Derived Categories

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[Gelfand-Manin: Methods of homological Algebra]

1. Motivation

a) Derived Functors

eg $\mathcal{F}$ sheaf of ab Gps on topol space X, $H^i(X,.) = R^i\Gamma(X,.)$

injective resolution $0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{G}^0 \xrightarrow{d^0} \mathcal{G}^1 \xrightarrow{d^1} \ldots$

apply $\Gamma(X,.)$: $0 \xrightarrow{d^{-1}} \Gamma(\mathcal{G}^0) \xrightarrow{d^0} \Gamma(\mathcal{G}^1) \xrightarrow{d^1} \ldots$

$H^i(X,\mathcal{F}) = \ker d^i / \operatorname{im} d^{i-1}$

Idea: replace sheaf $\mathcal{F}$ by complex of sheaves $\mathcal{G}^0 \to \mathcal{G}^1 \to \ldots$ which are acyclic for Γ, i.e. $R^i\Gamma(\mathcal{G}^j) = 0 \ \forall i \neq 0$

One should only apply Γ to those

b) Poincaré-Duality

2. Derived Categories

$\mathcal{C}$ = abelian category

$Kom(\mathcal{C})$ = category of complexes $K^\bullet : \ldots \to K^i \xrightarrow{d^i} K^{i+1} \to \ldots$ with $K^i \in Ob(\mathcal{C})$, $d^{i+1} \circ d^i = 0 \ \forall i$

Morphisms:

$$\begin{array}{ccccccc}
K^\bullet : & \ldots \to & K^i & \xrightarrow{d^i} & K^{i+1} & \to \ldots \\
f\downarrow & & f\downarrow & /\!/\!/ & \downarrow f & \\
L^\bullet : & \ldots \to & L^i & \xrightarrow{d^i} & L^{i+1} & \to \ldots
\end{array}$$

$Kom^+(\mathcal{C})$: $K^i = 0 \ \forall i \ll 0$

$Kom^-(\mathcal{C})$: $K^i = 0 \ \forall i \gg 0$ — full subcategories

$Kom^b(\mathcal{C})$: $K^i = 0 \ \forall i \gg 0 \ \forall i \ll 0$

They are abelian categories

Def: a) $H^i(K^\bullet) := \ker d^i / \operatorname{im} d^{i-1}$ <u>i-th cohomology of $K^\bullet$</u>

b) $f: K^\bullet \to L^\bullet$ is called a <u>quasi-isomorphism</u> if it induces an isom $f: H^i(K^\bullet) \to H^i(L^\bullet)\ \forall i$

Eg:
$$\begin{array}{c} 0 \to \mathcal{F} \to 0 \to \dots \\ \downarrow \varepsilon \\ 0 \to \mathcal{G}^0 \xrightarrow{d^1} \mathcal{G}^1 \to \dots \end{array}$$
ε is a <u>qis</u>

Def: The <u>derived category</u> $D(\mathcal{C})$ is obtained from $\operatorname{Kom}(\mathcal{C})$ by keeping the objects and formally inverting all qis

i.e. $\operatorname{Hom}_{D(\mathcal{C})}(K^\bullet, L^\bullet) = \{ K^\bullet = M_1^\bullet \xrightarrow{f_1^{\pm 1}} M_2^\bullet \xrightarrow{f_2^{\pm 1}} \dots \to M_n^\bullet = L^\bullet$ where f_i^{-1} means that $M_i \xleftarrow{f_i} M_{i+1}$ is a qis $\}$

Analogous: $D^*(\mathcal{C})$ from $\operatorname{Kom}^*(\mathcal{C})$ for $* = +, -, b$

<u>Theorem:</u> $D(\mathcal{C})$ (and $D^*(\mathcal{C})$, $*=+,-,b$) are <u>additive categories</u> but in general not abelian (i.e. $\operatorname{Hom}_{D(\mathcal{C})}$ are ab gps, $\exists\, 0^\bullet$ zero object, $\exists\, K^\bullet \oplus L^\bullet$ direct sums, BUT not ker, coker)

Proof: $f_1^{\pm 1} \circ f_2^{\pm 1} \dots \circ f_n^{\pm 1} + g_1^{\pm 1} \circ \dots \circ g_m^{\pm 1}$ have to find „common denominator": pull all inverses to the left (or right)

<u>Caution:</u> not possible in $\operatorname{Kom}(\mathcal{C})$

work in homotopy category $K(\mathcal{C})$ (there $f = g$ if they are homotopic)

then $\operatorname{Kom}(\mathcal{C}) \to K(\mathcal{C}) \to D(\mathcal{C})$ and everything works $\square$

<u>Problem:</u> $\nexists$ exact sequences in $D(\mathcal{C})$

$\exists\, h^i: K^i \to L^{i-1}$ s.t.
$f^i - g^i = d_L^{i-1} \circ h^i + h^{i+1} \circ d_K^i$

3. Distinguished Triangles

$f: K^\bullet \to L^\bullet$ in $Kom(\mathcal{C})$

Def a) Translation by $n \in \mathbb{Z}$: $K[n]^\bullet$ is the complex

$$K[n]^i = K^{n+i}, \quad d_{K[n]} = (-1)^n d_K$$

b) Cone of f: $C(f)$ is the complex

$$C(f)^i = K[1]^i \oplus L^i = K^{i+1} \oplus L^i, \quad d_{C(f)} = \begin{pmatrix} d_{K[1]} & 0 \\ f[1] & d_L \end{pmatrix}$$

i.e. $d_{C(f)}(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$

c) Cylinder of f: $Cyl(f)$ is the complex

$$Cyl(f)^\bullet = K^\bullet \oplus K[1]^\bullet \oplus L^\bullet, \quad d_{Cyl(f)} = \begin{pmatrix} d_K & -1 & 0 \\ 0 & d_{K[1]} & 0 \\ 0 & f[1] & d_L \end{pmatrix}$$

$$d_{Cyl(f)}(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$$

Prop: a) $\alpha: L^\bullet \xrightarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} Cyl(f)$, $l^i \mapsto (0,0,l^i)$ is a quasi-isom

b) $0 \to K^\bullet \xrightarrow{\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)} Cyl(f)^\bullet \xrightarrow{\left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)} C(f)^\bullet \to 0$ is exact in $Kom(\mathcal{C})$

c) $\exists\ \delta: C(f)^\bullet \xrightarrow{(1,0)} K[1]^\bullet$ Morphism

Def: A triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$
$\left(\begin{tikzcd} A \arrow[r] & B \arrow[dl] \\ C \arrow[u, "+1"] \end{tikzcd}\right)$

in $D(\mathcal{C})$ is called distinguished if

$$\exists \quad \begin{tikzcd} K^\bullet \arrow[r, "\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)"] \arrow[d, "\alpha"] \arrow[dr, phantom, "///"] & Cyl(f) \arrow[r, "\left(\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right)"] \arrow[d, "\beta"] \arrow[dr, phantom, "///"] & C(f) \arrow[r, "(1,0)"] \arrow[d, "\gamma"] \arrow[dr, phantom, "///"] & K[1]^\bullet \arrow[d, "\alpha[1]"] \\ A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & A[1]^\bullet \end{tikzcd} \qquad \alpha, \beta, \gamma \text{ qis}$$

All that is really needed for cohomology are distinguished triangles:

Theorem: a) If $0 \to K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \to 0$ is exact in $Kom(\mathcal{C})$

then
$$\begin{tikzcd} K^\bullet \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "///"] & Cyl(f) \arrow[r] \arrow[d, "\alpha^{-1}"] \arrow[dr, phantom, "///"] & C(f) \arrow[r, "(1,0)"] \arrow[d, "\gamma"] \arrow[dr, phantom, "///"] & K[1]^\bullet \arrow[d, equal] \\ K^\bullet \arrow[r, "f"] & L^\bullet \arrow[r, "g"] & M^\bullet \arrow[r, "(1,0)\circ\gamma^{-1}"] & K[1]^\bullet \end{tikzcd}$$
with $\gamma(k^{i+1}, l^i) = g(l^i)$ qis

b) If $K^\bullet \xrightarrow{f} L^\bullet \xrightarrow{g} M^\bullet \xrightarrow{h} K[1]^\bullet$ is a distinguished triangle

then $\ldots \to H^i(K^\bullet) \xrightarrow{H^i(f)} H^i(L^\bullet) \xrightarrow{H^i(g)} H^i(M^\bullet) \xrightarrow{H^i(h)} H^i(K[1]^\bullet)$
$= H^{i+1}(K^\bullet) \to \ldots$

is exact in $\mathcal{C}$ (long exact cohomology sequence)

$D(\mathcal{C})$ with dist triangles is a triangulated category
For cohomology this is as good as an abelian category

4. Derived Functors

Classically

Eg: a) X topol space, $\Gamma(X,.)$: (sheaves of ab gps on X) $\to$ (ab gps)
left exact derived functor $H^i(X,\bullet)$ via injective resolutions

b) R commut ring, $\mathcal{C} = (R\text{-mod})$

$$\otimes_R : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad (\mathcal{F},\mathcal{G}) \mapsto \mathcal{F} \otimes_R \mathcal{G}$$

$$\mathrm{Hom}_R : \mathcal{C}^\circ \times \mathcal{C} \to \mathcal{C}, \quad (\mathcal{F},\mathcal{G}) \mapsto \mathrm{Hom}_R(\mathcal{F},\mathcal{G})$$

are not exact and have derived functors
Tor_i^R, Ext_R^i classically defined via injective, projective resolutions.

Want to extend them to $D^*(\mathcal{C})$
Idea: apply them to complexes componentwise
But: only well behaved for injective, projective objects

<u>Def</u>: $F: \mathcal{C} \to \mathcal{C}'$ additive left exact functor between ab cat.
Then a full additive subcategory $\mathcal{A} \subseteq \mathcal{C}$ is called <u>adapted to F</u> if

a) Every $K^\bullet \in \mathrm{Kom}^+(\mathcal{C})$ is quasi-isomorphic to an $A^\bullet \in \mathrm{Kom}^+(\mathcal{A})$

b) For every $A^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ which is quasi-isom to $0^\bullet$ (i.e. $H^i(A^\bullet)=0\ \forall i$)
$F(A^\bullet): \ldots \to F(A^i) \xrightarrow{F(d^i)} F(A^{i+1}) \to \ldots \in \mathrm{Kom}^+(\mathcal{C}')$ is quasi-is to $0^\bullet$

If F is right exact replace Kom^+ by Kom^-

<u>Eg</u>: a) (injective sheaves) is adapted to $\Gamma(X,.)$ see page 5
(flabby sheaves) — " —

b) (projective R-mod), (flat R-mod) are adapted to $\bullet \otimes_R M$

c) (injective R-mod) is adapted to $\mathrm{Hom}_R(M,\bullet)$

<u>Theorem</u> Let $F: \mathcal{C} \to \mathcal{C}'$ be left exact and assume $\exists\ \mathcal{A} \subseteq \mathcal{C}$ adapted to F

Define $RF: D^+(\mathcal{C}) \to D^+(\mathcal{C}')$ by

$RF(K^\bullet) := F(A^\bullet)$ if $A^\bullet \in \mathrm{Kom}^+(\mathcal{A})$ is quasi-isom to $K^\bullet$

Then
a) RF is independent of the choice of $\mathcal{A} \subseteq \mathcal{C}$ and $A^\bullet \in \mathrm{Kom}^+(\mathcal{A})$
b) RF is <u>exact</u> (i.e. maps dist triangles into dist triangles)
c) the classical $R^iF(\cdot) \cong H^i(RF(\cdot))$ on $\mathcal{A}$

<u>Def:</u> $H^i(RF(K^\bullet))$ is called the i-th <u>hypercohomology</u> of the functor F wrt $K^\bullet$

For a right (left) exact bifunctor $F: \mathcal{C} \times \mathcal{C}' \to \mathcal{C}''$ one defines $F: \mathrm{Kom}^{-(+)}(\mathcal{C}) \times \mathrm{Kom}^{-(+)}(\mathcal{C}') \to \mathrm{Kom}^{-(+)}(\mathcal{C}'')$

$$F(K^\bullet, L^\bullet) := \bigoplus_{j \in \mathbb{Z}}^{\text{finite}} F(K^{i-j}, L^j)$$

To define LF (RF) use $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{A}' \subseteq \mathcal{C}$ s.t. $\forall\ X \in \mathcal{A}$ $\mathcal{A}'$ is adapted to $F(X, \cdot)$ and $\forall X' \in \mathcal{A}'$, $\mathcal{A}$ is adapted to $F(\cdot, X')$

define $LF: D^{-(+)}(\mathcal{C}) \times D^{-(+)}(\mathcal{C}') \to D^{-(+)}(\mathcal{C}'')$ via

$$LF(K^\bullet, K'^\bullet) = F(A^\bullet, A'^\bullet) \qquad (\text{resp } RF \ldots)$$
$$= F(K^\bullet, A'^\bullet) = F(A^\bullet, K'^\bullet)$$

Have $L^iF \cong H^i(LF)$

<u>Eg:</u> $F = \cdot \otimes_B \cdot$, $LF =: \cdot \overset{L}{\otimes}_B \cdot$, $L^iF = \mathrm{Tor}_i^B(\cdot, \cdot)$

<u>Theorem</u> Let $\mathcal{C} \xrightarrow{F} \mathcal{C}' \xrightarrow{G} \mathcal{C}''$ be left exact. Assume $\exists\ \mathcal{A} \subseteq \mathcal{C}$ adapted to F, $\mathcal{A}' \subseteq \mathcal{C}'$ adapted to G with $F(\mathcal{A}) \subseteq \mathcal{A}'$ then $\mathcal{A}$ is adapted to $G \circ F$ and

$$R(G \circ F) = RG \circ RF : D^+(\mathcal{C}) \to D^+(\mathcal{C}'')$$

(composition of functors spectral sequence)

<u>5. Grothendieck's 6 functors</u>

Let X be a topol space, $\mathcal{S}$ a sheaf of commut rings on X (stalks: finite global dim)

$f: Y \to X$ cont map of topol spaces

$(\mathcal{S}\text{-mod}/X) = $ (sheaves of $\mathcal{S}$-modules on X)

1. $R\mathcal{H}om_{\mathcal{S}}(\cdot,\cdot) : D^-(\mathcal{S}\text{-mod}/X)^\circ \times D^+(\mathcal{S}\text{-mod}/X) \longrightarrow D^+(\mathcal{S}\text{-mod}/X)$

derived of sheaf-Hom

2. $\cdot \overset{L}{\underset{\mathcal{S}}{\otimes}} \cdot : D^-(\mathcal{S}\text{-mod}/X) \times D^-(\mathcal{S}\text{-mod}/X) \to D^-(\mathcal{S}\text{-mod}/X)$

$R\mathcal{H}om$ is right adjoint to $\overset{L}{\otimes}$, even

<u>Thm</u> $R\mathcal{H}om_{\mathcal{S}}(\mathcal{F} \overset{L}{\underset{\mathcal{S}}{\otimes}} \mathcal{G}, \mathcal{H}) \cong R\mathcal{H}om_{\mathcal{S}}(\mathcal{F}, R\mathcal{H}om_{\mathcal{S}}(\mathcal{G},\mathcal{H}))$

$\forall\, \mathcal{F},\mathcal{G} \in D^-(\mathcal{S}\text{-mod}/X)$, $\mathcal{H} \in D^+(\mathcal{S}\text{-mod}/X)$

3. $f_* : (ab/Y) \to (ab/X)$, $f_*\mathcal{F}(\mathcal{U} \subseteq X) := \mathcal{F}(f^{-1}\mathcal{U})$

<u>direct image</u> left exact

$Rf_* : D^+(ab/Y) \to D^+(ab/X)$

4. $f^* : (ab/X) \to (ab/Y)$

$f^*\mathcal{F} = \text{sheaf}\left(\mathcal{U} \subseteq Y \mapsto \varinjlim_{V \subseteq X,\, f(\mathcal{U}) \subseteq V} \mathcal{F}(V)\right)$

<u>inverse image</u>

stalks $(f^*\mathcal{F})_y = \mathcal{F}_{f(y)} \Rightarrow f^*$ is exact

$\Rightarrow Rf^* = f^* = Lf^* : D(ab/X) \to D(ab/Y)$

also $f^* : D(\mathcal{S}\text{-mod}/X) \to D(f^*\mathcal{S}\text{-mod}/Y)$

and $Rf_* : D(f^*\mathcal{S}\text{-mod}/Y) \to D(\mathcal{S}\text{-mod}/X)$

f^* is left adjoint to Rf_*, even

<u>Thm</u> $R\mathcal{H}om_{\mathcal{S}}(\mathcal{F}, Rf_*\mathcal{G}) \cong Rf_* R\mathcal{H}om_{f^*\mathcal{S}}(f^*\mathcal{F}, \mathcal{G})$

5. Let X, Y be locally compact

$f_! : (f^*\mathcal{S}\text{-mod}/Y) \to (\mathcal{S}\text{-mod}/X)$

$f_!\mathcal{F}(\mathcal{U} \subseteq X) := \{ s \in \mathcal{F}(f^{-1}\mathcal{U}) : \operatorname{supp}(s) \xrightarrow{f} \mathcal{U} \text{ is proper}\}$

$\operatorname{supp}(s) = \{ y \in f^{-1}\mathcal{U} : s_y \neq 0\} \subseteq f^{-1}\mathcal{U}$ closed

proper = preimage of compacta are compact

<u>direct image with proper support</u>

$f_! \mathcal{F} \subseteq f_* \mathcal{F}$ sub-sheaf , $f_!$ left exact

$Rf_! : D^+(f^*\mathcal{S}\text{-mod}/Y) \longrightarrow D^+(\mathcal{S}\text{-mod}/X)$

6. $\mathcal{S} = A_X$ constant sheaf , A = commut ring with 1 (finite global dim)
X, Y loc comp
$\exists\, r \in \mathbb{N} : R^i f_! = 0 \quad \forall\, i > r$

<u>Theorem</u> Under the above assumptions there exists an exact functor $f^! : D^+(A_X\text{-mod}/X) \to D^+(A_Y\text{-mod}/Y)$ <u>inverse image with compact support</u> which is right adjoint to $Rf_!$. Even

$$R\mathcal{H}om_{A_X}(Rf_! \mathcal{F}, \mathcal{G}) \cong Rf_* R\mathcal{H}om_{A_Y}(\mathcal{F}, f^! \mathcal{G})$$

Various isomorphisms

eg:
$$\begin{tikzcd} Z \times_X Y \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"] & X \end{tikzcd}$$
then $g^* \circ Rf_! \cong Rf'_! \circ g'^*$

$D^+(f^*\mathcal{S}\text{-mod}/Y) \to D^+(g^*\mathcal{S}\text{-mod}/X)$

[Kashiwara – Shapira: Sheaves on Mfd, § II.6]

Remark: a) If f is proper then $f_! = f_*$, $Rf_! = Rf_*$
(not $f^* = f^!$)

b) If $X = \{pt\}$ then $f_* = \Gamma(Y, .)$
$f_! = \Gamma_c(Y, .)$

↘

6. Poincaré - Duality

Apply the last Thm to

$Y =$ n-dim oriented topol manifold

$X = \{pt\}$, $f : Y \to X$

$A = \mathbb{Q}$

$\mathcal{F} = \mathbb{Q}_Y[n] \;:\; \ldots \to 0 \to \underset{\stackrel{\uparrow}{n}}{\mathbb{Q}_Y} \to 0 \to \ldots$

$\mathcal{G} = \mathbb{Q}_X$

__Theorem__ $\quad f^! \mathbb{Q}_X \cong \mathbb{Q}_Y[n]$

Thus

$$\underbrace{R\mathcal{H}om_{\mathbb{Q}_X}}_{\substack{\| \\ \mathrm{Hom}_{\mathbb{Q}} \text{ exact}}}(\underbrace{R\Gamma_c(Y, \mathbb{Q}_Y)[n]}_{\mathbb{Q}-vs}, \underset{\substack{\| \\ \mathbb{Q}}}{\mathbb{Q}_X}) \cong R\Gamma(Y, \underbrace{R\mathcal{H}om_{\mathbb{Q}_Y}(\mathbb{Q}_Y[n], \mathbb{Q}_Y[n])}_{= \mathbb{Q}_Y})$$

with $\mathbb{Q}_X = \mathbb{Q}$ (subscript) $\| \; \mathbb{Q}$

$\Rightarrow \quad \mathrm{Hom}_{\mathbb{Q}}(R\Gamma_c(Y, \mathbb{Q}_Y)[n], \mathbb{Q}) \cong R\Gamma(Y, \mathbb{Q}_Y)$

Take H^i

$\Rightarrow \quad \mathrm{Hom}_{\mathbb{Q}}(H_c^{n-i}(Y, \mathbb{Q}_Y), \mathbb{Q}) \cong H^i(Y, \mathbb{Q}_Y)$ Poincaré-Duality

__Moral__: Theorem $f^! A_X = \ldots$
gives Duality-Theorem

Proof for (injective sheaves) is adapted to $\Gamma(X,.)$

$$
\begin{tikzcd}
- \quad A^\bullet : & 0 \arrow[r] & A^0 \arrow[r, "\partial^0"] \arrow[d, "\varepsilon^0"] & A^1 \arrow[r, "\partial^1"] \arrow[d, "\varepsilon^1"] & A^2 \arrow[r, "\partial^2"] \arrow[d, "\varepsilon^2"] & \dots \\
& & I^{00} \arrow[r, "d_I"] \arrow[d, "d_{II}"] & I^{01} \arrow[r] \arrow[d, "d_{II}"] & I^{02} \arrow[r] \arrow[d] & \dots \\
& & I^{10} \arrow[r, "d_I"] \arrow[d, "d_{II}"] & I^{11} \arrow[r] \arrow[d, "d_{II}"] & I^{12} \arrow[r] \arrow[d] & \dots \\
& & I^{20} \arrow[r, "d_I"] \arrow[d] & I^{21} \arrow[r] \arrow[d] & I^{22} \arrow[r] \arrow[d] & \dots \\
& & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
S^\bullet : 0 \to \underbrace{I^{00}}_{S^0} \xrightarrow{D^0} \underbrace{I^{01} \oplus I^{10}}_{S^1} \xrightarrow{D^1} \underbrace{I^{02} \oplus I^{11} \oplus I^{20}}_{S^2} \to \dots
$$

$$b^{00} \longmapsto (d_I b^{00}, d_{II} b^{00})$$

$$(b^{01}, b^{10}) \longmapsto (d_I b^{01}, d_{II} b^{01} + d_I b^{10}, d_{II} b^{10})$$

$\varepsilon \uparrow$

$$\varepsilon^0(a^0) \;\leftarrow\; a^0 \qquad (\varepsilon^1(a^1), 0) \;\leftarrow\; a^1 \qquad (\varepsilon^2(a^2), 0, 0) \;\leftarrow\; a^2$$

$$A^\bullet : 0 \longrightarrow A^0 \xrightarrow{\partial^0} A^1 \xrightarrow{\partial^1} A^2 \longrightarrow \dots$$

ε is a quasi-isom:

$\underline{\varepsilon^0}$: $\ker \partial^0 \hookrightarrow \ker D^0$ bc $(\underbrace{d_I\, \varepsilon^0(a^0)}_{\varepsilon^1 \circ \partial^0(a^0)}, \underbrace{d_{II}\, \varepsilon^0(a^0)}_{=0}) = (0,0)$

surj: $b^{00} \in \ker D^0$

$d_{II} b^{00} = 0 \Rightarrow \exists\, a^0 \in A^0 : \varepsilon^0(a^0) = b^{00}$

$0 = d_I b^{00} = d_I\, \varepsilon^0(a^0) = \underset{\substack{\uparrow \\ \text{inj}}}{\varepsilon^1} \circ \partial^0(a^0) \Rightarrow a^0 \in \ker \partial^0$ ✓

$\underline{\varepsilon^1}$: $\ker \partial^1 / \operatorname{im} \partial^0 \longrightarrow \ker D^1 / \operatorname{im} D^0$

inj: $a^1 \in \ker \partial^1$ with $(\varepsilon^1(a^1), 0) = D^0(b^{00}) = (d_I b^{00}, d_{II} b^{00})$

$\Rightarrow b^{00} = \varepsilon^0(a^0)$, $\varepsilon^1(a^1) = d_I b^{00} = d_I\, \varepsilon^0(a^0) = \varepsilon^1(\partial^0(a^0))$

$\Rightarrow a^1 = \partial^0(a^0)$

surj: $D^1(b^{01}, b^{10}) = 0 \Rightarrow d_{II} b^{10} = 0$

$\Rightarrow \exists\, b^{00} : b^{10} = d_{II}(b^{00})$, $(b^{01}, b^{10}) = (b^{01} - d_I b^{00}, 0) + D^0(b^{00})$

$\Rightarrow$ WLOG $b^{10} = 0 \Rightarrow d_{II} b^{01} = 0 \Rightarrow \exists\, a^1 : b^{01} = \varepsilon^1(a^1)$

$\varepsilon^2 \partial^1(a^1) = d_I\, \varepsilon^1(a^1) = d_I b^{01} = 0 \Rightarrow a^1 \in \ker \partial^1$ □