A Duality theorem for smooth representations of $\mathrm{PGL}_{d+1}(K)$

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1 The duality theorem

We set $G := \mathrm{Gl}_{d+1}(K)$, $\bar{G} := \mathrm{PGl}_{d+1}(K)$, and we equip both groups with the topology induced from the valuation topology on K. A (not necessarily finite-dimenaional) $\mathbb{C}$-linear representation of $\bar{G}$ is called *smooth* if all its isotropy groups are open. The category $\mathrm{Rep}\,\bar{G}$ of smooth $\mathbb{C}$-linear representations of $\bar{G}$ is abelian.

Given $V \in \mathrm{Rep}\,\bar{G}$, let $H_0(\mathbb{C})$ denote the maximal $\bar{G}$-trivial quotient of V, e.g. the quotient modulo the subspace generated by elements of the form $v - gv$ for $v \in V$, $g \in \bar{G}$. Then H_0 defines a right-exact functor from $\mathrm{Rep}\,\bar{G}$ to the category of complex vector spaces.

Recall that

$$V_\emptyset(\mathbb{C}) := C^\infty(G/P_\emptyset, \mathbb{C}) / \sum_{i\in\Delta} C^\infty(G/P_{\{i\}}, \mathbb{C}) \quad ,$$

is a smooth $\bar{G}$-space, where for $I \subseteq \Delta = \{1, \ldots, d\}$, $P_I \subseteq G$ is the stabilizer of the standard flag τ_I in $(K^{d+1})^*$.

Theorem. 1.1 *There is a natural isomorphism*

$$\mathrm{Ext}^{d-*}(V_\emptyset(\mathbb{C}), \cdot) \cong H_*(\cdot) \quad ,$$

where Ext *is taken in* $\mathrm{Rep}\,\bar{G}$.

The *proof* of Theorem 1.1 will occupy the rest of this section. The key tool is an explicit projective resolution of $V_\emptyset(\mathbb{C})$ in terms of alternating cochains on the Bruhat-Tits building $\mathcal{BT}$ of G.

1.1 Oriented simplices

We consider $\mathcal{BT}$ as a *simplicial complex* (or, in the older terminology of Godement, as a *simplicial scheme*), that is, as a set together with a set of nonempty subsets (simplices) such that every nonempty subset of a simplex is a simplex. An *ordered q-simplex* σ of $\mathcal{BT}$ is a simplicial map

$$s : \Delta_q \to \mathcal{BT} \quad .$$

An ordered q-simplex σ of $\mathcal{BT}$ is called *degenerate* if the underlying map s is not injective. The set of ordered simplices on $\mathcal{BT}$ is *simplicial* in the sense that a simplicial map $\Delta_p \to \Delta_q$ of standard simplicial complexes induces a map from ordered q-simplices to ordered p-simplices. In particular, the natural action of the permutation group S_{q+1} on Δ_q induces an action on the set of ordered (non-degenerate) q-simplices; we let

$$\mathcal{BT}_{(q)} := \{s : \Delta_q \hookrightarrow \mathcal{BT}\}/\sim \quad ,$$

where two nondegenerate ordered q-simplices are identified if they lie in the same orbit under the induced action of the alternating group A_{q+1}. The set $\mathcal{BT}_{(q)}$ is called the set of oriented q-simplices on $\mathcal{BT}$.

1.2 Oriented chains

The space $C_q(\mathcal{BT})_{\mathbb{C}}$ of $\mathbb{C}$-valued q-chains on $\mathcal{BT}$ is defined to be the complex vector space freely generated by the ordered q-simplices $s : \Delta_q \to \mathcal{BT}$ of $\mathcal{BT}$. We view $C_*(\mathcal{BT})_{\mathbb{C}}$ as a *simplicial* object, in the sense that every simplicial map $\Delta_p \to \Delta_q$ induces a homomorphism $C_q(\mathcal{BT})_{\mathbb{C}} \to C_p(\mathcal{BT})_{\mathbb{C}}$.

We define the space $C_q^{\mathrm{or}}(\mathcal{BT})_{\mathbb{C}}$ of *oriented* $\mathbb{C}$-valued q-chains on $\mathcal{BT}$ to be the quotient of $C_q(\mathcal{BT})_{\mathbb{C}}$ modulo the following relations:

(i) $c \circ f = 0$, where $p < q$, $c \in C_p(\mathcal{BT})$ and $f : \Delta_q \to \Delta_p$;

(ii) $c \circ \tau - \mathrm{sgn}(\tau)c$, where $\tau \in S_{q+1}$ is viewed as an automorphism of Δ_q.

The quotient $C_*^{\mathrm{or}}(\mathcal{BT})_{\mathbb{C}}$ is in general not a simplicial object. However, the differential on $C_*(\mathcal{BT})_{\mathbb{C}}$ (given by the alternating sum over the faces) descends to $C_*^{\mathrm{or}}(\mathcal{BT})_{\mathbb{C}}$ such that we obtain a chain complex $(C_*^{\mathrm{or}}(\mathcal{BT})_{\mathbb{C}}, \partial_*)$. In the following, we omit the subscript $\mathbb{C}$.

1.3 Alternating cochains

We let $C^q_{\mathrm{alt}}(\mathcal{BT})$ denote the $\mathbb{C}$-linear dual of $C_q^{\mathrm{or}}(\mathcal{BT})$, that is, the space of functions φ on the set of ordered q-simplices $s : \Delta_q \to \mathcal{BT}$ with the properties that

(i) for $p < q$, $\varphi(\Delta_q \to \Delta_p \to \mathcal{BT}) = 0$.

(ii) for $\tau \in S_{q+1}$, $\varphi(s \circ \tau) = \mathrm{sgn}(\tau)\varphi(s)$.

The space $C^q_{\mathrm{alt}}(\mathcal{BT})$ is called the space of alternating cochains on $\mathcal{BT}$. Let $C^q_{\mathrm{alt},c}(\mathcal{BT})$ denote the subspace of finitely supported alternating cochains. The codifferential d^* on $C^*_{\mathrm{alt}}(\mathcal{BT})$ restricts to $C^*_{\mathrm{alt},c}(\mathcal{BT})$ since $\mathcal{BT}$ is locally finite: If φ is a finitely supported q-cochain, the support of φ lies in only finitely many $(q+1)$-simplices. We obtain a cochain complex $(C^*_{\mathrm{alt},c}(\mathcal{BT}), d^*)$ of compactly supported alternating cochains on $\mathcal{BT}$. Each space $C^q_{\mathrm{alt},c}(\mathcal{BT})$ is a smooth representation of $\bar{G}$ via the natural $\bar{G}$-action on $\mathcal{BT}$.

1.4 A finite projective resolution for $V_\emptyset(\mathbb{C})$

We will use the following fact: There exists a natural surjective (augmentation) morphism of smooth $\bar{G}$-representations $\varepsilon : C^d_{\mathrm{alt},c}(\mathcal{BT}) \to V_\emptyset(\mathbb{C})$ such that

$$0 \to C^0_{\mathrm{alt},c}(\mathcal{BT}) \xrightarrow{d^0} \dots \xrightarrow{d^{d-1}} C^d_{\mathrm{alt},c}(\mathcal{BT}) \xrightarrow{\varepsilon} V_\emptyset(\mathbb{C}) \to 0 \quad , \quad (*)$$

is exact, that is, is a resolution of $V_\emptyset(\mathbb{C})$. The proof uses results by Borel and Serre on the cohomology of the p-adically topologized Tits building, in particular on the contractibility of certain subspaces of this object. We will simply accept the existence of $(*)$.

In the remainder of this subsection, we will prove the following

Lemma. 1.2 *The smooth $\bar{G}$-modules $C^q_{\mathrm{alt},c}(\mathcal{BT})$ are projective.*

For each oriented q-simplex σ of $\mathcal{BT}$, we define the compactly supported alternating cochain ω_σ by setting

$$\omega_\sigma \, : \, \tau \mapsto \begin{cases} 1 & \text{for } \tau = \sigma \\ -1 & \text{for } \tau = \bar{\sigma} \\ 0 & \text{else} \end{cases} \quad ,$$

where $\sigma \mapsto \bar{\sigma}$ is the natural map $\mathcal{BT}_{(q)} \to \mathcal{BT}_{(q)}$ which inverts the orientation and where $\tau = \sigma$ means that τ projects to σ under the projection which forgets the ordering up to orientation.

Let $C(\omega_\sigma) \subseteq C^q_{\mathrm{alt},c}(\mathcal{BT})$ denote the (smooth) $\bar{G}$-subrepresentation which is generated by ω_σ. The ω_σ modulo orientation form a $\mathbb{C}$-basis for $C^q_{\mathrm{alt},c}(\mathcal{BT})$, and $\bar{G}$ acts on this basis with finitely many orbits (on the level of edges, the action is even transitive). Hence, $C^q_{\mathrm{alt},c}(\mathcal{BT})$ decomposes as a finite direct sum of representations of type $C(\omega_\sigma)$. It thus suffices to prove that the $C(\omega_\sigma)$ are projective objets in $\mathrm{Rep}^{\mathrm{sm}}_{\bar{G}}(\mathbb{C})$.

Remark. Let $B \subseteq \bar{G}$ be open and compact. Then $V \mapsto V^B$ is an exact functor on smooth $\bar{G}$-representations. Indeed, if $V \to V'' \to 0$ is a surjection of smooth $\mathbb{C}[\bar{G}]$-modules and if v'' is a B-invariant element in V'', let v be any preimage of v'' in V. Let $P_v \subseteq \bar{G}$ denote the stabilizer of v. Then $P_v \cap B$ is open (smoothness), and $B/(P_v \cap B)$ is finite of some nonzero cardinality $g \in \mathbb{N}$ (compactness of B). The average $(1/g)\sum_{\tau \in B/(P_v \cap B)} \tau(v)$ is a B-invariant lift of v''.

Let $B_\sigma \subseteq G$ denote the stabilizer of σ (as an oriented simplex). Then the image of B_σ in $\bar{G}$ is open and compact. We distinguish two cases:

Case 1. There exists no $h \in G$ such that $h\sigma = \bar{\sigma}$. In this case, $C(\omega_\sigma) \cong C_c^\infty(G/B_\sigma, \mathbb{C})$. Indeed, B_σ is open in G, so the right hand side is just the complex vector space generated by the elements of G/B_σ. On the other hand, G/B_σ can be identified with the G-*orbit* of σ in $\mathcal{BT}_{(q)}$. By our assumption, this orbit can be identified with the G-orbit of the underlying non-oriented simplex $\sigma \subseteq \mathcal{BT}$. Hence, we obtain the desired isomorphism. Now

$$\mathrm{Hom}_G(C(\omega_\sigma), V) \cong \mathrm{Hom}_G\Big(\bigoplus_{G/B_\sigma} \mathbb{C}, V\Big) \cong \mathrm{Hom}_{B_\sigma}(\mathbb{C}, V) \cong V^{B_\sigma} \quad ,$$

functorially in V; hence $\mathrm{Hom}_{\bar{G}}(C(\omega_\sigma), \cdot)$ is an exact functor on smooth $\bar{G}$-representations. That is, $C(\omega_\sigma)$ is projective.

Case 2. There exists an element $h \in G$ such that $h\sigma = \bar{\sigma}$. Then $h^2 \in B_\sigma$, and $hB_\sigma h^{-1} \subseteq B_\sigma$, that is, h normalizes B_σ. Now $C(\omega_\sigma) \subseteq C_c^\infty(G/B_\sigma, \mathbb{C})$ is the subspace of functions χ which satisfy $\chi(gh) = -\chi(g)$ for all $g \in G$. This is clear from the description above. It follows that $\mathrm{Hom}_{\bar{G}}(C(\omega_\sigma), V) \cong (V^{B_\sigma})^{h=-1}$, functorially in V. We are taking invariants of the group generated by $h \circ (\mathrm{mult}_{-1})^{-1}$ within the finite group $\bar{G}/\mathrm{im}\, B_\sigma$, so $\mathrm{Hom}_{\bar{G}}(C(\omega_\sigma), \cdot)$ is again exact, and $C(\omega_\sigma)$ is again projective.

We now have the projective resolution $(*)$ of $V_\emptyset(\mathbb{C})$ at our disposal. We may use it to calculate $\mathrm{Ext}^*(V_\emptyset(\mathbb{C}), \cdot)$. In particular, we see that $\mathrm{Ext}^r(V_\emptyset(\mathbb{C}), \cdot) = 0$ for $r > d$.

1.5 A functorial projective resolution for smooth represenations

We will need to find, for any smooth $\bar{G}$-module V, a surjection $V_0 \to V$ of smooth $\bar{G}$-modules such that $\mathrm{Ext}^r(V_\emptyset(\mathbb{C}), V_0) = 0$ for all $r < d$. We will construct V_0 in a way such that V_0 depends functorially on V. We will even construct a projective resolution $V_* \to V \to 0$ which is functorial in V and which has the property that $\mathrm{Ext}^r(V_\emptyset, V_i) = 0$ for all $i \in \mathbb{N}$ and all $r < d$.

We set $\mathcal{H} := C_c^\infty(\bar{G}, \mathbb{C})$. Let μ be a Haar measure on $\bar{G}$ with respect to which the measure of compact open sets is rational. Convolution with respect to μ defines on $\mathcal{H}$ the structure of an associative non-unital ring:

$$(\varphi * \psi)(g) := \int_G \varphi(h)\psi(h^{-1}g)\, dh$$

The integral is a finite sum since φ and ψ are locally constant and compactly supported. Note that the Dirac Delta function is not locally constant. Every smooth $\bar{G}$-module V becomes a module over $\mathcal{H}$ via

$$\varphi \cdot v := \int_G \varphi(h)h(v)\,dh \quad .$$

Again the integral is a finite sum, since v is fixed by a compact open subgroup and since φ is compactly supported. One obtains an equivalence between smooth $\bar{G}$-modules and certain modules over $\mathcal{H}$. The natural left $\mathcal{H}$-module structure of $\mathcal{H}$ corresponds to the left-regular representation: $L_h : \psi \mapsto L_h\psi$, where $L_h\psi : g \mapsto \psi(h^{-1}g)$. Let us show that this representation is indeed smooth. Let $\psi \in C_c^\infty(\bar{G}, \mathbb{C})$ be given; since ψ is locally constant and compactly supported, we may assume that $\psi = \chi_H$ for some compact open H in $\bar{G}$. After translation, we may assume that $1 \in H$. Then H contains a compact open subgroup K of $\bar{G}$. The isotropy group I_K of K with respect to the left translation action of $\bar{G}$ on $\mathcal{P}(\bar{G})$ contains K, hence it is open. Now H is covered by finitely many translates g_iK of K. The isotropy group of g_iK is conjugate to I_K, hence open. The intersection of these isotropy groups is nonempty open and contained in the isotropy group of H.

One has the following important result by Blanc:

Lemma. 1.3 *The $\bar{G}$-module $\mathcal{H}$ is projective.*

Proof. We choose a locally constant compactly supported function $\varepsilon \in C_c^\infty(\bar{G}, \mathbb{C})$ such that

$$\int_G \varepsilon(g)dg \;=\; 1 \quad .$$

Moreover, for $\varphi \in C_c^\infty(\bar{G}, \mathbb{C})$ and $g \in \bar{G}$, we define $\varphi_g : h \mapsto \varphi(h)\varepsilon(g^{-1}h)$. The map $g \mapsto \varphi_g$ is locally constant on G, by smoothness of the left regular representation. Its support is compact since the supports of φ and $L_g\varepsilon$ are disjoint for g outside a compact set. Now $\int_g \varphi_g dg = \varphi$, and $L_h(\varphi_g) = (L_h\varphi)_{hg}$. Let $p : U \to V \to 0$ be a surjective morphism of smooth $\bar{G}$-modules, and let $f : C_c^\infty(\bar{G}, \mathbb{C}) \to V$ be a $\bar{G}$-morphism. Let $s : V \hookrightarrow U$ be any $\mathbb{C}$-linear section of p. We define $f' : C_c^\infty(\bar{G}, \mathbb{C}) \to U$ by

$$f' \;:\; \varphi \mapsto \int_G gsg^{-1}f(\varphi_g)dg \quad .$$

The integral is a finite sum. Indeed, $f(\varphi_g)$ is locally constant and compactly supported on G, so we may assume it is constant. By smoothness of V, $g^{-1}f(\varphi_g)$ is locally constant on g, so we may also assume that $sg^{-1}(\varphi_g)$ is constant. By smoothness of U, $gsg^{-1}(\varphi_g)$ is locally constant on G. Since φ_g has finite support, the support of the integrand is finite.

It is clear that f' extends f: Indeed, $p(f'(\varphi)) = \int_g f(\varphi_g)dg = f(\int_g \varphi_g)dg = f(\varphi)$. Moreover, we see that f' is $\bar{G}$-equivariant: Let h be an element of $\bar{G}$; then

$$\begin{aligned} hf'(\varphi) &= \int_G (hg)s(hg)^{-1}hf(\varphi_g)dg \\ &= \int_G (hg)s(hg)^{-1}f((L_h\varphi)_{hg})d(hg) \\ &= f'(L_h\varphi) \quad . \end{aligned}$$

$\square$

Given $V \in \operatorname{Rep}\bar{G}$, we let $\tilde{V}$ denote V equipped with the *trivial* $\bar{G}$-action. Then $\mathcal{H} \otimes_{\mathbb{C}} \tilde{V}$ is a direct sum of projective modules, hence projective. We have a natural equivariant surjective homomorphism $\varrho_V : \mathcal{H} \otimes_{\mathbb{C}} \tilde{V} \to V$ sending $\varphi \otimes v$ to $\varphi \cdot v$. Surjectivity follows from the fact that for every $v \in V$, there is a $\varphi \in \mathcal{H}$ such that $\varphi \cdot v = v$. Indeed, it suffices to take $\varphi = \mu(I_v)^{-1}\chi_{I_v}$, where $I_v \subseteq G$ is the isotropy group of v and where χ_{I_v} is its characteristic function. We set $V_0 := \mathcal{H} \otimes_{\mathbb{C}} \tilde{V}$, and we let $\varrho_0 := \varrho_V$. Inductively, we set $V_i := \mathcal{H} \otimes_{\mathbb{C}} (\ker \varrho_{i-1})^{\sim}$, $\varrho_i := \varrho_{\ker \varrho_{i-1}}$. We thereby obtain the desired functorial projective resolution of V.

1.6 An acyclicity statement

We found, for every V, a functorial projective resolution $V_* \to V$, where the projectives V_i are of the form $\mathcal{H} \otimes_{\mathbb{C}} \tilde{W}$, $\tilde{W}$ denoting a trivial $\mathbb{C}[\bar{G}]$-module. In this section, we prove that $\operatorname{Ext}^r(V_\emptyset(\mathbb{C}), \mathcal{H} \otimes_{\mathbb{C}} \tilde{W}) = 0$ in degrees $r < d$. Projectivity of $\mathcal{H}$ is irrelevant for this business. We use the canonical projective resolution $(*)$ of $V_\emptyset(\mathbb{C})$ in order to calculate the Ext groups. The following Lemma says that $\mathcal{H}$ plays the role of a dualizing object to pass from cochains to chains:

Lemma. 1.4 *There is a natural isomorphism*

$$\varphi : \operatorname{Hom}_{\bar{G}}(C^q_{\mathrm{alt},c}(\mathcal{BT}), \mathcal{H}) \xrightarrow{\sim} C^{\mathrm{or}}_q(\mathcal{BT})$$

sending f *to* $(f(\omega_\sigma)(1))_\sigma$.

Proof. A $\bar{G}$-morphism f is defined by its values on finitely many ω_σ, and each $f(\omega_\sigma)$ has finite support on G; it follows that $f(\omega_\sigma)(1) \neq 0$ for just finitely many σ. The inverse $\bar{G}$-map is given as follows: we compose the double duality map $C^{\mathrm{or}}_q(\mathcal{BT}) \to C^{\mathrm{or}}_q(\mathcal{BT})^{**}$ with the projection onto $C^q_{\mathrm{alt},c}(\mathcal{BT})^* = \operatorname{Hom}_{\mathbb{C}}(C^q_{\mathrm{alt},c}(\mathcal{BT}), \mathbb{C})$ and with the natural map to $\operatorname{Hom}_{\bar{G}}(C^q_{\mathrm{alt},c}, C^\infty(\bar{G}, \mathbb{C}))$. This composition has image in $\operatorname{Hom}_{\bar{G}}(C^q_{\mathrm{alt},c}(\mathcal{BT}), C^\infty_c(G, \mathbb{C})$. $\square$

The above duality carries d^* to δ_*. More generally, we obtain

$$\operatorname{Hom}_{\bar{G}}(C^*_{\mathrm{alt},c}(\mathcal{BT}), \mathcal{H} \otimes \tilde{W}) \cong C^{\mathrm{or}}_*(\mathcal{BT}) \otimes \tilde{W} \quad .$$

Since $\mathcal{BT}$ is *contractible*, this complex is a resolution of $\tilde{W}$. Hence, the desired acyclicity statement follows.

1.7 The left derived functor of $\mathbf{Ext}^d(V_\emptyset(\mathbb{C}),\cdot)$

We have proved the following:

(i) $\mathrm{Ext}^r(V_\emptyset(\mathbb{C}),\cdot)$ vanishes in degrees $r > d$.

(ii) For every smooth $\bar{G}$-module V, there is a surjective morphism $V_0 \to V$ such that $\mathrm{Ext}^r(V_\emptyset(\mathbb{C}), V_0) = 0$ for all $r < d$.

The first statement implies that $\mathrm{Ext}^d(V_\emptyset(\mathbb{C}),\cdot)$ is right exact, and together with the second statement, general arguments from homological algebra imply that $\mathrm{Ext}^{d-*}(V_\emptyset(\mathbb{C}),\cdot)$ is the left derived functor of $\mathrm{Ext}^d(V_\emptyset(\mathbb{C}),\cdot)$.

1.8 Proof of the duality theorem

Since $H_*(\cdot)$ is the left derived functor of $H_0(\cdot)$, it suffices to find a functorial isomorphism

$$\mathrm{Ext}^d(V_\emptyset(\mathbb{C}),\cdot) \cong H_0(\cdot) \quad .$$

We will again use the natural projective resoultion $(*)$ of $V_\emptyset(\mathbb{C})$. It will now become clear why we used ordered chains and alternating cochains.

We fix a vertex σ of $\mathcal{BT}$. Oriented vertices are just vertices, and $\bar{G}$ acts transitively on the vertices of $\mathcal{BT}$. Hence, we have seen that $\mathrm{Hom}_{\bar{G}}(C^0_{\mathrm{alt},c}(\mathcal{BT}), V)$ is naturally isomorphic to V^{B_σ}. Moreover, the natural surjection $V \to H_0(V)$ induces, by exactness of taking B_σ-invariants, a surjection $V^{B_\sigma} \to H_0(V)^{B_\sigma} = H_0(V)$.

We claim that the image of $(d^0)^*$ lies in the kernel of this surjection. If we can show this, we obtain a surjective natural transformation $\mathrm{Ext}^d(V_\emptyset(\mathbb{C}),\cdot) \to H_0(\cdot)$. To justify the claim, it suffices to prove the following Lemma. Indeed, the Lemma implies that if $f : C^0 \to V$ lies in the image of $(d^0)^*$, it factors over $C^0 \to C^1$ and a map $C^1 \to V$. That is, every element in the image is the image of a combination of type $\sum_i c_i - g_i c_i$ and, hence, is itself such a combination and lies in the kernel of the map to the maximal trivial quotient.

Lemma. 1.5 *The image of d^0 is contained in the kernel of $C^1_{\mathrm{alt},c}(\mathcal{BT}) \to H_0(C^1_{\mathrm{alt},c}(\mathcal{BT}))$.*

Proof. We pick our generating element ω_σ in the domain $C^0_{\mathrm{alt},c}(\mathcal{BT})$ of d^0. The differential $d^0\omega_\sigma$ is $\sum \omega_{\langle\tau,\sigma\rangle}$, where τ varies among the neighboring vertices of σ and where $\langle\tau,\sigma\rangle$ denotes the oriented 2-simplex associated to the pair of neighboring vertices (τ,σ). Let $g \in G$ be such that $\tau = g\sigma$. Then also $\sigma^{-1}\tau$ is a neighbor of σ. Indeed, $\mathcal{BT}$ carries a translation invariant metric which has the property that two vertices are neighbors if and only if their distance with respect to that metric is ≤ 1. Let p denote the projection map onto the maximal $\bar{G}$-trivial quotient of $C^1_{\mathrm{alt},c}(\mathcal{BT})$. Then

$$p(\omega_{\langle\tau,\sigma\rangle} = p(\omega_{\langle g\sigma,\sigma\rangle}) = p(\omega_{\langle\sigma,g^{-1}\sigma\rangle}) = -p(\omega_{\langle g^{-1}\sigma,\sigma\rangle}) \quad ,$$

hence $p(\omega_{\langle\tau,\sigma\rangle} + p(\omega_{\langle g^{-1}\sigma,\sigma\rangle}) = 0$. If $g^{-1}\sigma = \sigma$, then already $p(\omega_{\langle\tau,\sigma\rangle} = 0$ must hold. It follows that $p(d^0(\omega_\sigma)) = 0$. □

It remains to prove that $\mathrm{Ext}^d(V_\emptyset(\mathbb{C}), V) \to H_0(V)$ is also *injective*. It suffices to prove this for $V = \mathcal{H}$. Indeed: We consider the functorial projective resolution $V_* \to V$ of V which we established above. We can use it to calculate $\mathrm{Ext}^{d-*}(V_\emptyset(\mathbb{C}), V)$ and $H_*(V)$ as the homology of the complexes $\mathrm{Ext}^d(V_\emptyset(\mathbb{C}), V_*)$ and $H_0(V_*)$. We have already established a surjective homomorphism of these complexes; if we can show that it is an isomorphism, we obtain an isomorphism on homology, and we are done. We note that $\mathrm{Ext}^d(V_\emptyset(\mathbb{C}), \mathcal{H} \otimes \tilde{W}) = \mathrm{Ext}^d(V_\emptyset(\mathbb{C}), \mathcal{H}) \otimes \tilde{W}$, similarly for H_0 (compatibility with direct sums). So indeed it suffices to establish injectivity for $V = \mathcal{H}$. Since we have already shown surjectivity, it suffices to prove that both $\mathrm{Ext}^d(V_\emptyset(\mathbb{C}), \mathcal{H})$ and $H_0(\mathcal{H})$ are one-dimensional. (The second statement slightly generalizes to the assertion that $H_0(\mathcal{H} \otimes \tilde{W}) = \tilde{W}$). We integrate to prove these statements:

Lemma. 1.6 *The function $\mathcal{H} \to \mathbb{C}$ that integrates over G factors over an isomorphism $H_0(\mathcal{H}) \xrightarrow{\sim} \mathbb{C}$.*

Proof. Integration is clearly surjective, and the integral over a locally constant function of type $f - L_g f$ is zero. It remains to see that every locally constant compactly supported function f of integral zero can be written as a linear combination of functions of type $f_i - L_{g_i} f_i$. We fix a compact open subgroup K in G with the property that $f = L_g f$ for every $g \in K$. This is possible since the isotropy group of f is open (smoothness of the left regular representation) and since the compact open sets form a basis for the topology on G. Now we can write

$$\begin{aligned} f &= \sum_{g \in K\backslash G} f(g) L_g \chi_K \\ &= \sum_{g \in K\backslash G} f(g)(L_g \chi_K - \chi_K) + \chi_K \cdot \mu(K)^{-1} \int_G f(g)dg \quad , \end{aligned}$$

so the statement is clear. The sums are finite since f has compact support. □

Lemma. 1.7 *There is a surjective homomorphism* $\mathrm{Hom}_G(C^0_{\mathrm{alt},c}(\mathcal{BT}), \mathcal{H}) \to \mathbb{C}$ *whose kernel is the image of* $(d^0)^*$.

Proof. We have identified the '$\mathcal{H}$-dual' of the alternating cochain complex with the oriented chain complex, so we need to prove the assertion for 0-chains on $\mathcal{BT}$ and the image of ∂_0.

Integration is given by summation, which is surjective. An oriented 1-chain is mapped to the difference of its endpoints. Clearly this lies in the kernel of summation. So we look at a 0-chain c with $\sum_\sigma c_\sigma = 0$. Looking at the resolution $C_*^{\mathrm{or}}(\mathcal{BT}) \to \mathbb{C} \to 0$ of $\mathbb{C}$, where augmentation is summation, the statement follows from the contractibility of $\mathcal{BT}$. $\square$

The duality theorem has been proved.

2 Consequences of the duality theorem

We can now calculate the remaining Ext groups which we need to understand the cohomology of X_Γ:

Proposition. 2.1

$$\mathrm{Ext}^r(V_I(\mathbb{C}), V_\emptyset(\mathbb{C})) = \begin{cases} \mathbb{C} & \text{if } r = \sharp I \\ 0 & \text{else} \end{cases}$$

Proof. We have that

$$\mathrm{Ext}^r(V_I(\mathbb{C}), V_\emptyset(\mathbb{C})) = \mathrm{Ext}^r(V_\emptyset(\mathbb{C}), V_{\bar{I}}(\mathbb{C})) \quad ,$$

where $\bar{I}$ is the set $\{d+1-i\,;\, i \in I\} \subseteq \Delta = \{1, \ldots, d\}$. Indeed, one can show that $V_{\bar{I}}(\mathbb{C})$ is the smooth contragredient of $V_I(\mathbb{C})$, that is, the smooth $\bar{G}$-representation given by the subspace of smooth vectors inside the $\mathbb{C}$-linear dual of $V_I(\mathbb{C})$. In particular, $V_\emptyset(\mathbb{C})$ is its own contragredient. We know that $V_\emptyset(\mathbb{C})$ is *admissible*, that is, for each open subgroup in G the space fixed under that group has finite dimension. This justifies the above equality. Now by the duality theorem,

$$\mathrm{Ext}^r(V_\emptyset(\mathbb{C}), V_{\bar{I}}(\mathbb{C})) = H_{d-r}(V_{\bar{I}}(\mathbb{C})) \quad .$$

The right hand side, however, is the $\mathbb{C}$-linear dual of $\mathrm{Ext}^{d-*}(\mathbb{C}, V_I(\mathbb{C}))$ (group cohomology of $V_I(\mathbb{C})$), and we have already determined this space in the previous talk. $\square$

In the following, let $\mu(\Gamma)$ denote the multiplicity of $V_\emptyset(\mathbb{C})$ in Ind_Γ. Collecting our results, we see that we have proved the following:

Theorem. 2.2 *Let A be a field of characteristic zero. Then*

$$\mathrm{Ext}^r_{A[\Gamma]}(V_I(A), A) \cong \begin{cases} A & \text{if } \sharp I = d - r \neq d/2 \\ A^{\mu(\Gamma)} & \text{if } \sharp I = r \neq d/2 \\ A^{\mu(\Gamma)+1} & \text{if } \sharp I = r = d/2 \\ 0 & \text{otherwise} \end{cases}$$

We can now compute the cohomology of X_Γ. Remember that we have a spectral sequence

$$H^r(\Gamma, H^s(\Omega^{d+1})) \Rightarrow H^{r+1}(X_\Gamma)$$

in the each of the following cases: Γ is torsion free, H is de Rham cohomology or the topology of our site is finer than the étale topology. Let us assume that we are in this situation and that A is a field of characteristic zero. Recall that $H^s(\Omega^{d+1}) = \mathrm{Hom}_A(V_{\{1,\ldots,d-s\}}(A), A)$ whenever $s \leq d$, hence the $E_2^{r,s}$-terms are given by the corresponding Ext-groups. We have shown that $E_2^{r,s} = 0$ unless $r = s$ or $r + s = d$. If d is even, then all differentials must be zero. (If d is odd, there are two possibly nonzero differentials.) Hence, if d is even, we can write down the cohomology:

$$H^t(X_\Gamma) \cong \begin{cases} A & \text{if } 0 \leq t \leq 2d,\ t \neq d,\ t \text{ even} \\ A^{(d+1)\mu(\Gamma)+1} & \text{if } t = d \\ 0 & \text{otherwise} \end{cases} .$$

Indeed, we have to sum up dimensions along the lines where $r + s = t$. The middle line is clear. To get something nonzero, $s + r$ cannot be greater than $2d$. If t is odd, we 'jump' over the diagonal $s = r$. If t is even, we don't, and we hit $E_2^{r,r}$ with $r = t/2$. If $r = d/2$, this is zero; if not, it is A.

The only cohomology group which carries a nontrivial extension structure is $H^d(X_\Gamma)$; the subquotients of the natural filtration are given by $A^{\mu(\Gamma)}$ (away from the middle) and $A^{\mu(\Gamma)+1}$ (in the middle).

If d is odd, we have (at most) two nonvanishing differentials. One reduces an E_2-term on the anti-diagonal to the kernel of a map to a one-dimensional space, the other reduces another E_2-term on the antidiagonal to the quotient modulo the image of a one-dimensional space. So the dimension drops at most by 2. Note that the middle term which gives $\mu(\Gamma)+1$ instead of $\mu(\Gamma)$ does not show up in the odd dimension case. Hence, the A-dimension of $H^d(X_\Gamma)$ lies between $(d+1)\mu(\Gamma) - 2$ and $(d+1)\mu(\Gamma)$.

Let us now specialize to l-adic étale cohomology, with l a prime number different from p. The cohomolgy groups are Galois modules $H^t_{\text{ét}}(\overline{X_\Gamma}, \mathbb{Q}_l)$, where $\overline{X}_\Gamma$ denote the base change of X_Γ to an algebraic closure of K. We know that

$$H^s(\overline{\Omega}^{d+1}, \mathbb{Q}_l(s)) = \mathrm{Hom}_{\mathbb{Q}_l}(V_{1,\ldots,d-s}(\mathbb{Q}_l), \mathbb{Q}_l)$$

is a Galois-equivariant identification. Hence, every $E_2^{r,s}$-term gets a twist of $-s$, viewed as a *weight*-term. There are no Galois-equivariant linear maps between different twists of $\mathbb{Q}_l$, so the spectral sequence degenerates. Again, we can read off $H^t_{\text{ét}}(\overline{X_\Gamma}, \mathbb{Q}_l)$: it vanishes outside the $2d$ range, and away from the middle dimension, $t \neq d$, we have $\mathbb{Q}(-t/2)$ if t is even (hit diaogonal), zero if t is odd. The subquotients of the middle cohomology are given by $\mathbb{Q}_l(d-r)^{\mu(\Gamma)}$ for $r \neq d/2$ and $\mathbb{Q}_l(d-r)^{\mu(\Gamma)+1}$ for $r = d/2$ (only occurs for even d).

Finally, we turn the the case of de Rham cohomology. We use the fact that the quotient X_Γ is algebraic. By the GAGA principle, coherent cohomology groups and de Rham cohomology groups may be formed on the analytic or on the algebraic level. Since K has characteristic zero, the de Rham spectral sequence (a hypercohomology spectral sequence) degenerates. This follows from the Hodge theorem. It follows that

$$\dim_K H^t_{\mathrm{DR}}(X_\Gamma) \;=\; \sum_{r+s=t} \dim_K H^r(X_\Gamma, \Omega^s) \quad .$$

Moreover, the hard Lefschetz theorem holds, that is, the k-fold product with the cohomology class of a hyperplane induces an isomorphism $H^{d-k} \cong H^{d+k}$, where H here is singular cohomology or, equivalently, de Rham-cohomology. It follows that $\dim_K H^s(X_\Gamma, \Omega^s) \geq 1$. It follows that the spectral sequence degenerates (since the dimension on the diagonal cannot drop, the two critical differentials must be zero). One can finally determine some of the coherent cohomology groups $H^s(X_\Gamma, \Omega^j)$: For $s = j \neq d/2$, they must be one-dimensional since $H^{2s}_{\mathrm{DR}}(X_\Gamma)$ is one-dimensional. For the same reason (low-dimensionality), they must vanish for $s \neq j$, $s + j \neq d$.