \epsilon_0$. As a slogan, one could phrase the lemma as follows: as soon as the thickness of the neighborhoods of two hyperplanes becomes larger than the deviation of their orthogonal vectors, they coincide.

An immediate consequence (already by the obvious direction) is

Lemma 2.

$$\bigcup_{H \in \mathscr{H}_n} H(|\pi^n|) \supseteq \bigcup_{H \in \mathscr{H}} H.$$

Now the road to triumph is paved: We show that increasing sequence of subsets

$$\Omega_n := \Omega(|\pi^n|) := \mathbf{P}_K^d - \bigcup_{H \in \mathscr{H}_n} H(|\pi^n|)$$

of $\Omega^{(d+1)}$ (by Lemma 2) constitute an admissible covering of $\Omega^{(d+1)}$ by admissible open subvarieties.

Proposition 2. *The Ω_n are admissible open subsets in $\mathbf{P}_K^d$ (and therefore rigid-analytic varieties).*

Proof. Ω_n is a finite intersection of subsets of the form $\mathbf{P}_K^d - H(|\pi^n|)$. It therefore suffices to show by the axioms of the G-topology to show that those subsets are admissible open. Now up to a linear isomorphism , $P_K^d - H$ is the standard open subset $U_0 = \{[z_0 : \cdots : z_d] : z_0 \neq 0\}$ of $P_K^d - H$. Explicitly, if $l_H = v^*$ for a vector $v \in L_0$, then we complete $\{v\}$ to a basis of L_0 and map this to the canonical basis of L_0, sending v to $e_0 := (1, 0, \ldots, 0) \in L_0$. Therefore we may assume that $P_K^d(H(|\pi^n|) = \{[z_0 : \cdots : z_d] \in P_K^d : |z_0| > |\pi^n|\} \subset U_0$. Under the usual identification of U_0 with $A_K^{d,\text{rig}}$ via $[z_0 : \cdots : z_d] \mapsto (\frac{z_1}{z_0}, \ldots, \frac{z_d}{z_0})$, this becomes the open polydisc of radius $|\pi^n|^{-1}$, which can be seen to be admissible open by choosing a sequence of n-th roots of unity of the value group of $\bar{K}$ converging to $|\pi^n|$ (cf. [BoschLec], §1.10 Prop.7 or even more explicitly [SnBN]). ■

We proceed by showing that these constitute an admissible covering of $\Omega^{(d+1)}$.

Proposition 3. *The family* $\{\Omega_n : n \in \mathbf{N}\}$ *is an admissible covering of* $\Omega^{(d+1)}$.

Proof. Unwinding the definition of an admissible covering, we have to show that any morphism f on a K-affinoid variety such that $\operatorname{im} f \subset \Omega^{(d+1)}$ already satisfies $\operatorname{im} f \subset \Omega_n$ for an $n \in \mathbf{N}$.
Now just as in the construction of $\mathbf{A}_K^{d,\mathrm{rig}}$, one sees that the family $\{P_K^d - H(|\pi^n|) : n \in \mathbf{N}\} \simeq \mathbf{B}^d_{<|\pi^n|^{-1},K}$ is even an admissible covering of $P_K^d - H \simeq \mathbf{A}_K^{d,\mathrm{rig}}$ (needs the maximum modulus principle). Therefore by the universal property in the definition of an admissible covering, if $f : Y \to \mathbf{P}_K^d$ is any K-morphism form a K-affinoid variety Y into $\mathbf{P}_K^d$ such that $\operatorname{im}(f) \subset \mathbf{P}_K^d - H$, then there exists an $n(H) \in N$ such that $\operatorname{im} f \subset \mathbf{P}^d - H(|\pi^{n(H)}|)$.

We hence see that $\operatorname{im} f \subset \mathbf{P}_K^d - \bigcup_{H \in \mathscr{H}} H(|\pi^{n(H)}|)$; thus if we could bound $\{n(H) : H \in \mathscr{H}\}$, we would be done since then $\operatorname{im} f \subset \mathbf{P}_K^d - \bigcup_{H \in \mathscr{H}_n} H(|\pi^n|)$, where n is a upper bound of the $n(H)$.

We now cover $\mathscr{H} = \bigcup_{H \in calH} = \{H' \in \mathscr{H} : H' \subset H(|\pi^{n(H)}|)\}$. As by Lemma 1 the sets $\{H' \in \mathscr{H} : H' \subset H(|\pi^{n(H)}|)\}$ are just the sets of hyperplanes congruent to $H \mod (\pi^n)$, these sets in this covering of $\mathscr{H}$ are by definition open in $\mathscr{H} = \varprojlim \mathscr{H}_n$. Thus the union above actually runs over a union of a family of open subsets in $\mathscr{H}$. As $\mathscr{H}(= \mathbf{P}(L_0^*))$ is compact (profinite topological spaces being the same as compact and totally disconnected ones), we find finitely many hyperplanes $H_1, \ldots, H_r \in \mathscr{H}$ such that

$$\bigcup_{H \in \mathscr{H}} H(|\pi^{n(H)}|) = \bigcup_{i=1,\ldots,r} H_i(|\pi^{n_i}|);$$

where $n_i = \min\{n(H') : H' \subset H_i(|\pi^{n(H_i)}|)\}$. In particular $n(H)$ is bounded by $\max\{n_1, \ldots, n_r\}$. ■

We have thus finished Proposition 1 (i) and move on by recalling the definition of a Stein space:

Definition 3. *An analytic space* X *is called* Steinsch, *if there is an admissible covering by an increasing sequence* $U_1 \subseteq U_2 \subseteq \ldots$ *of open affinoid subdomains such that* U_i *is a Weierstrass-domain of a diameter in the value group of* K *less than* 1, *i.e.*

$$U_i = U_{i+1}(f_1^{(i)} \le |a_i|, \ldots, f_{r_i}^{(i)} \le |a_i|) \subseteq U_{i+1} = \operatorname{Spec}(A_{i+1})$$

for an element $a_i \in K$ *of norm less than* 1.

As a slogan, a Stein space if filtered by a sequence of Weierstrass-domains such that the previous one is obtained by cutting out points of norm less than 1 in the value group of K (identifying a system of affinoid generators with the coordinate functions).

We now proceed showing that those $\bar{\Omega}_n$ actually witness that $\Omega^{(d+1)}$ is Steinsch.

Here and later on we will always identify the elements of a reduced affinoid algebra as functions on the orbits of the absolute Galois group of K *in* $\bar{K}$.

Proposition 4. $\Omega^{(d+1)}$ *is Steinsch.*

Proof. We have

$$\bar{\Omega}_n = \{z \in P^d(\hat{\hat{K}}) : |l_H(z)| \geq |\pi^n| \text{ for all } H \in \mathscr{H}\}.$$

For any pair $H, H' \in \mathscr{H}$, we have the well-defined function

$$f_{H,H'} := \frac{l_H}{l_{H'}} \in \mathscr{O}(P^d_K - H).$$

It therefore restricts to a function in $\mathscr{O}(\Omega^{(d+1)})$ as we have (platonistically spoken) just *discovered* $\Omega^{(d+1)}$ to be admissible open. For each $n \in \mathbf{N}$ we choose a set $\bar{\mathscr{H}}_n$ of representatives for the equivalence classes of hyperplanes in $\mathscr{H}_{n+1}$ in such a way that it contains the coordinate hyperplanes $H_i = \{z_i = 0\}$ for $i = 0, \ldots, d$. Note that it suffices to demand $H \in \bar{\mathscr{H}}_n$ in the definition of $\bar{\Omega}_n$ as $l_H(z) < |\pi^n|$ exactly if $l_{H'}(z) < |\pi^n|$ for H, H' lying in the same equivalence class of $\mathscr{H}_{n+1}$ (cf. Lemma 1). We see that

$$\begin{aligned}
\bar{\Omega}_n &= \{z \in \Omega^{(d+1)} : |f_{H,H'}| \leq |\pi|^{-n} \text{ for all } H, H' \in \bar{\mathscr{H}}_n\} \\
&= \{z \in \Omega^{(d+1)} : |f_{H_i,H'}| \leq |\pi|^{-n} \text{ for all } i = 0, \ldots, d \text{ and } H' \in \bar{\mathscr{H}}_n\} \\
&= \{z \in \bar{\Omega}_{n+1} : \left|\pi^{n+1} f_{H_i,H'}\right| \leq |\pi| \text{ for all } H, H' \in \bar{\mathscr{H}}_n\}.
\end{aligned}$$

Regarding the first equality, note that by our conventions all linear forms are bounded by norm 1 on $\mathbf{P}^d_K$ and all $z \in \mathbf{P}^d_K$ have unit length, i.e. $1 = \max\{|z_0|, \ldots, |z_d|\}$; this is just emphasized in the second equality; for the third one, note that, by the original definition of $\bar{\Omega}_n$, we surely find $\bar{\Omega}_n \subset \bar{\Omega}_{n+1}$ and that $\bar{\mathscr{H}}_{n+1}$ just as well constitutes (a finer) system of representatives of $\mathscr{H}_{n+1}$ containing $H_0, \ldots, H_d$.

Now by the last characterization of $\bar{\Omega}_n$, we would be done if we could generally show that these $\pi^n f_{H,H'}$ for $H, H' \in \bar{\mathscr{H}}_n$ constitute a system of affinoid generators of $\mathscr{O}(\bar{\Omega}_n)$. We thus have to determine $\mathscr{O}(\bar{\Omega}_n)$ explicitly. Our candidate is the following: Define the affinoid K-algebra A_n to be the free Tate algebra over K in the in indeterminates $T_{H,H'}$ for $H, H' \in \bar{\mathscr{H}}_n$ divided by the closed ideal generated by

$$T_{H,H'} - \pi^n \text{ for } H \in \bar{\mathscr{H}}_n, \tag{1}$$

$$T_{H,H'} \cdot T_{H',H''} - \pi^n T_{H,H''} \text{ for } H, H', H'' \in \bar{\mathscr{H}}_n, \tag{2}$$

$$T_{H,H_j} - \sum_{i=0,\ldots,d} \lambda_i T_{H_i,H_j} \text{ if } l_H(z) = \sum_{i=0,\ldots,d} \lambda_i z_i \text{ for } H \in \bar{\mathscr{H}}_n \text{ and } j = 0, \ldots, d. \tag{3}$$

We then have the K-morphisms

$$\phi_n : \bar{\Omega}_n \to \operatorname{Spec}(A_n) \text{ given by } A_n \ni T_{H,H'} \mapsto \pi^n f_{H,H'} \in \mathscr{O}(\bar{\Omega}_n)$$

and

$$\begin{array}{rcl} \tilde{\psi}_n : \mathrm{Spec}(A_n) & \to & \mathbf{P}^d_K \\ x & \mapsto & [T_{H_0,H_j}(x) : \cdots : T_{H_d,H_j}(x)] \text{ (not necessarily unimodular),} \end{array}$$

the latter map being independent of the particular choice of $j \in \{0, \ldots, d\}$. (Note that a priori $\mathrm{Spec}(A_n)$ is an affinoid subdomain of a polydisc of rather large dimension).

First of all note that $\mathrm{im}\,\tilde{\psi}_n$ does not intersect any of the hyperplanes of $\bar{\mathscr{H}}_{n+1}$ (particularly $\tilde{\psi}_n$ is well-defined) and therefore none of the coordinates of the points in $\mathrm{im}\,\tilde{\psi}_n$ vanishes as $T_{H,H_j}(x) \cdot T_{H,H_i} = \pi^{2n} \neq 0$ for any $H \in \bar{\mathscr{H}}_n$ by (1).

We check that indeed $\mathrm{im}\,\tilde{\psi}_n \subset \bar{\Omega}_n$. To begin with, recall that $f_{H_k,H_l} = \frac{e_k^*}{e_l^*}$, where e_i^* is the i-th coordinate evaluation map. Therefore $f_{H_k,H_l}(\tilde{\psi}_n(x)) = \frac{T_{H_k,H_j}(x)}{T_{H_l,H_j}(x)} = \pi^n T_{H_k,H_l}(x)$ by (2) and consequently $f_{H,H_l}(\tilde{\psi}_n(x)) = \pi^n T_{H,H_l}(x)$ for arbitrary $H \in \bar{\mathscr{H}}_n$ by (3). Finally $f_{H,H'}(\tilde{\psi}_n(x)) = \pi^n T_{H,H'}(x)$ for $H, H' \in \bar{\mathscr{H}}_n$, again by (2). It follows that $\left| f_{H,H'}(\tilde{\psi}_n(x)) \right| \leq |\pi|^{-n}$ for all $H, H' \in \bar{\mathscr{H}}_n$ and therefore $\mathrm{im}\,\tilde{\psi}_n \subset \bar{\Omega}_n$.

Therefore $\tilde{\psi}_n$ factors through a K-morphism $\Psi_n : \mathrm{Spec}(A_n) \to \bar{\Omega}_n$. To prove $\mathscr{O}(\bar{\Omega}_n) \simeq A_n$, it suffices to show that the K-morphisms ψ_n and ϕ_n are inverse to each other on their point sets as these uniquely determine those in the case of (reduced) varieties (for example $\psi_n = \mathscr{O}(\bar{\Omega}_n) \ni f \mapsto f \circ \psi_n \in \mathscr{O}(\mathrm{Spec}(A_n))$ if we use the identification as remarked above).

Now let $x \in \mathrm{Spec}(A_n)$. Then for arbitrary $H, H' \in \bar{\mathscr{H}}_n$ we find

$$\begin{array}{rcr} T_{H,H'}(\phi_n(\psi_n(x))) & = & T_{H,H'} \circ \phi_n(\psi_n(x)) \\ & = & \pi^{-n} f_{H,H'}(\psi_n(x)) \\ & = & T_{H,H'}(x) \end{array}$$

as we have just proven the last equality. Vice versa let $z \in \bar{\Omega}_n$. Then

$$\begin{array}{rcr} \psi_n(\phi_n(z))_i & = & \psi_n((\pi^n f_{H,H'}(z))_{H,H' \in \bar{\mathscr{H}}_n})_i \\ & = & [\pi^n f_{H_0,H_j}(z) : \cdots : \pi^n f_{H_d,H_j}(z)] \\ & = & z. \end{array}$$

■

There are two other facts that need to be mentioned. Recall that an abstract simplical complex is merely a family of subsets of a fixed point set closed under taking subsets. The first one is that there is a natural map from $\Omega^{(d+1)}$ to the geometric realization of a simplicial complex whose vertices are given by homothety classes of lattices of o-lattices in K^{d+1}.

Definition 4. *$\mathscr{BT}$ is the simplicial complex whose vertices are the homothety classes $[L]$ of o-lattices in K^{d+1} and whose q-simplices are given by families $\{[L_0], \ldots, [L_q]\}$ of homothety classes such that*

$$L_0 \subsetneq L_1 \subsetneq \ldots \subsetneq L_q \subsetneq \pi^{-1} L_0.$$

Remark 1. *As modulo L_0 this yields a sequence of proper inclusions of vector spaces over the residue field of K, we find $q \leq d$.*

Recall that the geometric realization of a simplicial complex is obtained by adjoining formal sums $\sum_{i=0,\ldots,q} \lambda_i L_i$ with $|\lambda_i| \leq 1$ and $\sum_{i=0,\ldots,q} \lambda_i = 1$ for all adjacent points of a face (an element of the family of subsets). Here this looks like a tree for $d = 1$ and like some wildly attached triangles in 3-dimensional space if $d = 2$. Now it is important that the topological space of homothety classes of real norms on K^{d+1} can be identified $\mathrm{GL}_{d+1}(K)$-equivariantly with the geometric realization $|\mathscr{BT}|$ of the simplical complex $\mathscr{BT}$. Ralph gave a taste of this for the case $d = 1$ in last term's Oberseminar. I will briefly recall this construction.

If L and L' are adjacent (i.e. $\pi L \subsetneq L' \subsetneq L$), then there exists a basis $e_1, e_2 \in K^2$ such that $L = oe_1 + oe_2$ and $L' = oe_1 + o\pi e_2$ (as $L' = \tau^{-1} l$ for a line over the residue field of K and τ denoting the canonical projection modulo L_0). Each lattice L defines a canonical norm on K^2 by putting $|a_1 e_1 + a_2 e_2|_L := \max\{|a_1|, |a_2|\}$. Then the map is given as follows:

$$\begin{aligned} |\mathscr{BT}| &\to && \{\text{ Homothety classes of norms on } K^2\} \\ s \cdot L + s' \cdot L' &\mapsto && [|a_1 e_1 + a_2 e_2|_{sL+s'L'} := \max |a_1|, q^{t'} |a_2|. \end{aligned}$$

This was seen to be a bijection. The other canonical (but only surjective) map from $\Omega^{(d+1)}$ to the set of homothety classes of norms is given by

$$\rho : z = [z_0 : \ldots : z_d] \mapsto [|w|_{\rho(z)} := \left| \sum_{i=0,\ldots,d} w_i z_i \right| \text{ for } w = (w_0, \ldots, w_d) \in K^{d+1}].$$

Ralph used the fibres of the composed map $\Omega^{(d+1)} \to |\mathscr{BT}|$ to construct another admissible covering of $\Omega^{(d+1)}$.

The second fact concerns the fibres of the projection from the complement of the common point set of finitely many ϵ-neighborhoods of hyperplanes $H_0(|\pi|^n), \ldots, H_r(|\pi|^n)$ in $\Omega^{(d+1)}$ onto $\mathbf{P}_K^s$ for a certain $s < d$. These are locally open polydiscs in $\mathbf{A}_K^{d-s,\mathrm{rig}}$:

Proposition 5. *Let $\bar{M} := \sum_{i=0,\ldots,r} (o/\pi^n o) l_{H_i} \subset L_0^*/\pi^n L_0^*$. Define $s = \operatorname{rank} \bar{M} - 1$ (the minimal number of generators of $\bar{M}$ as an $o/\pi^n o$-module). We have a (in the rigid sense) locally trivial fibration*

$$\mathbf{P}_K^d - (H_0(|\pi|^n) \cap \ldots \cap H_r(|\pi|^n)) \to \mathbf{P}_K^s$$

over K with fibres open polydiscs in $\mathbf{A}_K^{d-s,\mathrm{rig}}$.

Recall that in this context a surjective map of rigid analytic varieties $\pi : E \to B$ is said to have a *locally trivial fibration* if for any point $b \in B$ there exists a so called *trivializing neighborhood* $U \ni b$ which is admissible open such that the its preimage $\pi^{-1}(U)$ is as a rigid analytic variety isomorphic to the product space $U \times D$ for an open polydisc $D \subset \mathbf{A}_K^{d-s,\mathrm{rig}}$.

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