Seminar Talk

Milnor's Counterexample to the Hauptvermutung

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This talk is about the classical article [Mil61b] by John W. Milnor, in which he disproves the Polyhedral Hauptvermutung. The Polyhedral Hauptvermutung is the conjecture, that for two simplicial complexes which are homeomorphic, there are subdivisions of those complexes which are piecewise-linearly homeomorphic.

1 A brief history of the Hauptvermutung

This section is based on the Hauptvermutung book [RCS+96] edited by Andrew Ranicki.

The word Hauptvermutung is a short form of the german term *Hauptvermutung der kombinatorischen Topologie* which means the main conjecture of combinatorial topology. It was first stated in 1908 by Tietze [Tie08] and by Steinitz [Ste08]. Their formulation was slightly different from the modern one, which is due to Rourke and Sanderson [RS70].

Before the appearing of Milnor's article, the Hauptvermutung had been proven for manifolds of dimension at most 3 and for polyhedra of dimension at most 2. Milnor then gave counterexamples for all dimensions ≥ 3. His construction was later generalized by Stallings [Sta65].

However, these counterexamples were not manifolds, so the question arose, whether the Hauptvermutung was true for manifolds. This was called the manifold Hauptvermutung (and the original one is often called polyhedral Hauptvermutung to distinguish the two). But also the manifold Hauptvermutung was proven not to hold, which was done by Kirby and Siebenmann in 1969 [KS77].

2 Milnor's approach to falsifying the Polyhedral Hauptvermutung

In his 1961 article [Mil61b], Milnor constructs simplicial complexes X_q as follows. For L_q denoting the lens manifold of type $(7, q)$ and dimension 3, X_q is the $L_q \times \Delta^n$ (Δ^n the standard n-simplex) with an adjoint cone over $L_q \times \partial\Delta^n$. He then proves the following two main theorems.

Theorem 1. *For $n \geq 3$ the complexes X_1 and X_2 are homeomorphic.*

Theorem 2. *There are no finite cell subdivisions of the simplicial complexes X_1 and X_2 which are isomorphic, so there is no piecewise-linear homeomorphism between those complexes.*

Additionally, Milnor's proofs gave two other interesting examples, which are stated in the following theorems.

Theorem 3. *The manifolds-with-boundary $L_1 \times D^5$ and $L_2 \times D^5$ are not diffeomorphic, but their interiors are.*

Theorem 4. *The manifolds $L_1 \times S^n$ and $L_2 \times S^n$ are h-cobordant, but not diffeomorphic.*

3 Lens manifolds

At first we give a construction of lens manifolds. Let p and q be relatively prime positive integers and $p > q$. We consider $S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2, |z_1|^2 + |z_2|^2 = 1\right\}$ as the unit sphere in $\mathbb{C}^2$. For $\omega = \exp\left(\frac{2\pi i}{p}\right)$ the cyclic group $\Pi := \mathbb{Z}/p\mathbb{Z}$ acts on S^3 for a generator T of $\mathbb{Z}/p\mathbb{Z}$ by

$$T(z_1, z_2) = (\omega z_1, \omega^q z_2).$$

This action is differentiable and has no fixed points, so the quotient S^3/Π is a manifold.

Definition 5. For p, q as above, the manifold S^3/Π is called the *lens manifold* $L(p, q)$.

The lens manifolds can be made both CW-complexes and simplicial complexes. As for the CW-structure, there is one with only four cells. Those are given as the images of the following subsets of S^3.

- The 0-cell $e_0 = (1, 0)$,
- the 1-cell $e_1 = \left\{\left(e^{i\theta}, 0\right), 0 < \theta < \frac{2\pi}{p}\right\}$,
- the 2-cell $e_2 = \left\{\left(z_1, \sqrt{1 - |z_1|^2}\right), |z_1| < 1\right\}$,
- the 3-cell $e_3 = \left\{\left(z_1, e^{i\theta}\sqrt{1 - |z_1|^2}\right), 0 < \theta < \frac{2\pi}{p}, |z_1| < 1\right\}$.

A simplicial structure for the lens manifold $L(p, q)$ is for example this one: Let P be the convex hull of the set

$$\{(\omega^j, 0), 0 \leq j < p\} \cup \{(0, \omega^k), 0 \leq k < p\}$$

in $\mathbb{C}^2$. The boundary ∂P of this solid polyhedron is a simplicial complex which is obviously homeomorphic to S^3. The simplicial structure for $L(p, q)$ is now obtained by subdividing barycentrically twice and then dividing out the action of Π.

These simplicial structures, which were constructed by Tietze [Tie08], are compatible with the differentiable structure and the CW-structure. Tietze also computed the fundamental group (which is Π) and the homology of the lens manifolds and showed the following lemma.

Lemma 6. *If the lens manifolds $L(p, q)$ and $L(p', q')$ are homotopically equivalent, then $p = p'$.*

A combinatorial classification of the lens manifolds is due to Kurt Reidemeister [Rei36].

Lemma 7. *The lens manifolds $L(p, q)$ and $L(p, q')$ are combinatorially equivalent if and only if*

$$\begin{aligned} \textit{either } q' &\equiv \pm q \quad \mod p \\ \textit{or } \pm qq' &\equiv 1 \quad \mod p. \end{aligned}$$

It was shown later that two lens manifolds are combinatorially equivalent if and only if they are homeomorphic.

The classification of the lens manifolds up to homotopy equivalence is due to Whitehead [Whi41]. He obtained the following lemma [Olu53].

Lemma 8. *The lens manifolds $L(p, q)$ and $L(p, q')$ are homotopically equivalent if and only if $\pm qq'$ is a quadratic residue modulo p.*

As a last fact on lens manifolds it should be noted, that all 3-dimensional ones are parallelizable. This is because all compact orientable 3-manifolds are [Sti36], which can for instance be seen by the Wu formula and obstruction theory.

4 Mazur's theorem

In 1961, Barry Mazur [Maz61] dealt with the question, whether given a homotopy equivalence $\phi\colon M_1 \to M_2$ of differentiable manifolds of the same dimension there is a diffeomorphism homotopic to ϕ. In this context he introduced a notion called k-equivalence and proved a theorem, which will be useful to prove our first main theorem.

From now on let M_1 and M_2 be two k-dimensional closed differential manifolds, which are parallelizable and homotopically equivalent.

Theorem 9. *For $n > k$, $M_1 \times \mathbb{R}^n$ is diffeomorphic to $M_2 \times \mathbb{R}^n$.*

To sketch a proof of Mazur's theorem, we first state the main ingredient.

Lemma 10. *For integers n, k satisfying $n > k > 1$, any embedding*

$$h\colon M_1 \times D^n \longrightarrow \mathrm{Int}(M_1 \times D^n)$$

which is homotopic to the identity can be extended to a diffeomorphism of pairs

$$\big(M_1 \times 2D^n, M_1 \times D^n\big) \longrightarrow \big(M_1 \times D^n, h(M_1 \times D^n)\big).$$

Sketch of proof. The main step is to show that h restricted to $M_1 \times \{0\}$ is differentiably isotopic to the inclusion $M_1 \times \{0\} \hookrightarrow M_1 \times D^n$ which follows from theorems of Haefliger [Hae61] and Whitney [Whi36]. □

Sketch of proof of Mazur's theorem. Let $f\colon M_1 \to M_2$ be a homotopy equivalence and choose a differential embedding $f'\colon M_1 \to \mathrm{Int}(M_2 \times D^n)$ approximating $x \mapsto \big(f(x), 0\big)$. Since M_1 and M_2 are parallelizable, we get from a theorem of Milnor [Mil61a], that (for $n > k$) the normal bundle of $f'(M_1)$ is trivial, so we can choose a tubular neighbourhood of $f'(M_1)$ in $\mathrm{Int}(M_2 \times D^n)$ which is diffeomorphic to $M_1 \times D^n$. Thus we get an embedding $i\colon M_1 \times D^n \to M_2 \times D^n$ and by taking a homotopy inverse, another embedding $j\colon M_2 \times D^n \to M_1 \times D^n$.

The direct limit of the infinite sequence

$$M_1 \times D^n \xrightarrow{i} M_2 \times D^n \xrightarrow{j} M_1 \times D^n \xrightarrow{i} \cdots$$

shall be called V. Lemma 10 now tells us, that V is diffeomorphic to $M_1 \times \mathbb{R}^n$ which is the union

$$M_1 \times D^n \subset M_1 \times 2D^n \subset M_1 \times 4D^n \subset \cdots$$

and to $M_2 \times \mathbb{R}^n$. So these two manifolds are diffeomorphic. □

Mazur's theorem enables us to prove our first main theorem for the case $n > 3$. First we state the following lemma which is easily obtained from the classification of the lens manifolds.

Lemma 11. *If $\pm qq'$ is a quadratic residue modulo p and $n > 3$, then $L(p, q) \times \mathbb{R}^n$ and $L(p, q') \times \mathbb{R}^n$ are diffeomorphic.*

Proof of theorem 1 for $n > 3$. If we remove the vertex of the cone in X_q, the remaining part is clearly homeomorphic to $L_q \times \mathbb{R}^n$. So X_q is homeomorphic to the one-point compactification of $L_q \times \mathbb{R}^n$. Lemma 11 now shows that X_1 and X_2 are homeomorphic. □

To proceed to the case of $n = 3$, we need another lemma, which in particular shows that $L_1 \times S^4$ and $L_2 \times S^4$ are h-cobordant and thus gives us half of the proof of theorem 4. First we define

$$W := \big(M_2 \times D^n\big) \setminus \mathrm{Int}\big(i(M_1 \times D^n)\big).$$

This compact differentiable manifold is bounded by $M_2 \times S^{n-1}$ and $i(M_1 \times S^{n-1})$.

Lemma 12. *If $n \geq 3$, then $M_2 \times S^{n-1}$ and $i(M_1 \times S^{n-1})$ are both deformation retracts of W.*

Sketch of proof. Let $W_1 := i(M_1 \times S^{n-1})$ and $W_2 := M_2 \times S^{n-1}$ be the boundary parts of W. Any map of a 2-dimensional complex into $M_2 \times D^n$ can be deformed into W, thus

$$\pi_1(W) \to \pi_1(M_2 \times D^n)$$
$$\pi_1(W_q) \to \pi_1(W)$$

are isomorphisms for $q = 1, 2$. Whitehead's theorem now yields that W_1 is a deformation retract of W, and by additionally using Poincaré duality one sees that W_2 is, too. □

Corollary 13. *For $n > k > 1$ the manifolds $M_1 \times S^{n-1}$ and $M_2 \times S^{n-1}$ are h-cobordant.*

Remark 14. In particular $L_1 \times S^4$ and $L_2 \times S^4$ are h-cobordant, so half of theorem 4 is now proved.

We are now able to finish proving our first main theorem.

Proof of theorem 1 for $n = 3$. It has been proven by Haefliger [Hae61], that any homotopy equivalence $L_1 \longrightarrow (L_2 \times D^3)$ is homotopic to an embedding f'. We get from obstruction theory that the normal bundle of $f'(L_1)$ is trivial, so lemma 12 tells us, that $L_2 \times S^2$ and $i(L_1 \times S^2)$ are deformation retracts of

$$W = \big(L_2 \times D^3\big) \setminus \operatorname{Int}\big(i(L_1 \times D^3)\big),$$

i. e. $L_1 \times S^2$ and $L_2 \times S^2$ are h-cobordant. A theorem of Stallings [Sta65] now yields

$$W \setminus (L_2 \times S^2) \approx i(L_1 \times S^2) \times [0, \infty),$$

which shows

$$\big(L_2 \times D^3\big) \setminus \big(L_2 \times S^2\big) \approx \big(L_1 \times D^3\big) \cup \big(L_1 \times S^2 \times [0, \infty)\big).$$

So $L_1 \times \mathbb{R}^3$ and $L_2 \times \mathbb{R}^3$ and thus X_1 and X_2 are homeomorphic. □

5 The torsion invariant

In this section we will construct an invariant which distinguishes simplicial complexes combinatorially. We will then use this invariant to find combinatorially distinct simplicial complexes which are nevertheless homeomorphic.

Definition 15. Let D be a principal ideal domain and M be a free D-module of rank q. A generator of $\Lambda^q_D(M)$ will be called a *volume* in M.

Remark 16. For $q > 0$, a volume can v can always be written as $v = b_1 \wedge \cdots \wedge b_q$ with a suitable basis $b_1, \ldots, b_q$ of M.

Definition 17. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of finitely generated free D-modules and let $v' = b'_1 \wedge \cdots \wedge b'_p$ and $v'' = b''_1 \wedge \cdots \wedge b''_r$ be volumes in M' and M'' respectively. Lift each basis element b''_i of M'' to a basis element b_i of M and define a volume on M by

$$v := b_1 \wedge \cdots \wedge b_r \wedge b'_1 \wedge \cdots \wedge b'_p.$$

In this situation write $v'' = \frac{v}{v'}$.

If we now take a long exact sequence

$$0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \longrightarrow 0$$

of free D-modules with a volume v_i on each C_i, we can splice the sequence and compute step by step the iterated quotient of the volumes which finally yields a unit

$$[v_0 v_1^{-1} v_2 v_3^{-1} \cdots v_n^{\pm 1}] \in D^\times.$$

Now consider a CW-complex K on which a group Π acts freely and cellurlaly such that the quotient K/Π has only finitely many cells. Then the cellular chain groups $C_i(K;\mathbb{Z})$ are modules over the group ring $\mathbb{Z}D$. Let $h\colon \Pi \to D^\times$ be a homomorphism and consider the chain complex

$$C_* := P \otimes_\Pi C_*(K;\mathbb{Z}).$$

In this complex, each i-cell determines a basis element in C_i and by taking the exterior product we obtain a volume on each C_i.

If we additionally assume the vanishing of all homology groups $H_i(C_*)$, so that the chain complex C_* yields a long exact sequence, we can define the torsion

$$\Delta_h(K) := [v_0 v_1^{-1} v_2 v_3^{-1} \cdots v_n^{\pm 1}] \in D^*.$$

For a CW-pair we can define a similar notion of torsion by using the relative chain complex. This yields a combinatorial invariant, but this fact will be proven later.

Theorem 18. *If the CW-pair (K', L') is an equivariant subdivision of (K, L) and $\Delta_h(K, L)$ is defined, then*

$$\Delta_h(K', L') = \Delta_h(K, L).$$

If we now let Π be the cyclic group of order p acting on S^3 as described in the introduction of the lens manifolds and as D the complex numbers we get (depending on the choice of h) for $L(7,1)$ as absolute value of Δ the possible values of approximately 1.33 or 0.41 or 0.26 and for $L(7,2)$ approximately 0.74 or 0.59 or 0.33, so these manifolds are combinatorially distinct. It should be noted here, that any homomorphism $\Pi \to \mathbb{C}^\times$ takes a generator of Π to a p-th root of unity.

We are now ready to prove our second main theorem.

Proof of theorem 2. Removing the top of the cone in X_q we obtain manifolds homeomorphic to $L(7,q)\times\mathbb{R}^n$ with fundamental group $\mathbb{Z}/7\mathbb{Z}$. Let K_q be the one-point compactification of the universal covering of $X_q \setminus x_0$. On those the fundamental group Π operates with a single fixed point and the quotient space K_q/Π is just X_q. We can lift any cell structure on X_q, x_0 to an equivariant one on K_q.

We get a cell structure on X_q by the four cells of $L(7,q)$ and the vertex x_0, which yields a cell structure on K_q with cells of the form $T^r e_i \times \mathbb{R}^n$ and one singe vertex k_0. The chain complex $C_*(K_q, k_0; \mathbb{Z})$ is free over the group ring $\mathbb{Z}\Pi$ and has as preferred generators $e_i \times \mathbb{R}^n$ Furthermore it is isomorphic to $C_*\big(\widetilde{L(7,q)}; \mathbb{Z}\big)$ with a dimension shift. Thus

$$\Delta_h(K-q, k_0) = \Delta_h\big(\widetilde{L(7,q)}; \mathbb{Z}\big)^{\pm 1}.$$

We conclude that $(K_1, k_0; \Pi)$ and $(K_2, k_0; \Pi)$ are combinatorially distinct and thus no CW-subdivisions of X_1 and X_2 are combinatorially equivalent. Since our given structures are such subdivisions, we are done. □

Invariance under subdivision

This subsection is devoted to a sketch of the proof of theorem 18. We first collect three lemmas without proof.

Lemma 19. *Let the following diagram of free D-modules of finite rank be commutative with all rows and columns exact.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{11} & \longrightarrow & M_{12} & \longrightarrow & M_{13} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{21} & \longrightarrow & M_{22} & \longrightarrow & M_{23} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{31} & \longrightarrow & M_{32} & \longrightarrow & M_{33} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then for volumes v_{ij} in M_{ij} with $i, j \leq 2$ the following identity holds.

$$\frac{\frac{v_{22}}{v_{12}}}{\frac{v_{21}}{v_{11}}} = \pm \frac{\frac{v_{22}}{v_{21}}}{\frac{v_{12}}{v_{11}}}$$

Lemma 20. *If Π operates freely on a CW-triple (K, L, M), then*

$$\Delta_h(K, M) = \Delta_h(K, L)\Delta_h(L, M),$$

i. e. if two of these invariants are defined, then the third is, too, and the equation holds.

Lemma 21. *If* Π *acts on* $K \setminus L$ *freely and cellularly, and if* $H_*(K, L; \mathbb{Z}) = 0$*, then* $\Delta_h(K, L) = 1$.

Proof of theorem 18. Let

$$L = K_0 \subset K_1 \subset \cdots \subset K_r = K$$

be an increasing sequence of subcomplexes of K so that each $K_{i+1} \setminus K_i$ consists of a single equivariant cell. Let I be the unit intervall (considered a CW-complex) on which Π acts trivially.

If K' is a subdivision of K let (A, B) denote the CW-pair obtained by subdividing $K \times \{1\}$ in $(K \times I, L \times I)$. We obtain an increasing sequence for A similar to the one above by subdividing $K_i \times \{1\}$ in $(K \times 0) \cup (K_i \times I)$.

Lemmas 20 and 21 now yield

$$\Delta_h(A_0, B) = \Delta_h(A_1, B) = \cdots = \Delta_h(A_r, B)$$

where $A_r = A$. So $\Delta_h(A, B) = \Delta_h(K, L)$.

Letting now $\bar{A}_i$ be the subcomplex of A obtained from $(K \times 1) \cup (K_i \times I)$ by subdividing $K \times 1$, we get $\Delta_h(K', L') = \Delta_h(\bar{A}_0, B)$ and

$$\Delta_h(\bar{A}_0, B) = \Delta_h(\bar{A}_1, B) = \cdots = \Delta_h(\bar{A}_r, B)$$

with $\bar{A}_r = A$. So

$$\Delta_h(K', L') = \Delta_h(A, B) = \Delta_h(K, L).$$

□

Lastly we state a theorem about the torsion of a product.

Theorem 22. *Let* A *be a finite CW-complex on which* Π *acts trivially. If* $\Delta_h(K)$ *is defined then* $\Delta_h(K \times A)$ *is defined and is equal to* $\Delta_h(K)^{\chi(A)}$.

We will not prove this theorem but instead state two corollaries which complete the proofs of the theorems 3 and 4.

Corollary 23. *For any* n *the differentiable manifold* $L_1 \times D^n$ *is not diffeomorphic to* $L_2 \times D^n$.

Corollary 24. *For* n *even the manifolds* $L_1 \times S^n$ *and* $L_2 \times S^n$ *are not diffeomorphic.*

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