INTRODUCTION TO BOUNDED COHOMOLOGY

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We will give an introduction to bounded cohomology for spaces as well as for discrete groups.

1. Definition of Bounded Cohomology

1.1. **Bounded Cohomology of Spaces.** Let X be a topological space and $S_n(X) = \{\sigma\colon \Delta^n \to X : \sigma \text{ continuous}\}$ the set of singular n-simplices in X. Denote by $C_\bullet(X;\mathbb{R}) = \mathbb{R}[S_\bullet(X)]$ the real chain complex of X and by

$$C^\bullet(X;\mathbb{R}) = \hom_{\mathbb{R}}\big(C_\bullet(X;\mathbb{R}),\mathbb{R}\big) = \operatorname{map}\big(S_\bullet(X),\mathbb{R}\big)$$

the real cochain complex of X.

Definition 1.1. A cochain $\varphi \in C^\bullet(X;\mathbb{R})$ is called *bounded* if there is a constant $C_\varphi > 0$ such that $|\varphi(\sigma)| < C_\varphi$ holds for all $\sigma \in S_\bullet(X)$. We set

$$\widehat{C}^\bullet(X) = \{\varphi \in C^\bullet(X;\mathbb{R}) : \varphi \text{ bounded}\}$$

and define the *bounded cohomology* of X as the cohomology of the subcomplex $\widehat{C}^\bullet(X)$. It will be denoted by $\widehat{H}^\bullet(X)$.

It is easy to see that bounded cohomology is homotopy invariant. But the excision axiom does not hold for bounded cohomology as we will see later. The proof for ordinary cohomology fails since a cochain need not stay bounded during the process of subdivision. It is a nice exercise to show that the first bounded cohomology $\widehat{H}^1(X)$ of each space X vanishes.

1.2. **Bounded Cohomology of Groups.** As in the case of topological spaces we define bounded cohomology of groups by attaching a boundedness condition to the cochain complex which defines ordinary group cohomology.

Definition 1.2. Let G be a discrete group and $B(G^\bullet) = \{\varphi\colon G^\bullet \to \mathbb{R} : \varphi \text{ bounded}\}$. Define coboundary maps

$$\mathbb{R} \xrightarrow{\delta_0} B(G^1) \xrightarrow{\delta_1} B(G^2) \xrightarrow{\delta_2} \cdots$$

by $\delta_0(t) = 0$ for all $t \in \mathbb{R}$ and

$$\begin{aligned} \delta_k(\varphi)(g_0,\dots,g_k) \;=\;& \varphi(g_1,\dots,g_k) \\ &+\sum_{i=1}^{k}(-1)^i\varphi(g_0,\dots,g_{i-1}g_i,\dots,g_k) \\ &+(-1)^{k+1}\varphi(g_0,\dots,g_{k-1}) \end{aligned} \tag{1}$$

for $k \geq 1$. The *bounded cohomology* of G is defined as the cohomology of the complex $B(G^\bullet)$ and is denoted by $\widehat{H}^\bullet(G)$.

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Remark.

(i) 1-cocycles are just group homomorphisms $\varphi\colon G \to \mathbb{R}$. Therefore $\widehat{H}^1(G) = 0$ for each group G since a bounded homomorphism into $\mathbb{R}$ is trivial.

(ii) The dual concept of bounded cohomology is l^1-homology: One defines $C^{l^1}_\bullet(X) = l^1\big(S_\bullet(X)\big)$ with the usual boundary for spaces and $C^{l^1}_\bullet(G) = l^1(G^\bullet)$ with boundary dual to the one in (1) for groups. There are Kronecker products

$$\begin{aligned} \widehat{H}^n(X) \otimes H^{l^1}_n(X) &\longrightarrow \mathbb{R} \\ \widehat{H}^n(G) \otimes H^{l^1}_n(G) &\longrightarrow \mathbb{R}. \end{aligned}$$

We will give another definition of bounded cohomology for discrete groups later. Next we will calculate the bounded cohomology in two special cases.

2. Two Examples

We will study the second bounded cohomology group of the free group with two generators and the bounded cohomology of amenable groups.

2.1. **Bounded Cohomology of Free Groups.** The following result appeared in [1], but the proof given there is not entirely correct, compare the remark in [3]. The proof given here is due to Mitsumatsu ([7]).

Proposition 2.1. *The second bounded cohomology group $\widehat{H}^2(\mathbb{Z} * \mathbb{Z})$ of the free group on two generators is not finitely generated as an $\mathbb{R}$-vector space.*

Proof. We give the idea of the proof. Let w be a reduced word of length $\ell(w) \geq 2$. The 1-cochain f_w given by

$$f_w(g) = (\text{number of times } w \text{ occurs in } g) - (\text{number of times } w^{-1} \text{ occurs in } g)$$

is clearly unbounded. But it is not hard to check that

$$\delta_1(f_w)(g_0, g_1) = f_w(g_1) - f_w(g_0 g_1) + f_w(g_0) \leq \ell(w)$$

holds for all $g_0, g_1 \in \mathbb{Z} * \mathbb{Z}$. Hence $\delta_1(f_w)$ is a bounded 2-cocycle. Consider $\delta_1(f_{[a^n,b^n]})$ for $n \geq 1$, where a, b are the generators of $\mathbb{Z} * \mathbb{Z}$. Now we define l^1-cycles

$$\begin{aligned} E_k &= \sum_{i=0}^{\infty} 2^{-i-1}\big([a^k, b^k]^{2^i}, [a^k, b^k]^{2^i}\big) \\ &\quad -(a^k, a^{-k}b^{-k}) + (b^{-k}, b^k a^{-k} b^{-k}) - (a^k, b^k a^{-k} b^{-k}) \end{aligned}$$

in $C^{l^1}_2(\mathbb{Z} * \mathbb{Z})$ for each $k \geq 1$ and check that

$$\big\langle [\delta_1(f_{[a^n,b^n]})], [E_k] \big\rangle = \delta_{n,k}.$$

This shows that the $\big([\delta_1(f_{[a^n,b^n]})]\big)_{n\in\mathbb{N}}$ is a linearly independent system. □

Brooks and Series ([2]) generalized this method to show that $\widehat{H}^2(G)$ is not finitely generated when G is the fundamental group of a compact oriented surface of genus at least two.

2.2. **Bounded Cohomology of Amenable Groups.** The situation in the case of amenable groups is quite different: One has the following result of Trauber (unpublished, for a proof see for example [4] or [5]).

Theorem 2.2. *Let G be an amenable group. Then $\widehat{H}^n(G) = 0$ for all $n \geq 1$.*

Since this result follows easily from the theory developed in later talks we will not present any proof here.

3. Bounded Cohomology via Injective Resolutions

The following approach to bounded cohomology of discrete groups is due to Ivanov ([5]). It contains our definition above as a special case. First we need some definitions.

3.1. **Strong Relatively Injective Resolutions.**

Definition 3.1. A *bounded left G-module* is a Banach space V with a left G-action by linear operators of norm not greater than one, in other words $\|g.v\| \leq \|v\|$ holds for all $g \in G, v \in V$. A morphism of bounded left G-modules is a bounded linear operator which commutes with the G-action.

If W is any Banach space then

$$B(G, W) = \{\varphi\colon G \to W : \exists c > 0 \, \forall g \in G \, \|\varphi(g)\| \leq c\}$$

is a bounded left G-module where the G-action is given by $g.\varphi(h) = \varphi(hg)$. The norm on $B(G, W)$ is given by $\|\varphi\|_\infty = \sup\{\|\varphi(g)\| : g \in G\}$. Note that

$$B(G^{n+1}, \mathbb{R}) \cong B\big(G, B(G^n, \mathbb{R})\big),$$

where the G-action on $B(G^{n+1}, \mathbb{R})$ is given by $g.\varphi(g_0, \dots, g_n) = \varphi(g_0, \dots, g_{n-1}, g_n g)$.
Often we will call bounded left G-modules simply G-modules.

Definition 3.2. An injective G-morphism $i\colon V_1 \to V_2$ of G-modules is called *strongly injective* if there is a bounded linear operator $p\colon V_2 \to V_1$ such that $p \circ i = \mathrm{id}_{V_1}$ and $\|p\| \leq 1$. Note that p is not assumed to be G-equivariant. A G-module U is called *relatively injective* if for all strongly injective G-morphisms $i\colon V_1 \to V_2$ and all G-morphisms $f\colon V_1 \to U$ there is a G-morphism $h\colon V_2 \to U$ such that $h \circ i = f$ and $\|h\| \leq \|f\|$. The following diagram should clarify the definition:

(2)
$$\begin{tikzcd} V_1 \arrow[r, shift left, "i"] \arrow[d, "f"'] & V_2 \arrow[l, shift left, "p"] \arrow[dl, dashed, "h"] \\ U & \end{tikzcd}$$

Lemma 3.3. *Let V be a Banach space. Then the G-module $B(G, V)$ is relatively injective. It follows that $B(G^n, \mathbb{R})$ is relatively injective for all $n \geq 1$.*

Proof. In the situation of (2) with $U = B(G, V)$ we define $h\colon V_2 \to B(G, V)$ as follows:

$$h(v)(g) = f\big(p(g.v)\big)(e),$$

where e is the unit element of G. The calculation which shows that h satisfies the desired properties is given in [5]. □

Definition 3.4. A *G-resolution* of a G-module V is an exact sequence of G-modules and G-morphisms

$$0 \to V \xrightarrow{d_{-1}} V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots .$$

A G-resolution is called *relatively injective* if the modules $V_0, V_1, \dots$ are relatively injective. A G-resolution is called *strong* if there is a sequence of operators

$$V \xleftarrow{k_0} V_0 \xleftarrow{k_1} V_1 \xleftarrow{k_2} \cdots$$

with $\|k_n\| \leq 1$ such that $d_{n-1}k_n + k_{n+1}d_n = \mathrm{id}_{V_n}$ for $n \geq 0$ and $k_0 d_{-1} = \mathrm{id}_V$. The operators k_i are not required to be G-equivariant.

The following lemma is a routine exercise in homological algebra.

Lemma 3.5. *Let U, V be two G-modules,*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, shift left, "d_{-1}"] & U_0 \arrow[l, shift left, "k_0"] \arrow[r, shift left, "d_0"] & U_1 \arrow[l, shift left, "k_1"] \arrow[r, shift left, "d_1"] & \cdots \arrow[l, shift left, "k_2"] \end{tikzcd}$$

a strong resolution of U and

$$0 \to V \xrightarrow{c_{-1}} V_0 \xrightarrow{c_0} V_1 \xrightarrow{c_1} V_2 \xrightarrow{c_2} \cdots .$$

a complex of relatively injective G-modules. Then any G-morphism $u\colon U \to V$ can be extended to a G-morphism of complexes, in other words there are G-morphisms $u_i\colon U_i \to V_i$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "d_{-1}"] \arrow[d, "u"] & U_0 \arrow[r, "d_0"] \arrow[d, dashed, "u_0"] & U_1 \arrow[r, "d_1"] \arrow[d, dashed, "u_1"] & \cdots \\ 0 \arrow[r] & V \arrow[r, "c_{-1}"] & V_0 \arrow[r, "c_0"] & V_1 \arrow[r, "c_1"] & \cdots \end{tikzcd}$$

Any two such extensions are G-chain homotopic.

3.2. **A new definiton of bounded cohomology.** Now we can state another definition of bounded cohomology which gives the same groups but turns out to be more flexible.

Definition 3.6. Let

$$0 \longrightarrow \mathbb{R} \rightleftarrows V_0 \rightleftarrows V_1 \rightleftarrows V_2 \rightleftarrows \cdots$$

be a strong relatively injective resolution of $\mathbb{R}$ considered as a trivial G-module. The *bounded cohomology* $\widehat{H}^*(G)$ of G is defined to be the cohomology of the induced complex

$$0 \to V_0^G \to V_1^G \to V_2^G \to \dots,$$

where $V_\bullet^G$ denotes the fixed points under the G-action in $V_\bullet$.

Lemma 3.5 ensures that this definition does not depend on the choice of the resolution. By the next Lemma our old definition of bounded cohomology fits into the world of injective resolutions.

Lemma 3.7. *The sequence*

$$0 \to \mathbb{R} \xrightarrow{d_{-1}} B(G) \xrightarrow{d_0} B(G^2) \xrightarrow{d_1} \cdots \tag{3}$$

with $d_{-1}(c)(g) = c$ *and*

$$\begin{aligned} d_n(\varphi)(g_0, \dots, g_{n+1}) &= (-1)^{n+1}\varphi(g_1, \dots, g_{n+1}) \\ &\quad + \sum_{i=0}^{n} (-1)^{n-i}\varphi(g_0, \dots, g_i g_{i+1}, \dots, g_{n+1}) \end{aligned}$$

is a strong relatively injective resolution of the trivial G-module $\mathbb{R}$. The G-action on $B(G^n)$ is given by $g.\varphi(g_1, \dots, g_n) = \varphi(g_1, \dots, g_{n-1}, g_n g)$.

Proof. The contracting homotopy is given by

$$\mathbb{R} \xleftarrow{k_0} B(G) \xleftarrow{k_1} B(G^2) \xleftarrow{k_2} \cdots$$

with $k_n(\varphi)(g_0, \dots, g_{n-1}) = \varphi(g_0, \dots, g_{n-1}, e)$. The induced complex after taking G-fixed points is just the complex used in the definition of section 1.2. □

The resolution (3) is called the *standard resolution* of the trivial G-module $\mathbb{R}$.

4. A Seminorm on Bounded Cohomology

4.1. **Definition of the Seminorm.**

Definition 4.1. Each $\widehat{C}^n(X)$ is a Banach space with respect to the norm

$$\|\varphi\|_\infty = \sup\{|\varphi(\sigma)| : \sigma \in S_n(X)\}.$$

Define a seminorm on $\widehat{H}^n(X)$ by

$$\|\psi\| = \inf\{\|\varphi\|_\infty : \varphi \text{ is a cocycle in } \widehat{C}^n(X) \text{ with } [\varphi] = \psi\}.$$

The situation for groups is slightly more difficult: Each strong relatively injective resolution induces its own seminorm on $\widehat{H}^n(G)$. This problem is resolved by the following definition.

Definition 4.2. For a given strong relatively injective resolution of the trivial G-module $\mathbb{R}$

$$0 \longrightarrow \mathbb{R} \rightleftarrows V_0 \rightleftarrows V_1 \rightleftarrows V_2 \rightleftarrows \cdots$$

define a seminorm

$$\|\psi\| = \inf\{\|\varphi\| : \varphi \text{ is a cocycle in } V_n^G \text{ with } [\varphi] = \psi\}$$

on $\widehat{H}^n(G)$. Define the *canonical seminorm* on $\widehat{H}^n(G)$ as the infimum over the seminorms given by all strong relatively injective resolutions of the trivial G-module $\mathbb{R}$.

The next result implies that this infimum is achieved by the standard resolution.

Theorem 4.3. *Let*

$$0 \longrightarrow \mathbb{R} \underset{k_0}{\overset{d_{-1}}{\rightleftarrows}} V_0 \underset{k_1}{\overset{d_0}{\rightleftarrows}} V_1 \underset{k_2}{\overset{d_1}{\rightleftarrows}} \cdots$$

be a strong resolution of the trivial G-module $\mathbb{R}$. There is a morphism of this resolution to the standard resolution

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{R}}"] & V_0 \arrow[r] \arrow[d, "v_0"] & V_1 \arrow[r] \arrow[d, "v_1"] & \cdots \\
0 \arrow[r] & \mathbb{R} \arrow[r] & B(G) \arrow[r] & B(G^2) \arrow[r] & \cdots
\end{tikzcd}$$

such that $\|v_n\| \leq 1$ for all $n \geq 0$.

Proof. We define v_n by

$$v_n(v)(g_0, \ldots, g_n) = k_0\big(g_0.k_1(\ldots k_{n-1}(g_{n-1}.k_n(g_n.v)) \ldots)\big).$$

v_n is G-equivariant and $\|v_n\| \leq 1$ holds (since $\|k_i\|, \|g.\_\| \leq 1$). For the proof that $v_\bullet$ is a chain map we refer to [5]. □

4.2. **When is the Seminorm a Norm?** It is a natural question if the seminorm on $\widehat{H}^n$ is in fact a norm. This is the case if and only if the image of the coboundary map in dimension n is closed.

In dimension zero the image is trivial and therefore closed, so the seminorm is a norm. The situation in dimension one is even easier but uninteresting: As mentioned above $\widehat{H}^1 = 0$ for each space or group, hence the seminorm is a norm. The question in dimension two is not that clear. Ivanov showed that the canonical seminorm on $\widehat{H}^2(G)$ is a norm for each group G by constructing a bounded operator $Q\colon B(G^2) \to B(G)$ with $\mathrm{im}(\delta_1) = \ker(Q)$ ([6]). For higher dimensions there is the following result of Soma ([8]).

Theorem 4.4. (i) *The canonical seminorm on $\widehat{H}^3(\mathbb{Z} * \mathbb{Z})$ is not a norm.*
(ii) *For each $n \geq 5$ there is a discrete group G such that the canonical seminorm on $\widehat{H}^n(G)$ is not a norm.*

5. The Bounded Cohomology of a Space and its Fundamental Group Coincide

In this section we prove that the bounded cohomology of of a space and the bounded cohomology of its fundamental group are isometrically isomorphic. The main technical work is done in the proof of the following striking result which is due to Gromov ([4]). We will sketch a proof of Ivanov ([5]).

Theorem 5.1. *Let X be a connected countable CW-complex with $\pi_1(X) = 1$. Then $\widehat{H}^n(X) = 0$ for all $n \geq 1$. Moreover there is a chain homotopy*

$$\mathbb{R} \xleftarrow{k_0} \widehat{C}^0(X) \xleftarrow{k_1} \widehat{C}^1(X) \xleftarrow{k_2} \widehat{C}^2(X) \xleftarrow{k_3} \cdots$$

between id *and* 0 *such that* $\|k_i\| \leq 1$.

Proof. (i) Let G be a topological Abelian group and $p\colon Y \to Z$ a principal G-bundle. Then there is a chain map $A^\bullet\colon \widehat{C}^\bullet(Y) \to \widehat{C}^\bullet(Z)$ with $A^\bullet \circ p^* = \mathrm{id}_{\widehat{C}^\bullet(Z)}$.

The construction of A^n uses a left invariant mean on $G^{(\Delta^n)} = \{f\colon \Delta^n \to G : f \text{ is continuous}\}$ which exists since $G^{(\Delta^n)}$ is Abelian and hence amenable.

(ii) Let G be a topological Abelian group which is a $K(\pi, n)$-space. Then there is a homotopy equivalence of fibrations

$$\begin{array}{ccc} EG & \longrightarrow & PK(\pi, n+1) \\ \downarrow & & \downarrow \\ BG & \xrightarrow{\simeq} & K(\pi, n+1). \end{array}$$

(iii) Dold-Thom construction: Let π be a countable Abelian group and $n \geq 1$. Then there is a topological Abelian group which is a model for $K(\pi, n)$.

(iv) Now we can construct a sequence

$$\cdots \xrightarrow{p_n} X_n \xrightarrow{p_{n-1}} X_{n-1} \xrightarrow{p_{n-2}} \cdots \xrightarrow{p_1} X_1 = X$$

with X_n n-connected, $\pi_i(X_n) = \pi_i(X)$ for $i > n$, and $p_n\colon X_{n+1} \to X_n$ a principal G_n-bundle, where G_n is a topological Abelian group which is a $K(\pi_{n+1}(X), n)$-space.

By point (i) we know that $p_n^*\colon \widehat{H}^\bullet(X_n) \to \widehat{H}^\bullet(X_{n+1})$ is injective. Hence we are done if some X_n is contractible.

(v) We will construct a partial contracting homotopy

$$\mathbb{R} \xleftarrow{k_0^n} \widehat{C}^0(X_n) \xleftarrow{k_1^n} \widehat{C}^1(X_n) \xleftarrow{k_2^n} \cdots \xleftarrow{k_n^n} \widehat{C}^n(X_n).$$

For this it suffices to construct

$$\{1\} \xrightarrow{L_{-1}} S_0(X_n) \xrightarrow{L_0} S_1(X_n) \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} S_n(X_n)$$

with $\partial L_i(\sigma) = \sigma - \sum_{j=0}^{i} L_{i+1}(\sigma_j)$, where σ_j is the j-face of σ. Then one defines $k_i^n = L_{i-1}^*$. Such an L will be constructed by induction making use of the fact that X_n is n-connected.

(vi) Now we can define a partial contracting homotopy

$$\mathbb{R} \xleftarrow{k_0} \widehat{C}^0(X) \xleftarrow{k_1} \widehat{C}^1(X) \xleftarrow{k_2} \cdots \xleftarrow{k_n} \widehat{C}^n(X)$$

by

$$k_i = A_1^i \circ \ldots \circ A_{n-1}^i \circ k_i^n \circ p_{n-1}^* \circ \ldots \circ p_1^*$$

with $A_n^\bullet\colon \widehat{C}^\bullet(X_{n+1}) \to \widehat{C}^\bullet(X_n)$ the map of point (i). It is possible to make the definition of k_i independent of n and to construct a complete contracting homotopy $(k_i)_{i\geq 0}$. □

Now we can prove the main result of this section. It is also due to Gromov, but the proof presented here is taken from Ivanov's paper [5].

Theorem 5.2. *Let X be connected countable CW-complex. Then there is an isometric isomorphism $\widehat{H}^\bullet(\pi_1(X)) \xrightarrow{\cong} \widehat{H}^\bullet(X)$.*

Proof. Let $p\colon \widetilde{X} \to X$ be the universal covering. Since $\widetilde{X}$ is simply connected

$$0 \longrightarrow \mathbb{R} \rightleftarrows \widehat{C}^0(\widetilde{X}) \rightleftarrows \widehat{C}^1(\widetilde{X}) \rightleftarrows \widehat{C}^2(\widetilde{X}) \rightleftarrows \cdots \tag{4}$$

is a strong resolution of $\mathbb{R}$ as a trivial $\pi_1(X)$-module by Theorem 5.1. The $\pi_1(X)$-action on $\widehat{C}^\bullet(\widetilde{X})$ is induced by $\pi_1(X)$ action on $\widetilde{X}$. The resolution is relatively injective by the following argument. Let $F \subset \widetilde{X}$ be a fundamental domain for the $\pi_1(X)$-action and $S_n(\widetilde{X}, F)$ be the set of those singular n-simplices which carry the first vertex of Δ^n into F. Then $\widehat{C}^n(\widetilde{X})$ is isometrically isomorphic to $B\big(\pi_1(X), B(S_n(\widetilde{X}, F), \mathbb{R})\big)$, and that is relatively injective by Lemma 3.3.

By definition the bounded cohomology of $\pi_1(X)$ is given by the cohomology of the complex

$$0 \to \widehat{C}^0(\widetilde{X})^{\pi_1(X)} \to \widehat{C}^1(\widetilde{X})^{\pi_1(X)} \to \widehat{C}^2(\widetilde{X})^{\pi_1(X)} \to \cdots .$$

But $p^*\colon \widehat{C}^\bullet(X) \to \widehat{C}^\bullet(\widetilde{X})$ induces an isometric chain isomorphism between $\widehat{C}^\bullet(X)$ and $\widehat{C}^\bullet(\widetilde{X})^{\pi_1(X)}$. It remains to prove that we really get an isometric isomorphism. If we denote the canonical seminorm by $\|\cdot\|$ and the one given by the resolution (4) by $\|\cdot\|_X$ we have to show that $\|\cdot\|_X \leq \|\cdot\|$ (the other inequality is obvious by the definition of the canonical seminorm). It suffices to construct a $\pi_1(X)$-morphism of the standard resolution into the resolution (4) of norm not exceeding one.

For each simplex $\sigma\colon \Delta^n \to \widetilde{X}$ define $S_n(\sigma) = (g_0, \ldots, g_n)$ where

$$\begin{aligned} g_n.\sigma(v_0) &\in F \\ g_{n-1}g_n.\sigma(v_1) &\in F \\ &\vdots \\ g_0g_1\ldots g_n.\sigma(v_n) &\in F \end{aligned}$$

Here v_i denotes the i-th vertex of Δ^n. Now define a map

$$\begin{aligned} u_n\colon B\big(\pi_1(X)^{n+1}\big) &\to \widehat{C}^n(\widetilde{X}) \\ f &\mapsto \Big(\sigma \mapsto f\big(S_n(\sigma)\big)\Big). \end{aligned}$$

One checks that $u_\bullet$ is a $\pi_1(X)$-equivariant chain map and $\|u_\bullet\| \leq 1$. This proves the theorem. □

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