Introduction to Generalized Sheaf Cohomology

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1 Right derived functors

Let $\mathcal{M}$ be a category and let S be a class of morphisms in $\mathcal{M}$. The localization $\mathcal{M} \xrightarrow{\gamma} \mathcal{M}_{\mathrm{loc}}$ of $\mathcal{M}$ with respect to S (see [GZ]) is characterized by the universal property that $\gamma(s)$ for any $s \in S$ is an isomorphism and that for any functor $F : \mathcal{M} \to \mathcal{B}$ which transforms the morphisms in S into isomorphisms in $\mathcal{B}$ there is a unique functor $\overline{F} : \mathcal{M}_{loc} \to \mathcal{B}$ such that $F = \overline{F} \circ \gamma$. Now let $F : \mathcal{M} \to \mathcal{B}$ be an arbitrary functor. We then want at least a functor $RF : \mathcal{M}_{\mathrm{loc}} \to \mathcal{B}$ such that $RF \circ \gamma$ is as "close" as possible to F.

Definition 1.1. *A right derivation of F is a functor*

$$RF : \mathcal{M}_{\mathrm{loc}} \longrightarrow \mathcal{B}$$

together with a natural transformation $\eta : F \to RF \circ \gamma$ such that, for any functor $G : \mathcal{M}_{\mathrm{loc}} \to \mathcal{B}$, the map

$$\begin{array}{rcl} \textit{natural transf.}\,(RF, G) & \xrightarrow{\sim} & \textit{natural transf.}\,(F, G \circ \gamma) \\ \epsilon & \longmapsto & (\epsilon * \gamma) \circ \eta \end{array}$$

is bijective.

Since the pair (RF, η) if it exists is unique up to unique isomorphism we usually will refer to RF as the right derived functor of F. There seems to be no completely general result about the existence of right derived functors; in various different situations one has different methods to construct them. In the following we will discuss an appropriate variant of the method of "resolutions" which will be applicable in all situations of interest to us.

From now on we always assume that the class S satisfies the condition that

in any commutative diagram $\overset{\nearrow\searrow}{\longrightarrow}$ in $\mathcal{M}$
(∗) in which two arrows represent morphisms in S
also the third arrow represents a morphism in S.

Suppose we are given:

1. a full subcategory $I \overset{i}{\hookrightarrow} \mathcal{M}$ such that $F(s)$ for any $s \in I \cap S$ is an isomorphism in $\mathcal{B}$; here $I \cap S$ denotes the class of morphisms in I which lie in S;

2. a "resolution", i.e.:
   a) maps $\mathrm{Ob}\,\mathcal{M} \overset{r}{\longrightarrow} \mathrm{Ob}\,I$ and $\mathrm{Hom}_{\mathcal{M}}(M, N) \overset{r}{\longrightarrow} \mathrm{Hom}_I(r(M), r(N))$ for any two objects M, N in $\mathcal{M}$ such that $\gamma \circ r : \mathcal{M} \to I_{loc}$ is a functor; here I_{loc} denotes the localization of I with respect to $I \cap S$;
   b) a morphism $\eta_M : M \to r(M)$ in S for any object M in $\mathcal{M}$ such that, for any morphism $m : M \to N$ in $\mathcal{M}$, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & r(M) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle r(m)} \\ N & \xrightarrow{\eta_N} & r(N) \end{array}$$

   is commutative.

According to 1. there is a unique functor $\overline{F} : I_{\mathrm{loc}} \to \mathcal{B}$ such that the diagram

$$\begin{array}{ccccc} I & \xrightarrow{i} & \mathcal{M} & \xrightarrow{F} & \mathcal{B} \\ & {\scriptstyle \gamma}\searrow & & \nearrow{\scriptstyle \overline{F}} & \\ & & I_{\mathrm{loc}} & & \end{array}$$

is commutative. Because of 2.a) we have that $\gamma \circ r : \mathcal{M} \to I_{\mathrm{loc}}$ and therefore also

$$F \circ i \circ r = \overline{F} \circ (\gamma \circ r) : \mathcal{M} \longrightarrow \mathcal{B}$$

are functors. And from 2.b) we deduce that η induces a natural transformation

$$F\eta : F \longrightarrow F \circ i \circ r$$

and a natural isomorphism

$$\gamma\eta : \gamma \overset{\sim}{\longrightarrow} \gamma \circ i \circ r \quad .$$

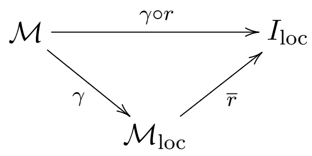
Remark 1.2. *We have $r(s) \in I \cap S$ for any $s \in S$.*

Proof. In the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & r(M) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle r(s)} \\ N & \xrightarrow{\eta_N} & r(N) \end{array}$$

all morphisms besides possibly $r(s)$ lie in S. The condition $(*)$ then implies that $r(s) \in I \cap S$. $\square$

Consequently there is a unique functor $\overline{r} : \mathcal{M}_{\text{loc}} \to I_{\text{loc}}$ such that the diagram



is commutative. We define

$$RF := \overline{F} \circ \overline{r} : \mathcal{M}_{\text{loc}} \longrightarrow \mathcal{B} \ .$$

Because of $RF \circ \gamma = F \circ i \circ r$ we can view $F\eta$ as a natural transformation

$$F\eta : F \longrightarrow RF \circ \gamma \quad .$$

Proposition 1.3. *The pair $(RF, F\eta)$ is a right derivation of F.*

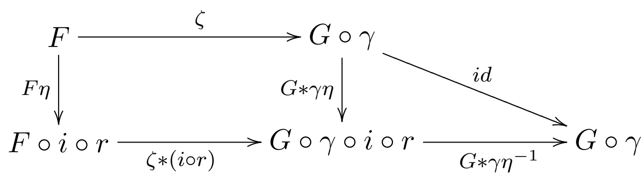
Proof. Let $G : \mathcal{M}_{\text{loc}} \to \mathcal{B}$ be a functor and let $\zeta : F \to G \circ \gamma$ be a natural transformation. The natural transformation

$$RF \circ \gamma = F \circ i \circ r \xrightarrow{\zeta * (i \circ r)} G \circ \gamma \circ i \circ r \xrightarrow{G * \gamma\eta^{-1}} G \circ \gamma$$

can be viewed as a natural transformation

$$\epsilon_0 : RF \longrightarrow G \quad ;$$

for that we only have to observe that any morphism in $\mathcal{M}_{\text{loc}}$ can be written as a finite composition of morphisms of the form $\gamma(m)$ and $\gamma(s)^{-1}$ with $s \in S$. The commutative diagram



shows that

$$(\epsilon_0 * \gamma) \circ F\eta = \zeta \quad .$$

Now let $\epsilon : RF \to G$ be any natural transformation such that $(\epsilon * \gamma) \circ F\eta = \zeta$. We have to show that $\epsilon = \epsilon_0$. Because of 2.b) every object in $\mathcal{M}_{\text{loc}}$ is isomorphic to an object of the form $\gamma(N)$ for some N in I. It therefore suffices to prove that

$$(\epsilon * \gamma)_N = (\epsilon_0 * \gamma)_N \text{ for any } N \text{ in } I \ .$$

But in that case, by 1. and 2.b), $(F\eta)_N$ is an isomorphism so that we get

$$(\epsilon * \gamma)_N = \zeta_N \circ (F\eta)_N^{-1} = (\epsilon_0 * \gamma)_N \ .$$

□

Remark 1.4. *1) $F\eta|I$ is a natural isomorphism.*

2) The functors $I_{loc} \xrightarrow{\bar{i}} \mathcal{M}_{loc}$ and $\mathcal{M}_{loc} \xrightarrow{\bar{r}} I_{loc}$ are quasi-inverse to each other.

In homological algebra one considers the following situation (compare [Har] and [Ver]). Given is an additive functor $f : \mathbf{A} \to \mathbf{A}'$ between two abelian categories where $\mathbf{A}$ is assumed to have enough injective objects. Put

$\mathcal{M}$:= the category $C^+(\mathbf{A})$ of bounded below complexes of objects in $\mathbf{A}$, and
S := the class of quasi-isomorphisms in $C^+(\mathbf{A})$.

Then $\mathcal{M}_{\mathrm{loc}} = D^+(\mathbf{A})$ is the usual derived category. The functor f induces a functor

$$F : C^+(\mathbf{A}) \longrightarrow D^+(\mathbf{A}')$$

to which the above considerations apply: Let I be the full subcategory in $\mathcal{M}$ of complexes of injective objects. Since in I quasi-isomorphisms already are homotopy equivalences 1. is granted by the additivity of f. Furthermore it is a basic fact that we can fix:

i. for any complex M in $C^+(\mathbf{A})$ an injective quasi-isomorphism

$$\eta_M : M \longrightarrow r(M)$$

into a complex $r(M)$ in I, and

ii. for any homomorphism $m : M \to N$ in $C^+(\mathbf{A})$ a homomorphism $r(m) : r(M) \to r(N)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & r(M) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle r(m)} \\ N & \xrightarrow{\eta_N} & r(N) \end{array}$$

is commutative.

Since $r(m)$ is unique up to homotopy these choices define a resolution in our sense. Therefore the right derived functor

$$R^+f := RF : D^+(\mathbf{A}) \longrightarrow D^+(\mathbf{A}')$$

of F exists. We mostly will be interested in the case where $\mathbf{A} := Shab(\mathcal{X})$ is the category of abelian sheaves on some site $\mathcal{X}$, $\mathbf{A}' := (ab)$ is the category of abelian groups, and $f := \Gamma_X$ is the section functor in some object X in $\mathcal{X}$. Then

$$\mathbb{H}^+(X, .) := R^+\Gamma_X(.) : D^+(Shab(\mathcal{X})) \longrightarrow D^+(ab)$$

is the usual hypercohomology of bounded below complexes of abelian sheaves.

In the next section we will briefly recall Quillen's generalization ([Qui]) of the concepts of homological algebra to what is called homotopical algebra.

2 Model categories

Let $\mathcal{M}$ be a category together with three distinguished classes of morphisms called weak equivalences, fibrations, and cofibrations. Furthermore, a fibration, resp. a cofibration, which also is a weak equivalence will be called a trivial fibration, resp. a trivial cofibration.

Definition 2.1. *([Qui]) $\mathcal{M}$ is a model category if it satisfies the following axioms:*

(M1) $\mathcal{M}$ is closed under finite projective and inductive limits.

(M2) In any commutative diagram $\overset{\nearrow\searrow}{\longrightarrow}$ in $\mathcal{M}$ in which two of the arrows represent weak equivalences also the third arrow represents a weak equivalence.

(M3) a. Fibrations and trivial fibrations are stable under composition and base change.
b. Cofibrations and trivial cofibrations are stable under composition and cobase change.
c. Any isomorphism is a trivial fibration and cofibration.

(M4) Given any commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & M \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & N \end{array}$$

in $\mathcal{M}$ where i is a cofibration and p is a fibration and either i or p is a weak equivalence then the dotted arrow exists making the diagram commute.

(M5) Any morphism f in $\mathcal{M}$ may be factored as
a. $f = p \circ i$, where p is a fibration and i is a trivial cofibration, and
b. $f = q \circ j$, where q is a trivial fibration and j is a cofibration.

The localization of $\mathcal{M}$ with respect to the class of weak equivalences is called the associated homotopy category and is denoted by $\mathrm{Ho}(\mathcal{M})$.

Example 2.2. $C^+(\mathbf{A})$ is a model category if we define a weak equivalence to be a quasi-isomorphism, a cofibration to be a monomorphism, and a fibration to be an epimorphism whose kernel is a complex of injective objects. The associated homotopy category $\mathrm{Ho}(C^+(\mathbf{A}))$ is the derived category $D^+(\mathbf{A})$.

By (M1) $\mathcal{M}$ has a final and an initial object. We call an object M in $\mathcal{M}$ fibrant, resp. cofibrant, if the canonical morphism from M into a final object, resp. from an initial object into M, is a fibration, resp. a cofibration. Now let $F : \mathcal{M} \to \mathcal{B}$ be any functor which transforms weak equivalences between fibrant objects into isomorphisms. By (M5) a. we can fix, for any object M in $\mathcal{M}$, a trivial cofibration

$$\eta_M : M \longrightarrow r(M) \text{ such that } r(M) \text{ is fibrant.}$$

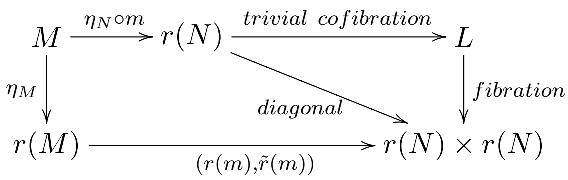
By (M4) we also can fix, for any morphism m in $\mathcal{M}$, a commutative diagram

$$\begin{CD} M @>{\eta_M}>> r(M) \\ @V{m}VV @VV{r(m)}V \\ N @>{\eta_N}>> r(N) \end{CD}\ .$$

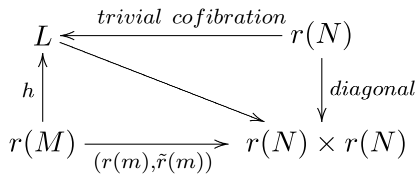
We claim that taking I to be the full subcategory of fibrant objects in $\mathcal{M}$ the above choices define a resolution in the sense of section 1. We have to show that whenever

$$\begin{CD} M @>{\eta_M}>> r(M) \\ @V{m}VV @VV{\tilde{r}(m)}V \\ N @>{\eta_N}>> r(N) \end{CD}$$

is another such commutative diagram then $\gamma(r(m)) = \gamma(\tilde{r}(m))$ holds true in I_{loc}. For that we consider the commutative diagram



where the right triangle arises from an application of (M5) a. to the diagonal morphism. Obviously we can apply (M4) to the outer rectangle getting a lifting $h : r(M) \to L$. Since according to (M3) a. the object L is fibrant we then have a commutative diagram



in which all objects are fibrant. Reading this diagram in I_{loc} we see that $\gamma(r(m)) = \gamma(\tilde{r}(m))$ in I_{loc}. We thus have established the following result.

Proposition 2.3. *([Qui]) If $\mathcal{M}$ is a model category then the right derived functor*

$$RF : \mathrm{Ho}(\mathcal{M}) \longrightarrow \mathcal{B}$$

exists for a functor $F : \mathcal{M} \to \mathcal{B}$ which transforms weak equivalences between fibrant objects into isomorphisms.

Later on we will need to know how the formation of the right derived functor is behaved with respect to the composition of functors. Let $F : \mathcal{M} \to \mathcal{B}$ be, as above, a functor which transforms weak equivalences between fibrant objects into isomorphisms. In addition let $G : \mathcal{M}' \to \mathcal{M}$ be a functor bet- ween model categories which respects fibrant objects and weak equivalences between fibrant objects. Then the right derived functors

$$\begin{array}{rcllll} RG & : & \mathrm{Ho}(\mathcal{M}') & \longrightarrow & \mathrm{Ho}(\mathcal{M}) & \text{of } \gamma \circ G\ , \\ RF & : & \mathrm{Ho}(\mathcal{M}) & \longrightarrow & \mathcal{B} & \text{of } F\ ,\ \text{and} \\ R(F \circ G) & : & \mathrm{Ho}(\mathcal{M}') & \longrightarrow & \mathcal{B} & \text{of } F \circ G \end{array}$$

exist and by the universal property of right derived functors there is a canonical natural transformation

$$R(F \circ G) \xrightarrow{\sim} RF \circ RG$$

which easily can be shown to be an isomorphism.

Next we gather a few more facts which will be of constant use later on.

Lemma 2.4. *(Brown) In a model category $\mathcal{M}$ we have:*

i. Any morphism f between fibrant objects can be factored as $f = p \circ i$ where p is a fibration and i is right-inverse to a trivial fibration;

ii. any morphism f between cofibrant objects can be factored as $f = q \circ j$ where q is left-inverse to a trivial cofibration and j is a cofibration;

iii. any base change of a weak equivalence between fibrant objects by a fibration is a weak equivalence;

iv. any cobase change of a weak equivalence between cofibrant objects by a cofibration is a weak equivalence.

Proof. [Bro] p. 421 and p. 428. □

Corollary 2.5. *Let $F : \mathcal{M}' \to \mathcal{M}$ be a functor between model categories. If F respects trivial fibrations, resp. trivial cofibrations, it transforms weak equivalences between fibrant, resp. cofibrant, objects into weak equivalences.*

Definition 2.6. *([Qui]) A model category $\mathcal{M}$ is closed if it satisfies the additional axiom:*

(CM) Weak equivalences, fibrations and cofibrations are stable under retracts in the category of arrows of $\mathcal{M}$.

Remark 2.7. *In a closed model category the axiom (M3) is a consequence of the others and consequently can be omitted.*

The main interest in this axiom (CM) lies in the fact that in a closed model category any two of the distinguished classes of morphisms determine the third class. We say that a morphism $i : A \to B$ in $\mathcal{M}$ has the LLP (left lifting property) with respect to a morphism $p : M \to N$, resp. that p has the RLP (right lifting property) with respect to i, if in any commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & M \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ B & \longrightarrow & N \end{array}$$

the dotted arrow (the lifting) exists.

Proposition 2.8. *In a closed model category we have:*

i. *A morphism is a fibration if and only if it has the RLP with respect to all trivial cofibrations;*

ii. *a morphism is a trivial fibration if and only if it has the RLP with respect to all cofibrations;*

iii. *a morphism is a cofibration if and only if it has the LLP with respect to all trivial fibrations;*

iv. *a morphism is a trivial cofibration if and only if it has the LLP with respect to all fibrations.*

Proof. [Qui] I §5. □

Combining that with Brown's lemma 2.4 we obtain a manageable criterion for when a functor respects weak equivalences between fibrant objects.

Proposition 2.9. *Let* $F : \mathcal{M}' \to \mathcal{M}$ *be a functor between model categories and let* $\mathcal{M}$ *be closed. If* F *has a left adjoint* $G : \mathcal{M} \to \mathcal{M}'$ *then we have:*

i. *If* G *respects cofibrations, then* F *transforms weak equivalences between fibrant objects into weak equivalences;*

ii. *if* G *respects trivial cofibrations, then* F *respects fibrations and, in particular, fibrant objects.*

Proof. i. By the Corollary 2.5 to Brown's lemma 2.4 it suffices to show that F respects trivial fibrations. This is a standard argument which we include for the convenience of the reader. Let $p : M \to N$ be a trivial fibration in

$\mathcal{M}'$. Since $\mathcal{M}$ is closed $F(p)$ will be a trivial fibration if in any commutative solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & F(M) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle F(p)} \\ B & \longrightarrow & F(N) \end{array}$$

in $\mathcal{M}$ where i is a cofibration the dotted arrow exists. Applying G to it we get the commutative diagram

$$\begin{array}{ccccc} G(A) & \longrightarrow & GF(M) & \xrightarrow{adjunction} & M \\ {\scriptstyle G(i)}\downarrow & & \downarrow{\scriptstyle GF(p)} & & \downarrow{\scriptstyle p} \\ G(B) & \longrightarrow & GF(N) & \xrightarrow{adjunction} & N \end{array}$$

where $G(i)$ by assumption again is a cofibration. Since the dotted arrow exists in the commutative diagram

$$\begin{array}{ccc} G(A) & \longrightarrow & M \\ {\scriptstyle G(i)}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ G(B) & \longrightarrow & N \end{array}$$

applying F to it gives the wanted dotted arrow in the original diagram

$$\begin{array}{ccccc} A & \xrightarrow{adjunction} & FG(A) & \longrightarrow & F(M) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle FG(i)} & \nearrow & \downarrow{\scriptstyle F(p)} \\ B & \xrightarrow{adjunction} & FG(B) & \longrightarrow & F(N)\ . \end{array}$$

ii. This is proved by an analogous argument. □

Finally we have to recall the "stabilization" of model categories — a construction due to Bousfield/Friedlander ([BF]).

Definition 2.10. *A category $\mathcal{M}$ with weak equivalences, fibrations, and cofibrations is proper if it satisfies (M1) and the axiom:*

(PM) a. Any base change of a weak equivalence by a fibration is a weak equivalence.
b. Any cobase change of a weak equivalence by a cofibration is a weak equivalence.

Now let $\mathcal{M}$ be a model category and suppose we are given a functor

$$Q : \mathcal{M} \longrightarrow \mathcal{M}$$

together with a natural transformation

$$\tau : \mathrm{id}_{\mathcal{M}} \longrightarrow Q \quad .$$

We call a morphism m in $\mathcal{M}$

a Q-weak equivalence	,	if Qm is a weak equivalence,
a Q-cofibration	,	if m is a cofibration, and
a Q-fibration	,	if m has the RLP with respect to all Q-trivial cofibrations.

Let $\mathcal{M}^Q$ denote the category $\mathcal{M}$ together with these three distinguished classes of morphisms.

Proposition 2.11. *If $\mathcal{M}$ is a proper closed model category such that*

a. Q respects weak equivalences and final objects,

b. τ_{QM} and $Q\tau_M : QM \to QQM$ are weak equivalences for any object M in $\mathcal{M}$, and

c. $\mathcal{M}^Q$ is proper,

then $\mathcal{M}^Q$ is a proper closed model category, too. Furthermore we have:

i. Weak equivalences are Q-weak equivalences and Q-fibrations are fibrations;

ii. a morphism is a Q-trivial Q-fibration if and only if it is a trivial fibration;

iii. an object M is Q-fibrant if and only if it is fibrant and $\tau_M : M \to QM$ is a weak equivalence;

iv. Q-weak equivalences between Q-fibrant objects are weak equivalences.

Proof. [BF] App. A. □

How does this construction behave with respect to derived functors? Assume that $\mathcal{M}$, Q, and τ fulfil the assumptions of the above Proposition 2.11 and assume that $F : \mathcal{M} \to \mathcal{B}$ is a functor which transforms weak equivalences between fibrant objects into isomorphisms. Then we have its right derived functor $RF : \mathrm{Ho}(\mathcal{M}) \to \mathcal{B}$. According to i. and iv. in Proposition 2.11 the functor F also transforms Q-weak equivalences between Q-fibrant objects into isomorphisms so that we have the right derived functor

$$R^Q F : \mathrm{Ho}(\mathcal{M}^Q) \longrightarrow \mathcal{B}$$

of $F : \mathcal{M}^Q \to \mathcal{B}$, too. In order to relate RF and $R^Q F$ we only have to observe that the functor $\mathrm{id} : \mathcal{M}^Q \to \mathcal{M}$ induced by the identity has the following properties:

- It transforms Q-weak equivalences between Q-fibrant objects into weak equivalences (by iv. in Prop. 2.11);
- it transforms Q-fibrations into fibrations (by i. in Prop. 2.11);
- the diagram $\begin{array}{ccc} \mathcal{M}^Q & \xrightarrow{F} & \mathcal{B} \\ & {\scriptstyle id}\searrow \quad \nearrow{\scriptstyle F} & \\ & \mathcal{M} & \end{array}$ is commutative.

Therefore the right derived functor $R\,\mathrm{id} : \mathrm{Ho}(\mathcal{M}^Q) \to \mathrm{Ho}(\mathcal{M})$ of $\gamma \circ \mathrm{id}$ exists and, according to our earlier discussion, we have

$$R^Q F \cong RF \circ R\,\mathrm{id} \quad .$$

It remains to compute $R\,\mathrm{id}$. Since the functor $Q : \mathcal{M}^Q \to \mathcal{M}$, by definition, transforms Q-weak equivalences into weak equivalences it induces a functor

$$Q : \mathrm{Ho}(\mathcal{M}^Q) \longrightarrow \mathrm{Ho}(\mathcal{M}) \quad .$$

By the universal property of right derived functors τ then induces a natural transformation

$$R\,\mathrm{id} \xrightarrow{\sim} Q$$

which is an isomorphism as we immediately conclude from iii. in Proposition 2.11. Altogether we therefore get a canonical natural isomorphism

$$R^Q F \cong RF \circ Q \quad .$$

In many applications one is in a slightly different type of situation where $F : \mathcal{M} \to \overline{\mathcal{M}}$ is a functor into another model category $\overline{\mathcal{M}}$ which also is equipped with a functor $\overline{Q} : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ and a natural transformation $\overline{\tau} : \mathrm{id}_{\overline{\mathcal{M}}} \to \overline{Q}$ fulfilling the assumptions of the above Proposition 2.11. We assume that F transforms weak equivalences between fibrant objects into weak equivalences so that we have the right derived functor

$$RF : \mathrm{Ho}(\mathcal{M}) \longrightarrow \mathrm{Ho}(\overline{\mathcal{M}})$$

of $\gamma \circ F$. Viewing F as a functor $F^Q : \mathcal{M}^Q \to \overline{\mathcal{M}}^{\overline{Q}}$ we also have, similarly as above, the right derived functor

$$RF^Q : \mathrm{Ho}(\mathcal{M}^Q) \longrightarrow \mathrm{Ho}(\overline{\mathcal{M}}^{\overline{Q}})$$

of $\gamma \circ F^Q$.

Lemma 2.12. *Under the assumption made above the diagram*

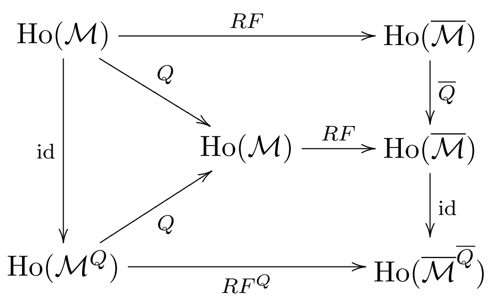
$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{M}) & \xrightarrow{RF} & \mathrm{Ho}(\overline{\mathcal{M}}) \\ {\scriptstyle Q}\uparrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Ho}(\mathcal{M}^Q) & \xrightarrow[RF^Q]{} & \mathrm{Ho}(\overline{\mathcal{M}}^{\overline{Q}}) \end{array}$$

is commutative up to a canonical natural isomorphism. If we assume in addition that Q respects fibrant objects and that there is a natural isomorphism $F \circ Q \cong \overline{Q} \circ F$, then the diagram

$$\begin{array}{ccc} \mathrm{Ho}(\mathcal{M}) & \xrightarrow{RF} & \mathrm{Ho}(\overline{\mathcal{M}}) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \overline{Q}} \\ \mathrm{Ho}(\mathcal{M}^Q) & \xrightarrow[RF^Q]{} & \mathrm{Ho}(\overline{\mathcal{M}}^{\overline{Q}}) \end{array}$$

is commutative up to a natural isomorphism.

Proof. In the diagram



the lower part is commutative up to a natural isomorphism by the above discussion and the upper part because of the additional assumptions. ☐

3 Simplicial sheaves

We fix a small site $\mathcal{X}$ which we assume has enough points. This assumption is not really necessary but it is technically very convenient and is fulfilled in most situations. Let $SSh(\mathcal{X})$ be the category of simplicial sheaves (of sets) on $\mathcal{X}$ together with the following three classes of morphisms:

- The weak equivalences are the morphisms which stalkwise induce weak equivalences between simplicial sets ([Qui] II §3 Prop. 4);

- the cofibrations are the monomorphisms;

- the fibrations are the morphisms with the RLP with respect to all trivial cofibrations.

Proposition 3.1. *(Joyal) $SSh(\mathcal{X})$ is a closed model category.*

Proof. [Ja1] Cor. 2.7. □

Now we fix an object X in $\mathcal{X}$ and we let

$$\begin{array}{rcrcl} \Gamma_X & : & SSh(\mathcal{X}) & \longrightarrow & sS \\ & & \mathcal{F} & \longmapsto & \mathcal{F}(X) \end{array}$$

be the section functor. Here $sS = SSh(*)$ denotes the usual model category of simplicial sets (compare [Qui] II §3 Prop. 2).

Lemma 3.2. *Γ_X transforms weak equivalences between fibrant simplicial sheaves into weak equivalences in sS.*

Proof. According to Proposition 2.9 it suffices to show that Γ_X has a left adjoint which respects monomorphisms. But it is well-known that the functor

$$\begin{array}{rcl} sS & \longrightarrow & SSh(\mathcal{X}) \\ A & \longmapsto & \underline{A}_X := \text{sheafification of the presheaf } U \mapsto \coprod_{U \to X} A \end{array}$$

is left adjoint to Γ_X. □

Therefore the right derived functor

$$\widetilde{\mathbb{H}}(X, .) : \mathrm{Ho}(SSh(\mathcal{X})) \longrightarrow \mathrm{Ho}(sS)$$

of $\gamma \circ \Gamma_X$ exists and is called <u>generalized cohomology</u> of simplicial sheaves. The immediate question which we have to answer of course is in which sense this functor generalizes the usual cohomology of abelian sheaves. Let $C^{\leq 0}(Shab(\mathcal{X}))$ be the category of (cohomological) complexes in degrees ≤ 0 of abelian sheaves on $\mathcal{X}$ together with the following three classes of homomorphisms:

- The weak equivalences are the quasi-isomorphisms;
- the cofibrations are the monomorphisms;
- the fibrations are the homomorphisms with the RLP with respect to all trivial cofibrations.

Proposition 3.3. *i. $C^{\leq 0}(Shab(\mathcal{X}))$ is a closed model category;*

ii. the section functor $\Gamma_X : C^{\leq 0}(Shab(\mathcal{X})) \longrightarrow C^{\leq 0}(ab)$ transforms quasi-isomorphisms between fibrant objects into quasi-isomorphisms.

Proof. Completely analogous to the proof of the corresponding results for $SSh(\mathcal{X})$. □

The homotopy category associated to $C^{\leq 0}(Shab(\mathcal{X}))$ is the derived category
$D^{\leq 0}(Shab(\mathcal{X}))$. The right derived functor of $\gamma \circ \Gamma_X$ will be denoted by

$$\mathbb{H}^{\leq 0}(X,.) : D^{\leq 0}(Shab(\mathcal{X})) \longrightarrow D^{\leq 0}(ab) \quad .$$

First we want to compare the functors $\widetilde{\mathbb{H}}$ and $\mathbb{H}^{\leq 0}$. Let $SShab(\mathcal{X})$ be the category of simplicial abelian sheaves on $\mathcal{X}$; a homomorphism in $SShab(\mathcal{X})$ is called a weak equivalence if it is a weak equivalence in $SSh(\mathcal{X})$. The Dold-Puppe equivalence (?) provides us with two natural functors

$$C^{\leq 0}(Shab(\mathcal{X})) \underset{N}{\overset{\tilde{s}}{\rightleftarrows}} SShab(\mathcal{X})$$

which are quasi-inverse to each other and under which quasi-isomorphisms correspond to weak equivalences. The faithful functor

$$s : C^{\leq 0}(Shab(\mathcal{X})) \xrightarrow{\tilde{s}} SShab(\mathcal{X}) \xrightarrow{\subseteq} SSh(\mathcal{X})$$

therefore transforms quasi-isomorphisms into weak equivalences and induces a functor

$$s : D^{\leq 0}(Shab(\mathcal{X})) \longrightarrow \mathrm{Ho}(SSh(\mathcal{X})) \quad .$$

Proposition 3.4. *The diagram*

$$\begin{array}{ccc} D^{\leq 0}(Shab(\mathcal{X})) & \xrightarrow{\mathbb{H}^{\leq 0}(X,.)} & D^{\leq 0}(ab) \\ \downarrow s & & \downarrow s \\ \mathrm{Ho}(SSh(\mathcal{X})) & \xrightarrow{\widetilde{\mathbb{H}}(X,.)} & \mathrm{Ho}(sS) \end{array}$$

is commutative (up to a canonical natural isomorphism).

Proof. From the definition of the Dold-Puppe equivalence it is easy to see that $s \circ \Gamma_X \cong \Gamma_X \circ s$ holds true. But this implies

$$s \circ \mathbb{H}^{\leq 0}(X,.) \cong R(s \circ \Gamma_X) \cong R(\Gamma_X \circ s) \cong \widetilde{\mathbb{H}}(X,.) \circ s$$

provided s respects fibrant objects. By Proposition 2.9 this will certainly be the case if s has a left adjoint which respects trivial cofibrations. The functor

$$\begin{array}{cccl} \mathbb{Z} & : & SSh(\mathcal{X}) & \longrightarrow & SShab(\mathcal{X}) \\ & & \mathcal{F} & \longmapsto & \text{simplicial sheaf of free} \\ & & & & \text{abelian groups over } \mathcal{F} \end{array}$$

obviously is left adjoint to the inclusion functor $SShab(\mathcal{X}) \xrightarrow{\subseteq} SSh(\mathcal{X})$. Therefore $N\mathbb{Z}$ is left adjoint to s; furthermore $N\mathbb{Z}$ clearly respects monomorphisms. It remains to show that $\mathbb{Z}$ respects weak equivalences. In case $\mathcal{X} = *$ this is a standard fact in homotopy theory. The general case then follows since $\mathbb{Z}$ commutes with the formation of stalks ([Mil] II. 3.20 (a)). $\square$

Next we have to discuss how the functor $\mathbb{H}^{\leq 0}$ is related to the usual hypercohomology functor $\mathbb{H}^{+}$. Both functors can be evaluated on bounded below complexes in $C^{\leq 0}(Shab(\mathcal{X}))$.

Lemma 3.5. *If the complex $\mathcal{F}^{\cdot}$ in $C^{\leq 0}(Shab(\mathcal{X}))$ is fibrant, then $\mathcal{F}^{n}$ is an injective abelian sheaf for any $n < 0$.*

Proof. Fix a $n < 0$ and an exact diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{j} & \mathcal{B} \\ & & & \searrow^{a} & \\ & & & & \mathcal{F}^{n} \end{array}$$

in $Shab(\mathcal{X})$. We then have the solid arrow diagram

$$\begin{array}{ccc} [\ldots \to 0 \to \mathcal{A} \xrightarrow{=} \mathcal{A} \to 0 \to \ldots] & \xrightarrow{(a,da)} & [\ldots \to \mathcal{F}^{n} \xrightarrow{d} \mathcal{F}^{n+1} \to \ldots] \\ \downarrow{\scriptstyle (j,j)} & \nearrow & \\ [\ldots \to 0 \to \mathcal{B} \xrightarrow{=} \mathcal{B} \to 0 \to \ldots] & & \end{array}$$

in $C^{\leq 0}(Shab(\mathcal{X}))$ in which (j,j) is a trivial cofibration. Since the complex $\mathcal{F}^{\cdot}$ is fibrant the dotted arrow making the diagram commutative exists by the RLP. In particular, the original diagram can be completed to a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathcal{A} & \xrightarrow{j} & \mathcal{B} \\ & & & \searrow^{a} \quad \swarrow & \\ & & & \mathcal{F}^{n} \ . & \end{array}$$

$\square$

Notation: For any (unbounded) complex $\mathcal{F}^{\cdot} = [\ldots \to \mathcal{F}^{m} \xrightarrow{d} \mathcal{F}^{m+1} \to \ldots]$ of objects in some abelian category we have the truncations

$$\begin{array}{ll} t_{\geq n}\mathcal{F}^{\cdot} & : \ldots \to 0 \to \mathcal{F}^{n}/\operatorname{im} d \to \mathcal{F}^{n+1} \to \ldots \quad \text{and} \\ t_{\leq n}\mathcal{F}^{\cdot} & : \ldots \to \mathcal{F}^{n-1} \to \ker d \to 0 \to \ldots \end{array}$$

Lemma 3.6. *A bounded below complex $\mathcal{F}^{\cdot}$ in $C^{\leq 0}(Shab(\mathcal{X}))$ is fibrant if and only if $\mathcal{F}^{n}$ is an injective abelian sheaf for any $n < 0$.*

Proof. Fix an $n_0 \leq 0$ such that $\mathcal{F}^n = 0$ for all $n < n_0$. We have to show that any diagram

$$\begin{array}{ccc} \mathcal{A}^\cdot & \longrightarrow & \mathcal{F}^\cdot \\ {\scriptstyle i}\downarrow & & \\ \mathcal{B}^\cdot & & \end{array}$$

in $C^{\leq 0}(Shab(\mathcal{X}))$ with a trivial cofibration i can be completed to a commutative diagram

$$\begin{array}{ccc} \mathcal{A}^\cdot & \longrightarrow & \mathcal{F}^\cdot \\ {\scriptstyle i}\downarrow & \nearrow & \\ \mathcal{B}^\cdot & & \end{array} .$$

One easily checks that with i its truncation $t_{\geq n_0} i$ is a trivial cofibration, too. We therefore can assume that $\mathcal{A}^n = \mathcal{B}^n = 0$ for all $n < n_0$. Furthermore if

$$0 \longrightarrow \mathcal{F}^0 \longrightarrow \widetilde{\mathcal{F}}^0 \longrightarrow \widetilde{\mathcal{F}}^1 \longrightarrow \ldots$$

is an exact injective resolution of the abelian sheaf $\mathcal{F}^0$ then we can replace the complex $\mathcal{F}^\cdot$ by the complex

$$\widetilde{\mathcal{F}}^\cdot : ... \to 0 \to \mathcal{F}^{n_0} \to ... \to \mathcal{F}^1 \to \widetilde{\mathcal{F}}^0 \to \widetilde{\mathcal{F}}^1 \to ...$$

which is a complex of injective abelian sheaves. In this way we end up with a solid arrow diagram

$$\begin{array}{ccc} \mathcal{A}^\cdot & \longrightarrow & \widetilde{\mathcal{F}}^\cdot \\ {\scriptstyle i}\downarrow & \nearrow & \\ \mathcal{B}^\cdot & & \end{array}$$

in the model category $C^+(Shab(\mathcal{X}))$ in which i is a trivial cofibration and $\widetilde{\mathcal{F}}^\cdot$ is a fibrant object. The wanted dotted arrow then exists by the RLP in $C^+(Shab(\mathcal{X}))$. $\square$

Proposition 3.7. *For any bounded below complex $\mathcal{F}^\cdot$ in $C^{\leq 0}(Shab(\mathcal{X}))$ we have a natural isomorphism*

$$\mathbb{H}^{\leq 0}(X, \mathcal{F}^\cdot) \cong t_{\leq 0}\mathbb{H}^+(X, \mathcal{F}^\cdot) \quad .$$

Proof. Let $\mathcal{F}^\cdot \to I^\cdot$ be a trivial cofibration into a fibrant complex $I^\cdot$ in $C^+(Shab(\mathcal{X}))$ so that we have $\mathbb{H}^+(X, \mathcal{F}^\cdot) \cong \gamma \circ \Gamma_X(I^\cdot)$. The induced homomorphism $\mathcal{F}^\cdot \to t_{\leq 0} I^\cdot$ then is a trivial cofibration in $C^{\leq 0}(Shab(\mathcal{X}))$ and, according to the above Lemma 3.6, the complex $t_{\leq 0} I^\cdot$ is fibrant in $C^{\leq 0}(Shab(\mathcal{X}))$. We therefore have $\mathbb{H}^{\leq 0}(X, \mathcal{F}^\cdot) \cong \gamma \circ \Gamma_X(t_{\leq 0} I^\cdot)$. The assertion then follows from the identity

$$\Gamma_X(t_{\leq 0} I^\cdot) = t_{\leq 0}\Gamma_X(I^\cdot)$$

which is an immediate consequence of the left exactness of Γ_X. $\square$

This result also shows that the cohomology theories $\widetilde{\mathbb{H}}$ and $\mathbb{H}^{\leq 0}$ are somewhat unsatisfactory. The reason for that will become more transparent in section 5 where we will discuss cohomology theories on the larger categories of spectra of sheaves, resp. of unbounded complexes of abelian sheaves.

We finish the present discussion about the relation between $\widetilde{\mathbb{H}}$ and $\mathbb{H}^{\leq 0}$ with a few more remarks on the characterization of those complexes $\mathcal{F}^{\cdot}$ in $C^{\leq 0}(Shab(\mathcal{X}))$ which are fibrant, resp. which have a fibrant image $s\mathcal{F}^{\cdot}$ in $SSh(\mathcal{X})$.

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