C*-pseudo-multiplicative unitaries and Hopf C*-bimodules

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Abstract

We introduce C*-pseudo-multiplicative unitaries and concrete Hopf C*-bimodules for the study of quantum groupoids in the setting of C*-algebras. These unitaries and Hopf C*-bimodules generalize multiplicative unitaries and Hopf C*-algebras and are analogues of the pseudo-multiplicative unitaries and Hopf–von Neumann-bimodules studied by Enock, Lesieur and Vallin. To each C*-pseudo-multiplicative unitary, we associate two Fourier algebras with a duality pairing, a C*-tensor category of representations, and in the regular case two reduced and two universal Hopf C*-bimodules. The theory is illustrated by examples related to locally compact Hausdorff groupoids. In particular, we obtain a continuous Fourier algebra for a locally compact Hausdorff groupoid.

1 Introduction

Multiplicative unitaries, which were first systematically studied by Baaj and Skandalis [3], are fundamental to the theory of quantum groups in the setting of operator algebras and to generalizations of Pontrjagin duality [36]. First, one can associate to every locally compact quantum group a multiplicative unitary [15, 16, 21]. Out of this unitary, one can construct two Hopf C*-algebras, where one coincides with the initial quantum group, while the other is the generalized Pontrjagin dual of the quantum group. The duality manifests itself by a pairing on dense Fourier subalgebras of the two Hopf C*-algebras. These Hopf C*-algebras can be completed to Hopf–von Neumann algebras and are reduced in the sense that they correspond to the regular representations of the quantum group and of its dual, respectively. Considering arbitrary representations, one can also construct out of the associated unitary two universal Hopf C*-algebras with morphisms onto the reduced ones. In the study of coactions of quantum groups on algebras, the unitary is an essential tool for the construction of dual coactions on
the reduced crossed products and in the proof of biduality [3] which generalizes the Takesaki–Takai duality.

Much of the theory of quantum groups has been generalized for quantum groupoids in a variety of settings, for example, for finite quantum groupoids in the setting of finite-dimensional C*-algebras by Böhm, Szlachányi, Nikshych and others [5, 6, 7, 22] and for measurable quantum groupoids in the setting of von Neumann algebras by Enock, Lesieur and Vallin [10, 11, 12, 19]. Fundamental for the second theory are the Hopf–von Neumann bimodules and pseudo-multiplicative unitaries introduced by Vallin [37, 38].

In this article, we introduce generalizations of multiplicative unitaries and Hopf C*-algebras that are suited for the study of locally compact quantum groupoids in the setting of C*-algebras, and extend many of the results on multiplicative unitaries that were obtained by Baaj and Skandalis in [3]. In particular, we associate to every regular C*-pseudo-multiplicative unitary two Hopf C*-bimodules and two Fourier algebras with a duality pairing, and construct universal Hopf C*-bimodules from a C*-tensor category of representations of the unitary. The theory presented here was applied already in [30] to the definition and study of compact C*-quantum groupoids, and will be applied in a forthcoming article to the study of reduced crossed products for coactions of Hopf C*-bimodules on C*-algebras and to an extension of the Baaj-Skandalis duality theorem; see also [32].

Our concepts are related to their von Neumann-algebraic counterparts as follows. In the theory of quantum groups, one can use the multiplicative unitary to pass between the setting of von Neumann algebras and the setting of C*-algebras and thus obtains a bijective correspondence between measurable and locally compact quantum groups. This correspondence breaks down for quantum groupoids — already for ordinary spaces, considered as groupoids consisting entirely of units, a measure does not determine a topology. In particular, one can not expect to pass from a measurable quantum groupoid in the setting of von Neumann algebras to a locally compact quantum groupoid in the setting of C*-algebras in a canonical way. The reverse passage, however, is possible, at least on the level of the unitaries and the Hopf bimodules.

Fundamental to our approach is the framework of modules, relative tensor products and fiber products in the setting of C*-algebras introduced in [31]. That article also explains in detail how the theory developed here can be reformulated in the setting of von Neumann algebras, where we recover Vallin’s notions of a pseudo-multiplicative unitary and a Hopf–von Neumann bimodule, and how to pass from the level of C*-algebras to the setting of von Neumann algebras by means of various functors.

The theory presented here overcomes several restrictions of our former generalizations of multiplicative unitaries and Hopf C*-algebras [34]; see also [33].

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Organization This article is organized as follows. We start with preliminaries, summarizing notation, terminology and some background on Hilbert C*-modules.

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In Section 2, we recall the notion of a multiplicative unitary and define $C^*$-pseudo-multiplicative unitaries. This definition involves $C^*$-modules over $C^*$-bases and their relative tensor product, which were introduced in [31] and which we briefly recall. As an example, we construct the $C^*$-pseudo-multiplicative unitary of a locally compact Hausdorff groupoid. We shall come back to this example frequently.

In Section 3, we associate to every well-behaved $C^*$-pseudo-multiplicative unitary two Hopf $C^*$-bimodules. These Hopf $C^*$-bimodules are generalized Hopf $C^*$-algebras, where the target of the comultiplication is no longer a tensor product but a fiber product that is taken relative to an underlying $C^*$-base. Inside these Hopf $C^*$-bimodules, we identify dense convolution subalgebras which can be considered as generalized Fourier algebras, and construct a dual pairing on these subalgebras. To illustrate the theory, we apply all constructions to the unitary associated to a groupoid $G$, where one recovers the reduced groupoid $C^*$-algebra of $G$ on one side and the function algebra of $G$ on the other side.

In Section 4, we study representations and corepresentations of $C^*$-pseudo-multiplicative unitaries. These (co)representations form a $C^*$-tensor category and lead to the construction of universal variants of the Hopf $C^*$-bimodules introduced in Section 3. For the unitary associated to a groupoid, we establish a categorical equivalence between corepresentations of the unitary and representations of the groupoid.

In Section 5, we show that every $C^*$-pseudo-multiplicative unitary satisfying a certain regularity condition is well-behaved. This condition is satisfied, for example, by the unitaries associated to groupoids and by the unitaries associated to compact quantum groupoids. Furthermore, we collect some results on proper and étale $C^*$-pseudo-multiplicative unitaries.

**Terminology and notation**

Given a subset $Y$ of a normed space $X$, we denote by $[Y] \subset X$ the closed linear span of $Y$. We call a linear map $\phi$ between normed spaces *contractive* or a linear contraction if $\|\phi\| \leq 1$.

All sesquilinear maps like inner products of Hilbert spaces are assumed to be conjugate-linear in the first component and linear in the second one. Let $H, K$ be Hilbert spaces. We canonically identify $L(H, K)$ with a subspace of $L(H \otimes K)$. Given subsets $X \subset L(H)$ and $Y \subset L(H, K)$, we denote by $X'$ the commutant of $X$ and by $[Y]$ the $\sigma$-weak closure of $Y$.

Given a $C^*$-subalgebra $A \subset L(H)$ and a $*$-homomorphism $\pi: A \to L(K)$, we put $L^\pi(H, K) := \{ T \in L(H, K) \mid Ta = \pi(a)T \text{ for all } a \in A \}$. \hfill (1)

We use the ket-bra notation and define for each $\xi \in H$ operators $|\xi\rangle: H \to C, \lambda \mapsto \lambda \xi$, and $\langle \xi| : H \to C, \xi' \mapsto \langle \xi| \xi'\rangle$.

We shall use some theory of groupoids; for background, see [26] or [24]. Given a groupoid $G$, we denote its unit space by $G^0$, its range map by $r$, its source map by $s$, and set $G_x := \{(x, y) \in G \mid r(x) = r(y)\}$, $G_{x,y} := \{(x, y) \in G \mid s(x) = r(y)\}$, and $G^u = r^{-1}(u), G^u = s^{-1}(u)$ for each $u \in G^0$.

We shall make extensive use of (right) Hilbert $C^*$-modules and the internal tensor product; a standard reference is [17]. Let $A$ and $B$ be $C^*$-algebras. Given Hilbert $C^*$-modules $E$ and $F$ over $B$, we denote by $L_B(E, F)$ the space of all adjointable operators.
from $E$ to $F$. Let $E$ and $F$ be $C^*$-modules over $A$ and $B$, respectively, and let $\pi: A \to \mathcal{L}_R(F)$ be a $*$-homomorphism. Recall that the internal tensor product $E \otimes_R F$ is the Hilbert $C^*$-module over $B$ which is the closed linear span of elements $\eta \otimes_R \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary and $(\eta \otimes_R \xi) (\eta' \otimes_R \xi') = (\eta \pi(\eta')) \xi) \xi'$. We denote the internal tensor product by “⊗” and drop the index $\pi$ if the representation is understood; thus, for example, $E \otimes F = E \otimes_R F = E \otimes_R F$.

We also define a flipped internal tensor product $F \otimes_R E$ as follows. Equipped the algebraic tensor product $F \otimes E$ with the structure maps $(\xi \otimes \eta) = (\xi \pi(\eta)) \xi' = (\xi \pi(\eta)) \xi'$ and $(\xi \otimes \eta) b = \xi b \otimes \eta$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, $b \in B$. As above, we drop the index $\pi$ and simply write “⊗” instead of “$\pi \otimes$” if the representation $\pi$ is understood. Evidently, the usual and the flipped internal tensor product are related by a unitary map $\Sigma: F \otimes E \overset{\cong}{\to} E \otimes F$, $\eta \otimes \xi \mapsto \xi \otimes \eta$.

For each $\xi \in E$, the maps

$$l_\xi^E(\xi): F \to E \otimes F, \eta \mapsto \xi \otimes \eta, \quad r_\xi^E(\xi): F \to F \otimes E, \eta \mapsto \eta \otimes \xi,$$

are adjointable operators, and for all $\eta \in F, \xi' \in E$,

$$l_\xi^E(\xi) (\eta \otimes \xi') = (\xi \pi(\eta)) \eta = r_\xi^E(\xi) (\eta \otimes \xi').$$

Again, we drop the subscript $\pi$ in $l_\xi^E(\xi)$ and $r_\xi^E(\xi)$ if this representation is understood.

Finally, let $E_1, E_2$ be Hilbert $C^*$-modules over $A$, let $F_1, F_2$ be Hilbert $C^*$-modules over $B$ with representations $\pi_i: A \to \mathcal{L}_R(F_i) (i = 1, 2)$, and let $S \in \mathcal{L}(E_1, E_2)$, $T \in \mathcal{L}(F_1, F_2)$ such that $T \pi_1(a) = \pi_2(a) T$ for all $a \in A$. Then there exists a unique operator $S \otimes T \in \mathcal{L}(E_1 \otimes F_1, E_2 \otimes F_2)$ such that $(S \otimes T) (\eta \otimes \xi) = S \eta \otimes T \xi$ for all $\eta \in E_1, \xi \in F_1$, and $(S \otimes T)^* = S^* \otimes T^*$ [9, Proposition 1.34].

### 2 \textbf{$C^*$-pseudo-multiplicative unitaries}

Recall that a multiplicative unitary on a Hilbert space $H$ is a unitary $V: H \otimes H \to H \otimes H$ that satisfies the pentagon equation $V_{12} V_{13} V_{23} = V_{23} V_{12}$ (see [3]). Here, $V_{12}, V_{13}, V_{23}$ are operators on $H \otimes H \otimes H$ defined by $V_{12} = V \otimes \text{id}$, $V_{23} = \text{id} \otimes V$, $V_{13} = (\Sigma \otimes \text{id}) V_{12} (\Sigma \otimes \text{id}) = (\Sigma \otimes \Sigma) V_{12} (\text{id} \otimes \Sigma)$, where $\Sigma \in \mathcal{L}(H \otimes H)$ denotes the flip $\eta \otimes \xi \mapsto \xi \otimes \eta$. As an example, consider a locally compact group $G$ with left Haar measure $\lambda$. Then the formula

$$(V f)(x, y) = f(x, x^{-1} y)$$

defines a linear bijection of $C_c(G \times G)$ that extends to a unitary on $L^2(G \times G, \lambda \otimes \lambda) \cong L^2(G, \lambda) \otimes L^2(G, \lambda)$. This unitary is multiplicative, and the pentagon equation amounts to associativity of the multiplication in $G$.

In this section, we generalize the notion of a multiplicative unitary so that it covers the example above if we replace the group $G$ by a locally compact Hausdorff groupoid.
In that case, formula (3) only makes sense for \((x, y) \in G_r \times_s G\) and defines a linear bijection from \(C_c(G_r \times_s G)\) to \(C_c(G_r \times_s G)\). If the groupoid \(G\) is finite, that bijection is a unitary from \(\mathcal{F}(G_r \times_s G)\) to \(\mathcal{F}(G_r \times_s G)\), and these two Hilbert spaces can be identified with tensor products of \(\mathcal{F}(G)\) with \(\mathcal{F}(G)\), considered as a module over the algebra \(C(G^0)\) with respect to representations that are naturally induced by the maps \(s, r: G \to G^0\). For a general groupoid, the simple algebraic tensor product of modules has to be replaced by a refined version. In the setting of von Neumann algebras, Vallin used the relative tensor product of Hilbert modules introduced by Connes, also known as Connes’ fusion of correspondences, to define pseudo-multiplicative unitaries [38] which include as a main example the unitary of a measurable groupoid. To take the topology of \(G\) into account, we shall work in the setting of \(C^*-\)algebras and use the relative tensor product of \(C^*\)-modules over \(C^*\)-bases introduced in [31].

### 2.1 The relative tensor product of \(C^*\)-modules over \(C^*\)-bases

In this subsection, we recall the relative tensor product of \(C^*\)-modules over \(C^*\)-bases which is fundamental to the definition of a \(C^*\)-pseudo-multiplicative unitary, and generalize the theory presented in [31, Section 2] in two respects. First, we introduce the notion of a semi-morphism between \(C^*\)-modules which will be important in subsection 4.4. Second, the definition of a \(C^*\)-pseudo-multiplicative unitaries forces us to consider \(C^*\)-\(n\)-modules for \(n \geq 2\) and not only \(C^*\)-bimodules. We shall not give separate proofs of statements that are only mild generalizations of statements found in [31]. For additional motivation and details, we refer to [31]; an extended example can be found in subsection 2.3.

**\(C^*\)-modules over \(C^*\)-bases** A \(C^*\)-base is a triple \((\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^!\) consisting of a Hilbert space \(\mathfrak{A}\) and two commuting nondegenerate \(C^*\)-algebras \(\mathfrak{B}, \mathfrak{B}^! \subseteq L(\mathfrak{A})\). A \(C^*\)-base should be thought of as a \(C^*\)-algebraic counterpart to pairs consisting of a von Neumann algebra and its commutant. As an example, one can associate to every faithful KMS-state \(\mu\) on a \(C^*\)-algebra \(B\) the \(C^*\)-base \((H_\mu, B, B^\mu)\), where \(H_\mu\) is the GNS-space for \(\mu\) and \(B\) and \(B^\mu\) act on \(H_\mu = H_{\mu\mathfrak{A}}\) via the GNS-representations [31, Example 2.9]. If \(b = (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^!)\) is a \(C^*\)-base, then so is \(b^1 := (\mathfrak{A}, \mathfrak{B}^!, \mathfrak{B})\) and \(M(b) := (\mathfrak{A}, M(\mathfrak{B}), M(\mathfrak{B}^!))\), where \(M(\mathfrak{B})\) and \(M(\mathfrak{B}^!\) are naturally represented of \(\mathfrak{A}\).

From now on, let \(b = (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^!\) be a \(C^*\)-base. We shall use the following notion of a \(C^*\)-module. A \(C^*\)-\(b\)-module is a pair \(H_\alpha = (H, \alpha)\), where \(H\) is a Hilbert space and \(\alpha \subseteq L(\mathfrak{A}, H)\) is a closed subspace satisfying \([\alpha \mathfrak{A}] = H\), \([\alpha \mathfrak{B}] = \alpha\), and \([\alpha^* \alpha] = \mathfrak{B} \subseteq L(\mathfrak{A})\). If \(H_\alpha\) is a \(C^*\)-\(b\)-module, then \(\alpha\) is a Hilbert \(C^*\)-module over \(B\) with inner product \((\xi, \xi') \mapsto \xi^* \xi'\) and there exist isomorphisms

\[
\alpha \otimes \mathfrak{A} \to H, \quad \xi \otimes \zeta \mapsto \xi \zeta, \\
\mathfrak{A} \otimes \alpha \to H, \quad \zeta \otimes \xi \mapsto \zeta \xi, \quad (4)
\]

and a nondegenerate representation

\[
\rho_\alpha: \mathfrak{B}^! \to L(H), \quad \rho_\alpha(b^1)(\xi \zeta) = \xi b^1 \zeta \quad \text{for all } b^1 \in \mathfrak{B}^!, \xi, \zeta \in \mathfrak{A}.
\]

A semi-morphism between \(C^*\)-\(b\)-modules \(H_\alpha\) and \(K_\beta\) is an operator \(T \in L(H, K)\) satisfying \(T \alpha \subseteq \beta\). If additionally \(T^* \beta \subseteq \alpha\), we call \(T\) a morphism. We denote the set of all
(semi-)morphisms by \( L_{(i)}(A, B) \). If \( T \in L_{(i)}(A, B) \), then \( T_\rho(b^i) = \rho(b^i)T \) for all \( b^i \in B^i \), and if additionally \( T \in L(A, B) \), then left multiplication by \( T \) defines an operator in \( L_{(\alpha)}(\alpha, \beta) \) which we again denote by \( T \).

We shall use the following notion of \( C^*-b_i \) and \( C^*-n \)-modules. Let \( b_1, \ldots, b_n \) be \( C^* \)-bases, where \( b_i = (\mathfrak{a}_i, B_i, B_i^0) \) for each \( i \). A \( C^*-(b_1, \ldots, b_n) \)-module is a tuple \((H, \alpha_1, \ldots, \alpha_n)\), where \( H \) is a Hilbert space and \((H, \alpha_i) \) is a \( C^*-b_i \)-module for each \( i \) such that \( [\rho_\alpha(B_i^0)] \alpha_j = \alpha_j \) whenever \( i \neq j \). In the case \( n = 2 \), we abbreviate \( \alpha H_\beta := (H, \alpha, \beta) \). We note that if \((H, \alpha_1, \ldots, \alpha_n) \) is a \( C^*-(b_1, \ldots, b_n) \)-module, then \([\rho_\alpha(B_i^0), \rho_\alpha(B_i^0)] = 0 \) whenever \( i \neq j \). The set of (semi-)morphisms between \( C^*-(b_1, \ldots, b_n) \)-modules \( \mathcal{H} = (H, \alpha_1, \ldots, \alpha_n) \), \( \mathcal{K} = (K, \gamma_1, \ldots, \gamma_n) \) is \( \mathcal{L}_{(\alpha)}(\mathcal{H}, \mathcal{K}) := \bigcap_{i=1}^n \mathcal{L}_{(i)}(H_{(i)}(\alpha_i, \beta_i) \subseteq L(H, K) \).

**The relative tensor product** Let \( b = (\mathfrak{a}, B, B^0) \) be a \( C^* \)-base, \( H_\beta \) a \( C^*-b \)-module, and \( K_\gamma \) a \( C^*-b^0 \)-module. The relative tensor product of \( H_\beta \) and \( K_\gamma \) is the Hilbert space

\[
H_\beta \mathcal{B}_{b} K := \beta \otimes \mathcal{B} \otimes \gamma.
\]

It is spanned by elements \( \xi \otimes \zeta \otimes \eta \), where \( \xi \in \beta, \zeta \in \mathcal{B}, \eta \in \gamma \), and

\[
\langle \xi \otimes \zeta \otimes \eta | \xi' \otimes \zeta' \otimes \eta' \rangle = \langle \xi \xi' \zeta \zeta' \eta \eta' \rangle = \langle \xi \eta \zeta \eta' \zeta' \xi' \rangle
\]

for all \( \xi, \xi' \in \beta, \zeta, \zeta' \in \mathcal{B}, \eta, \eta' \in \gamma \). Obviously, there exists a unitary flip

\[
\Sigma: H_\beta \mathcal{B}_{b} K \rightarrow K_\gamma \mathcal{B}_{b} H, \quad \xi \otimes \zeta \otimes \eta \mapsto \eta \otimes \zeta \otimes \xi.
\]

Using the unitaries in (4) on \( H_\beta \) and \( K_\gamma \), respectively, we shall make the following identifications without further notice:

\[
H_\beta \mathcal{B}_{b} K \cong H_\beta \mathcal{B}_{b} K \cong \beta \otimes \mathcal{B} \otimes \gamma, \quad \xi \otimes \eta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \otimes \eta \zeta.
\]

For all \( S \in \rho_\beta(\mathcal{B}^0)' \) and \( T \in \rho_\gamma(\mathcal{B})' \), we have operators

\[
S \otimes \text{id} \in L(H_\beta \mathcal{B}_{b} K), \quad \text{id} \otimes T \in L(H_\beta \mathcal{B}_{b} K).
\]

If \( S \in \mathcal{L}_{(1)}(H_\beta) \) or \( T \in \mathcal{L}_{(1)}(K_\gamma) \), then \((S \otimes \text{id})(X \otimes \eta \zeta) = S \xi \otimes \eta \zeta \) or \((\text{id} \otimes T)(\xi \otimes \eta) = \xi \otimes T \eta \), respectively, for all \( \xi \in \beta, \zeta \in \mathcal{B}, \eta \in \gamma \), so that we can define

\[
S \otimes T := (S \otimes \text{id})(\text{id} \otimes T) \in L(H_\beta \mathcal{B}_{b} K)
\]

for all \( S, T \in \mathcal{L}_{(1)}(H_\beta) \times \mathcal{L}_{(1)}(K_\gamma) \).

For each \( \xi \in \beta \) and \( \eta \in \gamma \), there exist bounded linear operators

\[
|\xi|_1: K \rightarrow H_\beta \mathcal{B}_{b} K, \quad \omega \mapsto \xi \otimes \omega, \quad |\eta|_2: H \rightarrow H_\beta \mathcal{B}_{b} K, \quad \omega \mapsto \omega \otimes \eta.
\]
whose adjoints $\langle \xi \rangle_1 := |\xi\rangle_1^*$ and $\langle \eta \rangle_2 := |\eta\rangle_2^*$ are given by
$$
\langle \xi \rangle_1 : \xi \otimes \omega \mapsto \rho_\gamma(\xi \xi^*) \omega,
\langle \eta \rangle_2 : \omega \otimes \eta^* \mapsto \rho_\beta(\eta^* \eta) \omega.
$$
We put $|\beta_1\rangle := \{ |\xi\rangle \ | \xi \in \beta \} \subseteq L(K, H_b \otimes \gamma K)$ and similarly define $|\beta_2\rangle, |\gamma_1\rangle, |\gamma_2\rangle$.

Let $\mathcal{H} = (H, \alpha_1, \ldots, \alpha_m, b)$ be a $C^\ast$-$(a_1, \ldots, a_m, b)$-module and $\mathcal{K} = (K, \gamma, \delta_1, \ldots, \delta_n)$ a $C^\ast$-$(b^1, c_1, \ldots, c_n)$-module, where $\alpha_i = (\delta_i, A_i, A_i^\dagger)$ and $\epsilon_j = (\xi_j, \mathcal{C}_j, \mathcal{C}_j^\dagger)$ are $C^\ast$-bases for all $i, j$. We put
$$
\alpha_i \cdot \gamma := |\gamma_2\rangle \alpha_i \subseteq L(\mathcal{H}_b, H_b \otimes \gamma K),
\beta \cdot \delta_j := |\beta_1\rangle \delta_j \subseteq L(\mathcal{K}_b, H_b \otimes \gamma K)
$$
for all $i, j$. Then $(H_b \otimes \gamma K, \alpha \cdot \gamma, \beta \cdot \delta, \delta_1, \ldots, \delta_n)$ is a $C^\ast$-$(a_1, \ldots, a_m, c_1, \ldots, c_n)$-module, called the relative tensor product of $\mathcal{H}$ and $\mathcal{K}$ and denoted by $\mathcal{H} \otimes \mathcal{K}$. For all $i, j$ and $a^1 \in \mathcal{A}_i^1, c^1 \in \mathcal{C}_j^1$,
$$
\rho_{(a_i, \gamma)}(a^1) = \rho_{a_i}(a^1) \otimes \text{id},
\rho_{(b_j, \delta_j)}(c^1) = \text{id} \otimes \rho_{b_j}(c^1).
$$

The relative tensor product has nice categorical properties:

- **Functionality** Let $\tilde{\mathcal{H}} = (\tilde{H}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, \tilde{b})$ be a $C^\ast$-$(a_1, \ldots, a_m, b)$-module, $\tilde{\mathcal{K}} = (\tilde{K}, \tilde{\gamma}, \tilde{\delta}_1, \ldots, \tilde{\delta}_n)$ a $C^\ast$-$(b^1, c_1, \ldots, c_n)$-module, and $S \in \mathcal{L}(j)(\mathcal{H}, \tilde{\mathcal{H}}), T \in \mathcal{L}(j)(\mathcal{K}, \tilde{\mathcal{K}})$. Then there exists a unique operator $S \otimes T \in \mathcal{L}(j)(\mathcal{H} \otimes \mathcal{K}, \tilde{\mathcal{H}} \otimes \tilde{\mathcal{K}})$ satisfying $(S \otimes T)(\xi \otimes \zeta \otimes \eta) = S\xi \otimes \zeta \otimes T\eta$ for all $\xi \in \beta, \zeta \in \mathcal{A}, \eta \in \gamma$.

- **Unitality** The triple $\mathcal{U} := (\mathcal{R}, \mathcal{B}^1, \mathcal{B})$ is a $C^\ast$-$(\mathcal{B}^1, \mathcal{B})$-module and the maps
$$
\begin{align*}
\lambda_{\mathcal{H}} : & H_b \otimes \mathcal{B}^1 \mathcal{R} \to \mathcal{H}, \quad \xi \otimes \zeta \otimes b^1 \mapsto \xi b^1 \zeta, \\
\lambda_{\mathcal{K}} : & \mathcal{R} \otimes \mathcal{B} \mathcal{K} \to \mathcal{K}, \quad b \otimes \zeta \otimes \eta \mapsto b\zeta \eta,
\end{align*}
$$
are isomorphisms of $C^\ast$-$(a_1, \ldots, a_m, b)$-modules and $C^\ast$-$(b^1, c_1, \ldots, c_n)$-modules $\mathcal{H} \otimes \mathcal{U} \to \mathcal{H}$ and $\mathcal{U} \otimes \mathcal{K} \to \mathcal{K}$, respectively, natural in $\mathcal{H}$ and $\mathcal{K}$.

- **Associativity** Let $a, b_1, \ldots, b_n$ be $C^\ast$-bases, $\hat{\mathcal{K}} = (K, \gamma, \delta_1, \ldots, \delta_n, \epsilon)$ a $C^\ast$-$(b^1, c_1, \ldots, c_n, \delta)$-module and $L = (L, \phi, \psi_1, \ldots, \psi_l)$ a $C^\ast$-$(b^1, c_1, \ldots, c_l)$-module. Then there exists a canonical isomorphism
$$
\alpha_{\mathcal{H}, \mathcal{K}, L} : (H_b \otimes \gamma K) \otimes \mathcal{B}^1 \otimes \mathcal{B}L \to \beta \otimes \rho_b K \otimes \phi \otimes H_b \otimes \gamma \phi (K \otimes \mathcal{B}L)
$$
which is an isomorphism of $C^\ast$-$(a_1, \ldots, a_m, c_1, \ldots, c_n, c_1, \ldots, c_l)$-modules $(\mathcal{H} \otimes \hat{\mathcal{K}}) \otimes \mathcal{L} \to \mathcal{H} \otimes (\hat{\mathcal{K}} \otimes \mathcal{L})$. From now on, we identify the Hilbert spaces in (6) and denote them by $H_b \otimes \gamma K \otimes \mathcal{B}L$. 

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Direct sums Let $a = (\delta, \mathfrak{A}, \mathfrak{A}^\dagger)$ and $b = (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^\dagger)$ be $C^*$-bases and let $(\mathcal{H}_i)_i$ be a family of $C^*$-(a, b)-modules, where $\mathcal{H}_i = (H_i, \alpha_i, \beta_i)$ for each $i$. Denote by $\bigoplus \alpha_i \subseteq L(\delta, \otimes H_i)$ the norm-closed linear span of all operators of the form $\zeta \leftarrow (\zeta_i)_i$, where $\zeta_i$ is contained in the algebraic direct sum $\bigoplus \alpha_i \otimes \alpha_j$, and similarly define $\bigoplus \beta_i \subseteq L(\mathfrak{A}, \otimes H_i)$. Then the triple $\bigoplus \mathcal{H}_i := (\otimes_i H_i, \bigoplus_i \alpha_i, \bigoplus_i \beta_i)$ is a $C^*$-(a, b)-module, and for each $j$, the canonical inclusions $t^i_j : H_j \to \otimes_i H_i$ and projection $\pi^i_j$; $\bigoplus_i H_i \to H_j$ are morphisms $\mathcal{H}_j \to \mathcal{H}_i$ and $\bigoplus_i \mathcal{H}_i \to \mathcal{H}_j$.

2.2 The definition of $C^*$-pseudo-multiplicative unitaries

Using the relative tensor product of $C^*$-modules introduced above, we generalize the notion of a multiplicative unitary as follows. Let $b = (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^\dagger)$ a $C^*$-base, $(H, \beta, \alpha)$ a $C^*$-(b, b, b)-module, and $V : H \otimes a H \to H \otimes b H$ a unitary satisfying

$$V(\alpha \otimes \alpha) = \alpha \otimes \alpha, \quad V(\beta \otimes \beta) = \beta \otimes \beta, \quad V(\beta \otimes \alpha) = \beta \otimes \beta$$

in $L(\mathfrak{B}, H \otimes b H)$. Then all operators in the following diagram are well defined,

$$
\begin{array}{c}
\xymatrix{ H_{b} \otimes a H \otimes a H \ar[r]^{V \otimes id} \ar[d]^{|id|_{b}} & H_{\alpha} \otimes \beta \otimes a H \ar[r]^{id \otimes V} & H_{\alpha} \otimes b \otimes a H \ar[d]^{V \otimes id} \\
H_{b} \otimes (\omega \otimes \alpha) (H \otimes b H) \ar[d]^{\Sigma_{23} & V \otimes id} & (H_{\beta} \otimes a H) (\alpha \otimes \beta) \otimes b H \\
H_{b} \otimes a H \otimes a H \ar[r]^{V \otimes id} & (H_{\alpha} \otimes \beta) \otimes b H }
\end{array}
$$

where $\Sigma_{23}$ denotes the isomorphism

$$(H_{b} \otimes b H) \otimes b H \cong (H_{\rho} \otimes \beta) \otimes \alpha \overset{\Sigma_{23}}{\cong} (H_{b} \otimes a H) (\alpha \otimes \beta) \otimes b H$$

given by $(\zeta \otimes \xi) \otimes \eta \mapsto (\zeta \otimes \eta) \otimes \xi$. We adopt the leg notation [3] and write

$$V_{12} \text{ for } V \otimes id, \quad V_{23} \text{ for } V \otimes id, \quad V_{13} \text{ for } \Sigma_{23}(V \otimes id)(id \otimes \Sigma).$$
Definition 2.1. A $C^*$-pseudo-multiplicative unitary is a tuple $(b, H, \hat{\beta}, \alpha, \beta, V)$ consisting of a $C^*$-base $b$, a $C^*$-module $(H, \hat{\beta}, \alpha, \beta)$, and a unitary $V: H_{b^1} \otimes_{a} H_{b} \to H_{a} \otimes_{b} H$ such that equation (7) holds and diagram (8) commutes. We frequently call just $V$ a $C^*$-pseudo-multiplicative unitary.

This definition covers the following special cases:

Remarks 2.2. Let $(b, H, \hat{\beta}, \alpha, \beta, V)$ be a $C^*$-pseudo-multiplicative unitary.

i) If $b$ is the trivial $C^*$-base $(C, C, C)$, then $H_{b} \otimes_{a} H \cong H \otimes H \cong H \otimes_{b} H$, and $V$ is a multiplicative unitary.

ii) If we consider $\rho_{b}$ and $\rho_{\hat{\beta}}$ as representations $\rho_{b}, \rho_{\hat{\beta}}: \mathcal{B} \to \mathcal{L}(H_{b}) \cong \mathcal{L}_{\mathcal{B}}(\alpha)$, then the map $\alpha \hat{\beta} \otimes \alpha \cong \alpha \otimes \alpha \leftrightarrow \alpha \otimes \alpha \otimes \alpha \cong \alpha \otimes \rho_{b} \alpha$ given by $\omega \rightarrow V\omega$ is a pseudo-multiplicative unitary on $C^*$-modules in the sense of [34].

iii) Assume that $b = b^1$; then $\mathcal{B} = \mathcal{B}^1$ is commutative. If $\hat{\beta} = \alpha$, then the pseudo-multiplicative unitary in ii) is a pseudo-multiplicative unitary in the sense of O'uchi [23]. If additionally $\hat{\beta} = \alpha = \beta$, then the unitary in ii) is a continuous field of multiplicative unitaries in the sense of Blanchard [4].

iv) Assume that $b$ is the $C^*$-base associated to a faithful proper KMS-weight $\mu$ on a $C^*$-algebra $B$ (see [31, Example 2.9]). Then $\mu$ extends to a n.s.f. weight $\tilde{\mu}$ on $[\mathcal{B}]$, and with respect to the canonical isomorphisms $H_{b} \otimes_{a} H \cong H_{b} \otimes_{b} \mathcal{B} \otimes_{a} H$ and $H_{a} \otimes_{b} H \cong H_{a} \otimes_{b} \mathcal{B} \otimes_{a} H$ (see [31, Corollary 2.21]), $V$ is a pseudo-multiplicative unitary on Hilbert spaces in the sense of Vallin [38].

Let us give some examples and easy constructions:

Examples 2.3. i) To every locally compact, Hausdorff, second countable groupoid with a left Haar system, we shall associate a $C^*$-pseudo-multiplicative unitary in the next subsection.

ii) In [35], a $C^*$-pseudo-multiplicative unitary is associated to every compact $C^*$-quantum groupoid.

iii) The opposite of a $C^*$-pseudo-multiplicative unitary $(b, H, \hat{\beta}, \alpha, \beta, V)$ is the tuple $(b, H, \hat{\beta}, \alpha, \hat{\beta}, V^{op})$, where $V^{op}$ denotes the composition $\Sigma^{op}\Sigma: H_{b} \otimes_{a} H \cong H_{b} \otimes_{b} V \otimes_{a} H \cong H_{b} \otimes_{a} H \otimes_{b} V \otimes_{a} H \cong H_{b} \otimes_{b} V \otimes_{a} H \cong H_{b} \otimes_{b} H$. A tedious but straightforward calculation shows that this is a $C^*$-pseudo-multiplicative unitary.

iv) The direct sum of a family $((b_{i}, H_{i}, \hat{\beta}_{i}, \alpha_{i}, \beta_{i}, V_{i}))_{i}$ of $C^*$-pseudo-multiplicative unitaries is defined as follows. Write $b^{i} = (\delta^{i}, \mathcal{B}, \mathcal{B}^{1})$ for each $i$, put $\delta^{i} := \bigoplus_{i}, \delta^{i}, H^{i} := \bigoplus_{i}, H^{i}$, denote by $\mathcal{B}^{(1)} := \bigoplus_{i}, \mathcal{B}^{(1)} \subseteq \mathcal{L}(\delta^{i})$ the $c_{0}$-direct sum of $C^*$-algebras, and by $\hat{\beta} := \bigoplus_{i}, \hat{\beta}_{i}, \alpha := \bigoplus_{i}, \alpha_{i}, \beta := \bigoplus_{i}, \beta_{i}$ the $c_{0}$-direct sum in $\mathcal{L}(\delta^{i}, H^{i})$. Then $b := (\delta, \mathcal{B}, \mathcal{B}^{(1)})$ is a $C^*$-base, there exist natural isomorphisms $H_{b} \otimes_{a} H \cong \bigoplus_{i}, H_{b}^{i} \otimes_{a} H^{i}$ and $H_{a} \otimes_{b} H \cong \bigoplus_{i}, H_{b}^{i} \otimes_{a} H^{i}$ [31, Proposition 2.17], and if $V$
denotes the unitary corresponding to \( \bigoplus_i V_i \) with respect to these isomorphisms, then the tuple \((b, H, \hat{\beta}, \alpha, \beta, V)\) is a \(C^*\)-pseudo-multiplicative unitary.

**v)** The tensor product of \(C^*\)-pseudo-multiplicative unitaries \((b, H, \hat{\beta}, \alpha, \beta, V)\) and \((\epsilon, K, \hat{\delta}, \gamma, \delta, W)\) is defined as follows. Denote by \(\mathcal{B}^{(1)}(\otimes \mathcal{C}^{(1)} \subseteq \mathcal{L}(\mathcal{S} \otimes \mathfrak{A})\) and \(\mathfrak{B} \otimes \hat{\delta}, \alpha \otimes \gamma, \beta \otimes \delta \subseteq \mathcal{L}(\mathcal{S} \otimes \mathfrak{A}, H \otimes K)\) the closed subspaces generated by elementary tensor products. Then \(b \otimes \epsilon := (\mathcal{S} \otimes \mathfrak{A}, \mathcal{B} \otimes \mathcal{C}^{(1)} \subseteq \mathcal{L}(\mathcal{S} \otimes \mathfrak{A}, H \otimes K)\) is a \(C^*\)-base, there exist natural isomorphisms \((H \otimes K)_{\mathfrak{B} \otimes \hat{\delta}_{\mathfrak{B} \otimes \hat{\delta}}} \otimes b \otimes \epsilon \mathcal{A} \otimes \epsilon \mathcal{A} = (H \otimes K)_{\mathfrak{B} \otimes \hat{\delta}_{\mathfrak{B} \otimes \hat{\delta}}} \otimes b \otimes \epsilon \mathcal{A} \otimes \epsilon \mathcal{A} \) and if \(U\) denotes the unitary corresponding to \(V \otimes W\) with respect to these isomorphisms, then \((b \otimes \epsilon, H \otimes K, \hat{\beta} \otimes \hat{\delta}, \alpha \otimes \gamma, \beta \otimes \delta, U)\) is a \(C^*\)-pseudo-multiplicative unitary.

### 2.3 The \(C^*\)-pseudo-multiplicative unitary of a groupoid

To every locally compact, Hausdorff, second countable groupoid with left Haar system, we shall associate a \(C^*\)-pseudo-multiplicative unitary. The underlying pseudo-multiplicative unitary was introduced by Vallin [38], and associated unitaries on \(C^*\)-modules were discussed in [23, 34]. We focus on the aspects that are new in the present setting.

Let \(G\) be a locally compact, Hausdorff, second countable groupoid with left Haar system \(\lambda\), and associated right Haar system \(\lambda^{-1}\), and let \(\mu\) be a measure on \(G^0\) with full support. Define measures \(v, v^{-1}\) on \(G\) by

\[
\int_G f \, dv := \int_{C_u(G)} \int_{C_u} f(x) \, d\lambda_u(x) \, d\mu(u), \quad \int_G f \, dv^{-1} := \int_{C_u(G)} \int_{C_u} f(x) \, d\lambda_u^{-1}(x) \, d\mu(u)
\]

for all \(f \in C_u(G)\). Thus, \(v^{-1} = i \cdot v\), where \(i: G \to G\) is given by \(x \mapsto x^{-1}\). We assume that \(\mu\) is quasi-invariant in the sense that \(v\) and \(v^{-1}\) are equivalent, and denote by \(D := dv/dv^{-1}\) the Radon-Nikodym derivative.

We identify functions in \(C_0(G^0, \mu)\) and \(C_0(G)\) with multiplication operators on the Hilbert spaces \(L^2(G^0, \mu)\) and \(L^2(G, v)\), respectively, and let

\[
\mathfrak{A} := L^2(G^0, \mu), \quad \mathcal{B} := \mathfrak{B}^1 := C_0(G^0) \subseteq \mathcal{L}(\mathfrak{A}), \quad b := (\mathfrak{A}, \mathcal{B}, \mathcal{B}^1), \quad H := L^2(G, v).
\]

Pulling functions on \(G^0\) back to \(G\) along \(r\) or \(s\), we obtain representations

\[
r^*: C_0(G^0) \to C_0(G) \to \mathcal{L}(H), \quad s^*: C_0(G^0) \to C_0(G) \to \mathcal{L}(H).
\]

We define Hilbert \(C^*\)-modules \(L^2(G, \lambda)\) and \(L^2(G, \lambda^{-1})\) over \(C_0(G^0)\) as the respective completions of the pre-\(C^*\)-module \(C_\epsilon(G)\), the structure maps being given by

\[
\langle \xi' | \xi \rangle(u) = \int_{C_u} \overline{\xi'(x)} \xi(x) \, d\lambda_u(x), \quad \xi f = r^*(f) \xi \quad \text{in the case of } L^2(G, \lambda),
\]

\[
\langle \xi' | \xi \rangle(u) = \int_{C_u} \overline{\xi'(x)} \xi(x) \, d\lambda_u^{-1}(x), \quad \xi f = s^*(f) \xi \quad \text{in the case of } L^2(G, \lambda^{-1})
\]

respectively, for all \(\xi, \xi' \in C_\epsilon(G), u \in G^0, f \in C_0(G^0)\).
**Lemma 2.4.** There exist isometric embeddings

\[ j: L^2(G, \lambda) \to L(\mathbb{R}, H), \quad \hat{j}: L^2(G, \lambda^{-1}) \to L(\mathbb{R}, H) \]

such that for all \( \xi \in C_c(G), \zeta \in C_c(G^0) \)

\[ \langle j(\xi) \zeta \rangle(x) = \xi(x) \zeta(r(x)), \quad \langle \hat{j}(\xi) \zeta \rangle(x) = \xi(x) D^{-1/2}(x) \zeta(s(x)). \]

**Proof.** Let \( E := L^2(G, \lambda), \hat{E} := L^2(G, \lambda^{-1}) \), and \( \xi, \xi' \in C_c(G), \zeta, \zeta' \in C_c(G^0) \). Then

\[
\begin{align*}
\langle j(\xi) \zeta \rangle | \langle j(\xi') \zeta' \rangle &= \int_{G^0} \int_{G^0} \xi(r(x)) \zeta'(s(x)) \zeta(r(x)) \xi(s(x)) \, d\mu(x) \, d\mu(u) = \langle \zeta' | j(\xi) \zeta \rangle_E, \\
\langle j(\xi) \zeta | j(\xi') \zeta' \rangle &= \int_{G^0} \int_{G^0} \xi(r(x)) \zeta(s(x)) \xi'(s(x)) \zeta'(s(x)) \, d\mu(x) \, d\mu(u) = \langle \xi | j(\xi') \zeta' \rangle_E.
\end{align*}
\]

\( \square \)

Let \( \alpha := \beta := j(L^2(G, \lambda)) \) and \( \hat{\beta} := \hat{j}(L^2(G, \lambda^{-1})). \) Easy calculations show:

**Lemma 2.5.** \((H, \hat{\beta}, \alpha, \beta)\) is a \( C^*-(b^1, b^2)\)-module, \( \rho_\alpha = \rho_\beta = r^* \) and \( \rho_\hat{\beta} = s^* \), and \( j \) and \( \hat{j} \) are unitary maps of Hilbert \( C^*\)-modules over \( C_0(G^0) \cong \mathbb{B}. \)

The Hilbert spaces \( H_{b^1 \otimes \phi}H \) and \( H_{a \otimes b}H \) can be described as follows. We define measures \( v_{s, r}^2 \) on \( G_s \times G \) and \( v_{s, r}^2 \) on \( G_s \times G \) by

\[
\int_{G_s \times G} f \, dv_{s, r}^2 := \int_{G_s} \int_{G^0} f(x, y) \, d\lambda_x(x) \, d\lambda_y(x) \, d\mu(u),
\]

\[
\int_{G_s \times G} g \, dv_{s, r}^2 := \int_{G_s} \int_{G^0} g(x, y) \, d\lambda_x(x) \, d\lambda_y(x) \, d\mu(u)
\]

for all \( f \in C_c(G_s \times G), g \in C_c(G_s \times G). \) Routine calculations show:

**Lemma 2.6.** There exist unique isomorphisms

\[ \Phi_{\beta, \alpha}: H_{b^1 \otimes \phi}H \to L^2(G_s \times G, v_{s, r}^2), \quad \Phi_{\alpha, \beta}: H_{a \otimes b}H \to L^2(G_s \times G, v_{s, r}^2) \]

such that for all \( \eta, \xi \in C_c(G) \) and \( \zeta \in C_c(G^0) \),

\[
\Phi_{\beta, \alpha}(j(\eta) \otimes \zeta \otimes \xi) = \eta(x) D^{-1/2}(x) \xi(s(x)) \zeta(y),
\]

\[
\Phi_{\alpha, \beta}(j(\eta) \otimes \zeta \otimes \xi) = \eta(x) \zeta(r(x)) \xi(y).
\]

\( \square \)

From now on, we identify \( H_{\hat{\beta} \otimes \phi}H \) with \( L^2(G_s \times G) \) and \( H_{a \otimes b}H \) with \( L^2(G_s \times G) \) via \( \Phi_{\hat{\beta}, \alpha} \) and \( \Phi_{\alpha, \beta}, \) respectively, without further notice.

**Theorem 2.7.** There exists a \( C^*\)-pseudo-multiplicative unitary \((b, H, \hat{\beta}, \alpha, \beta, V)\) such that \( (V \omega)(x, y) = \omega(x, x^{-1}y) \) for all \( \omega \in C_c(G_s \times G) \) and \((x, y) \in G_s \times G). \)
Proof. Straightforward calculations show that \((H, \hat{\beta}, \alpha, \beta)\) is a \(C^*-\langle b^1, b, b^\dagger \rangle\)-module.

The homeomorphism \(G_x \times_r G \to G_x \times_r G, (x, y) \mapsto (x, x^{-1} y)\), induces an isomorphism \(V_0 : C_c(G_x \times_r G) \to C_c(G_x \times_r G)\) such that \((V_0 \omega)(x, y) = \omega(x, x^{-1} y)\) for all \(\omega \in C_c(G_x \times_r G)\) and \((x, y) \in G_x \times_r G\). Using left-invariance of \(\lambda\), one finds that \(V_0\) extends to a unitary \(V : H^0_b \otimes_a H \cong L^2(G_x \times_r G) \to L^2(G_x \times_r G) \cong H_a \otimes_b H\).

We claim that \(V\) is a \(C^*\)-pseudo-multiplicative unitary. First, we show that \(V(\hat{\beta} \triangleright \hat{\beta}) = \alpha \triangleright \hat{\beta}\). For each \(\xi, \xi' \in C_c(G), \xi \in C_c(G^0)\), and \((x, y) \in G_x \times_r G\),

\[
(V \hat{\beta} \triangleright \hat{\beta})(\xi)(x, y) = (\hat{\beta})(\xi) \bar{c}(x, x^{-1} y) = \xi(x) \xi'(x^{-1} y)D^{-1/2}(x)D^{-1/2}(1^{-1} y)\xi(y),
\]

\[
(\hat{\beta} \triangleright \hat{\beta})(\xi)(x, y) = \xi(x) \xi'(y)D^{-1/2}(1^{-1} y)\xi(y).
\]

Using standard approximation arguments and the fact that \(D(x)D(1^{-1} y) = D(y)\) for \(x, y\)-almost all \(x, y \in G_x \times_r G\) (see [13] or [24, p. 89]), we find that \(V(\hat{\beta} \triangleright \hat{\beta}) = [T(C_c(G_x \times_r G))] = \alpha \triangleright \hat{\beta}\), where for each \(\alpha \in C_c(G_x \times_r G)\),

\[
(T(\alpha))(x, y) = \omega(x, y)D^{-1/2}(1^{-1} y)\xi(y) \quad \text{for all } \xi \in C_c(G^0), (x, y) \in G_x \times_r G.
\]

Similar calculations show that the remaining relations in (7) hold.

Tedious but straightforward calculations show that diagram (8) commutes; see also [38]. Therefore, \(V\) is a \(C^*\)-pseudo-multiplicative unitary.

\[\square\]

3 Hopf \(C^*\)-bimodules and the legs of a \(C^*\)-pseudo-multiplicative unitary

To every regular multiplicative unitary \(V\) on a Hilbert space \(H\), Baaj and Skandalis associate two Hopf \(C^*\)-algebras \((\hat{A}_V, \Delta_V)\) and \((A_V, \Delta_V)\) as follows [3]. They show for every multiplicative unitary \(V\), the subspaces \(\hat{A}_V^0\) and \(A_V^0\) of \(L(\mathcal{L})\) defined by

\[
\hat{A}_V^0 := \{(id \otimes \omega)(V) \mid \omega \in \mathcal{L}(\mathcal{H})_+\}, \quad A_V^0 := \{(v \otimes id)(V) \mid v \in \mathcal{L}(\mathcal{H})_+\}
\]

are closed under multiplication. In the regular case, their norm closures \(\hat{A}_V\) and \(A_V\), respectively, are \(C^*\)-algebras, and the \(*\)-homomorphisms \(\Delta_V : \hat{A}_V \to \mathcal{L}(H \otimes H)\) and \(\Delta_V : A_V \to \mathcal{L}(H \otimes H)\) given by

\[
\hat{A}_V: \hat{\alpha} \mapsto V^*(1 \otimes \hat{\alpha}), \quad \Delta_V: a \mapsto V(a \otimes 1)V^*,
\]

map \(\hat{A}_V\) to \(M(\hat{A}_V \otimes \hat{A}_V) \subseteq \mathcal{L}(H \otimes H)\) and \(A_V\) to \(M(A_V \otimes A_V) \subseteq \mathcal{L}(H \otimes H)\), respectively, and form comultiplications on \(A_V\) and \(\hat{A}_V\). Finally, there exists a perfect pairing

\[
\hat{A}_V^0 \times A_V^0 \to \mathbb{C}, \quad (\omega \otimes \omega)(V), (\varphi \otimes \varphi)(V) \to \langle \omega \otimes \omega, V \rangle,
\]

which expresses the duality between \((\hat{A}_V, \Delta_V)\) and \((A_V, \Delta_V)\).
3.1 The fiber product of \( C \)-motivation and details, we refer to [31]; two examples can be found in subsection 3.5.

The fiber product is also functorial with respect to these generalized morphisms. For additional details and context, see [31].

In this subsection, we recall the fiber product of \( C \)-sites are then used to define Hopf \( C \)-bases and systematically study slice maps and related constructions. These prerequisites are then used to define Hopf \( C \)-bimodules and associated convolution algebras. Finally, we adapt the constructions of Baaj and Skandalis to \( C \)-pseudo-multiplicative unitaries and apply them to the unitary associated to a groupoid.

Throughout this section, let \( b = (f, \mathcal{B}, \mathfrak{B}^1) \) be a \( C^* \)-base, \( (H, \hat{\beta}, \alpha, \beta) \) a \( C^* \)-\((b^1, b, \mathfrak{B}^1)\)-module and \( V : H_{\hat{\beta}} \otimes_{\hat{\alpha}} H_{\hat{\beta}} \otimes_{\hat{\beta}} H \) a \( C^* \)-pseudo-multiplicative unitary.

3.1 The fiber product of \( C^* \)-algebras over \( C^* \)-bases

In this subsection, we recall the fiber product of \( C^* \)-algebras over \( C^* \)-bases [31], introduce several new notions of a morphism of such \( C^* \)-algebras, and show that the fiber product is also functorial with respect to these generalized morphisms. For additional motivation and details, we refer to [31]; two examples can be found in subsection 3.5.
Let \( b_1, \ldots, b_n \) be \( C^* \)-bases, where \( b_i = (\mathcal{R}_i, \mathcal{B}_i, \mathcal{B}'_i) \) for each \( i \). A (nondegenerate) \( C^* \)-algebra consists of a \( C^* \)-module \( \langle H, \alpha_1, \ldots, \alpha_n \rangle \) and a (nondegenerate) \( C^* \)-algebra \( A \subseteq \mathcal{L}(H) \) such that \( \rho_a(\mathcal{B}_i) A \) is contained in \( A \) for each \( i \). We shall only be interested in the cases \( n = 1, 2 \), where we abbreviate \( A^0_H := (H, \alpha, A) \), \( A^0_{\alpha, \beta} := (\alpha H\beta, A) \).

We need several natural notions of a morphism. Let \( A = (\mathcal{H}, A) \) and \( C = (\mathcal{K}, C) \) be \( C^* \)-algebras, where \( \mathcal{H} = (H, \alpha_1, \ldots, \alpha_n) \) and \( \mathcal{K} = (K, \gamma_1, \ldots, \gamma_m) \). A *-homomorphism \( \pi: A \rightarrow C \) is called a (semi-)normal morphism or jointly (semi-)normal morphism from \( A \) to \( C \) if \( \mathcal{L}_\pi^\mathcal{H}(H, \mathcal{K}) \alpha_i = \gamma_i \) or \( \mathcal{L}_\pi^\mathcal{K}(\mathcal{H}, \mathcal{K}) \alpha_i = \gamma_i \), respectively, for each \( i \), where

\[
\mathcal{L}_\pi^\mathcal{H}(H, \mathcal{K}) = \mathcal{L}(H, K) \cap \mathcal{L}\mathcal{L}_\pi^\mathcal{K}(H, \mathcal{K}) \quad \text{and} \quad \mathcal{L}_\pi^\mathcal{K}(\mathcal{H}, \mathcal{K}) = \mathcal{L}(H, K) \cap \mathcal{L}\mathcal{L}_\pi^\mathcal{K}(\mathcal{H}, \mathcal{K})
\]

One easily verifies that every semi-normal morphism \( \pi \) between \( C^* \)-algebras \( A^0_H \) and \( C^0_K \) satisfies \( \pi(\rho_a(b^i)) = \rho_a(\pi(b^i)) \) for all \( b^i \in \mathcal{B}^i \).

We construct a fiber product of \( C^* \)-algebras over \( C^* \)-bases as follows. Let \( b \) be a \( C^* \)-base, \( A^0_H \) a \( C^* \)-algebra, and \( B^0_K \) a \( C^* \)-base. A fiber product of \( A_H^0 \) and \( B_K^0 \) is the \( C^* \)-algebra

\[
A^0_{\alpha, \beta} \ast B^0_{\gamma, \delta} := \{ x \in \mathcal{L}(H \circledast_b K) \mid x[B]_1, x^* [B]_1 \subseteq \zeta \}_{\mathcal{L}(K, H \circledast_b K)}, \quad \zeta \mathcal{L}(K, H \circledast_b K) \}
\]

If \( A \) and \( B \) are unital, so is \( A^0_{\alpha, \beta} \ast B^0_{\gamma, \delta} \), but otherwise, \( A^0_{\alpha, \beta} \ast B^0_{\gamma, \delta} \) may be degenerate. Clearly, conjugation by the flip \( \Sigma: H \circledast_b K \rightarrow K \circledast_b H \) yields an isomorphism

\[
\text{Ad}_\Sigma: A^0_{\alpha, \beta} \ast B^0_{\gamma, \delta} \rightarrow B^0_{\gamma, \delta} \ast A^0_{\alpha, \beta}
\]

If \( a, c \) are \( C^* \)-bases, \( A^0_{\alpha, \beta} \) a \( C^* \)-algebra and \( B^0_{\gamma, \delta} \) a \( C^* \)-algebra, then

\[
A^0_{\alpha, \beta} \ast B^0_{\gamma, \delta} = (a H \circledast_b K, \alpha \ast \beta, B)
\]

is a \( C^* \)-algebra, called the fiber product of \( A^0_{\alpha, \beta} \) and \( B^0_{\gamma, \delta} \).

The fiber product construction is functorial with respect to normal morphisms [31, Theorem 3.23], but also with respect to (jointly) semi-normal morphisms. For the proof, we slightly modify [31, Lemma 3.22].

**Lemma 3.1.** Let \( \pi \) be a semi-normal morphism of \( C^* \)-algebras \( A^0_H \) and \( C^0_K \), let \( B^0_K \) be a \( C^* \)-algebra, and let \( I := \mathcal{L}(H, L) \circledast_b \text{id} \).

i) \( II' I' \subseteq I \) and there exists a unique *-homomorphism \( \rho_I: (I')' \rightarrow (I')' \) such that \( \rho_I(x) y = xy \) for all \( x \in (I')' \) and \( y \in I \).

ii) There exists a linear contraction \( j_x \) from the subspace \( [\gamma_2 A] \subseteq \mathcal{L}(H, H \circledast_b K) \to [\gamma_2 C] \subseteq \mathcal{L}(L, L \circledast_b K) \) given by \( x \pi(\alpha) \to \eta_2 \alpha \pi(\alpha) \).
Second, \( \rho_j(A_b \times_B B) \subseteq C_{a^\lambda b} \) and \( \rho_j(x)[\eta]_2 = j_a(x)[\eta]_2 \) for all \( x \in A_b \times_B B, \eta \in \gamma \).

\textbf{Proof.} i) The assertion on \( I \) is evident and the assertion on \( \rho_j \) is [31, Proposition 2.2].

ii) The existence of \( j_a \) follows from the fact that we have \( (\rho_j(\eta)[\pi(a)])^* (\rho_j(\eta)[\pi(d)]) = \pi(a)^* \rho_j(\eta)[\pi(d' \pi(a))] \) for all \( \eta, a, d' \in A \).

iii) Let \( J := \mathcal{L}^\gamma(\mathcal{H}_b, L_\lambda) \). First, we have \( \rho_j(A_b \times_B B)[\gamma]_2 \subseteq \lbrack [\gamma]_2 C \rbrack \) because \( \rho_j(x)[\eta]_2 = j_a(x)[\eta]_2 \) for all \( x \in A_b \times_B B, \eta \in \gamma \). Indeed, for all \( S \in J \),

\[
\rho_j(x)[\eta]_2 S = \rho_j(x)(S \otimes \text{id})[\eta]_2 = (S \otimes \text{id})x[\eta]_2 = j_a(x)[\eta]_2 S.
\]

\textbf{Theorem 3.2.} Let \( a, b, c \) be \( C^* \)-bases, \( \Phi \) a semi-normal morphism of \( C^* \cdot (a, b) \)-algebras \( \mathcal{A} = A_{bL}^a \) and \( C = C_{cL}^a \), and \( \Psi \) a semi-normal morphism of \( C^* \cdot (b^*, c) \)-algebras \( \mathcal{B} = B_{cR}^{b^*} \) and \( D = D_{bR}^{c^*} \). Then there exists a unique semi-normal morphism of \( C^* \cdot (a, b, c) \)-algebras \( \Phi \cdot \Psi : \mathcal{A} \cdot \mathcal{B} \to \mathcal{C} \cdot \mathcal{D} \) such that

\[
(\Phi \cdot \Psi)(x)R = Rx \quad \text{for all } x \in A_b \times_B B \text{ and } R \in I_MH + J_IK,
\]

where \( I_K = \mathcal{L}(H, L) \otimes 1_b \) and \( J_F = 1_b \otimes \mathcal{L}(M, K) \) for \( X \in \{K, M\}, Y \in \{H, L\} \). If both \( \Phi \) and \( \Psi \) are normal, jointly semi-normal or jointly normal, then also \( \Phi \cdot \Psi \) is normal, jointly semi-normal or jointly normal, respectively.

\textbf{Proof.} This follows from Lemma 3.1 and a similar argument as in the proof of [31, Theorem 3.13].

Unfortunately, the fiber product need not be associative, but in our applications, it will only appear as the target of a comultiplication whose coassociativity will compensate the non-associativity of the fiber product.

### 3.2 Spaces of maps on \( C^* \)-algebras over \( C^* \)-bases

To define convolution algebras of Hopf \( C^* \)-bimodules and generalized Fourier algebras of \( C^* \)-pseudo-multiplicative unitaries, we need to consider several spaces of maps on \( C^* \)-algebras over \( C^* \)-bases.

Let \( a = (\mathcal{A}, \mathcal{X}_\lambda) \) and \( b = (\mathcal{B}, \mathcal{X}_\lambda) \) be \( C^* \)-bases, \( H \) a Hilbert space, \( H_a \) a \( C^* \)-module, \( H_b \) a \( C^* \)-b-module, and \( A \subseteq \mathcal{L}(H) \) a \( C^* \)-algebra. We denote by \( \alpha_a^\beta \) the space of all sequences \( \eta = (\eta_k)_{k \in \mathbb{N}} \) in \( \alpha \) for which the sum \( \sum_k \eta_k^* \eta_k \) converges in norm, and put \( \|\eta\| := \|\sum_k \eta_k^* \eta_k\|^{1/2} \) for each \( \eta \in \alpha_a^\beta \). Similarly, we define \( \beta_{\alpha_a^\beta} \). Then standard arguments show that for all \( \eta \in \beta_{\alpha_a^\beta}, \eta' \in \beta_{\alpha_a^\beta} \), there exists a bounded linear map

\[
\omega_{\eta, \eta'} : A \to \mathcal{L}(\mathcal{A}, \mathcal{B}), \quad T \mapsto \sum_{k \in \mathbb{N}} \eta_k^* T \eta_k,
\]
where the sum converges in norm and \( \|\omega_{\eta,\eta'}\| \leq \|\eta\|\|\eta'\| \). We put

\[
\Omega_{\beta,\alpha}(A) := \{ \omega_{\eta,\eta'} \mid \eta \in \beta^o, \eta' \in \alpha^o \} \subseteq L(A, L(\bar{S}, R)),
\]

where \( L(A, L(\bar{S}, R)) \) denotes the space of bounded linear maps from \( A \) to \( L(\bar{S}, R) \). If \( \beta = \alpha \), we abbreviate \( \Omega_{\beta}(A) := \Omega_{\beta,\alpha}(A) \). It is easy to see that \( \Omega_{\beta,\alpha}(A) \) is a subspace of \( L(A, L(\bar{S}, R)) \) and that the following formula defines a norm on \( \Omega_{\beta,\alpha}(A) \):

\[
\|\omega\| := \inf \{ \|\eta\|\|\eta'\| \mid \eta \in \beta^o, \eta' \in \alpha^o, \omega = \omega_{\eta,\eta'} \} \text{ for all } \omega \in \Omega_{\beta,\alpha}(A).
\]

**Lemma 3.3.** \( \Omega_{\beta,\alpha}(A) \) is a Banach space.

**Proof.** Let \( (\omega^k)_k \) be a sequence in \( \Omega_{\beta,\alpha}(A) \) such that \( \|\omega^k\| \leq 4^{-k} \) for all \( k \in \mathbb{N} \). We show that the sum \( \sum_k \omega^k \) converges in norm in \( \Omega_{\beta,\alpha}(A) \). For each \( k \in \mathbb{N} \), we can choose \( \eta^k \in \beta^o \) and \( \eta'^k \in \alpha^o \) such that \( \omega^k = \omega_{\eta^k,\eta'^k} \) and \( \|\eta^k\|\|\eta'^k\| \leq 4^{-k} \). Without loss of generality, we may assume \( \|\eta^k\| \leq 2^{-k} \) and \( \|\eta'^k\| \leq 2^{-k} \). Choose a bijection \( i: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) and let \( \eta_{i(k,n)} = \eta^k \) and \( \eta'_{i(k,n)} = \eta'^k \) for all \( k, n \in \mathbb{N} \). Routine calculations show that \( \eta \in \beta^o, \eta' \in \alpha^o \), and that the sum \( \sum \omega^k \) converges in norm to \( \omega_{\eta,\eta'} \in \Omega_{\beta,\alpha}(A) \).

We have the following straightforward result:

**Proposition 3.4.** There exists a linear isometry \( \Omega_{\beta,\alpha}(A) \to \Omega_{\alpha,\beta}(A) \), \( \omega \mapsto \omega^* \), such that \( \omega^*(a) = \omega(a^*)^* \) for all \( a \in A \) and \( (\omega_{\eta,\eta'})^* = \omega_{\eta',\eta} \) for all \( \eta \in \beta^o, \eta' \in \alpha^o \).

We can pull back maps of the form considered above via morphisms as follows:

**Proposition 3.5.**

1. Let \( \pi \) be a normal morphism of \( C^*\)-b-algebras \( A^\alpha_H \) and \( B^\beta_K \). Then there exists a linear contraction \( \pi^\alpha: \Omega_\pi(B) \to \Omega_\alpha(A) \) given by \( \omega \mapsto \omega \circ \pi \).

2. Let \( \pi \) be a jointly normal morphism of \( C^*\)-b-algebras \( A_H^\alpha \) and \( B_K^\beta \). Then there exists a linear contraction \( \pi^\prime: \Omega_{\pi,\beta}(B) \to \Omega_{\beta,\alpha}(A) \) given by \( \omega \mapsto \omega \circ \pi \).

**Proof.** We only prove ii), the proof of i) is similar. Let \( I := L(\pi(H), K) \) and \( \eta \in \delta^o \). \( \eta' \in \gamma^p \). Then there exists a closed separable subspace \( I_0 \subseteq I \) such that \( \eta_n \in [I_0] \) and \( \eta'_n \in [I_0] \) for all \( n \in \mathbb{N} \). We may also assume that \( I_0H_n \subseteq I_0 \), and then \( [I_0] \) is a \( \sigma \)-unital \( C^* \)-algebra and has a bounded sequential approximate unit \( \{u_k\}_k \) of the form \( u_k = \sum_{i \in I} T_i^* T_i \), where \( \{T_i\}_i \) is a sequence in \( \mathcal{H} \) [17, Proposition 6.7]. We choose a bijection \( i: \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) and let \( \xi_{i(n,m)} := T_i^* \eta_n \in \beta \) and \( \xi_{i(n,m)} := T_i^* \eta_n \in \alpha \) for all \( i, n, m \in \mathbb{N} \).

Then the sum \( \sum_i \xi_{i(n,m)} \xi_{i(n,m)}^* \) converges to \( \eta_n^* \eta_n \) for each \( n \in \mathbb{N} \) in norm because \( \eta_n \in [I_0] \). Therefore, \( \xi \in \beta^o \) and \( \|\xi\| = \|\eta\| \), and a similar argument shows that \( \xi^* \in \alpha^o \) and \( \|\xi^*\| = \|\eta\| \). Finally,

\[
\omega_{\eta,\eta'}(a) = \sum_{i,n} \eta_n^* a T_i T_i^* \eta_n' = \sum_{i,n} \eta_n^* \pi(a) T_i T_i^* \eta_n' = \sum_n \eta_n^* \pi(a) \eta_n = \omega_{\eta,\eta'}(\pi(a))
\]

for each \( a \in A \), where the sum converges in norm, and hence \( \omega_{\eta,\eta'} \circ \pi = \omega_{\eta',\eta} \in \Omega_{\beta,\alpha}(A) \) and \( \|\omega_{\eta,\eta'} \circ \pi\| \leq \|\xi\|\|\xi^*\| = \|\eta\|\|\eta\| \).

For each map of the form considered above, we can form a slice map as follows.
Proposition 3.6. Let $A_H^b$ be a $C^*\text{-}b$-algebra and $B_K^{\delta}$ a $C^*\text{-}b^1$-algebra.

i) There exist linear contractions

$$\Omega_\phi(A) \to \Omega_\phi(B), \quad \phi \mapsto \phi \ast \id,$$

such that for all $\xi, \xi' \in \beta^\omega$ and $\eta, \eta' \in \gamma^\omega$,

$$\omega_{\xi, \xi'} \ast \id = \omega_{\xi_{\beta^\omega}, \xi'}, \quad \text{where } \xi_{\beta^\omega} = [\xi_n]_1, \quad \xi'_n = [\xi'_n]_1 \text{ for all } n \in \mathbb{N},$$

$$\id \ast \omega_{\eta, \eta'} = \omega_{\eta_{\gamma^\omega}, \eta'}, \quad \text{where } \eta_{\gamma^\omega} = [\eta_n]_2, \eta'_n = [\eta'_n]_2 \text{ for all } n \in \mathbb{N}.$$  

ii) We have $\psi \circ (\phi \ast \id) = \phi \circ (\id \ast \psi)$ for all $\phi \in \Omega_\phi(A)$ and $\psi \in \Omega_\psi(B).$

Proof. i), ii) Straightforward; see [31, Proposition 3.30].

Finally, we need to consider fiber products of the linear maps considered above. We denote by “$\otimes$” the projective tensor product of Banach spaces.

Proposition 3.7. Let $A^{a,b}$ be a $C^*\text{-}(a, b)$-algebra and $B^{\delta}_K$ a $C^*\text{-}(b^1, c)$-algebra.

i) There exist linear contractions

$$\Omega_\phi(A) \otimes \Omega_\psi(B) \to \Omega_{(\omega \bowtie \phi)}(A_b^{*\ast}B), \quad \omega \otimes \omega' \mapsto \omega \bowtie \omega' := \omega \circ (\id \ast \omega'),$$

$$\Omega_\phi(A) \otimes \Omega_\psi(B) \to \Omega_{(\eta \bowtie \phi)}(A_b^{*\ast}B), \quad \omega \otimes \omega' \mapsto \omega \bowtie \omega' := \omega' \circ (\id \ast \omega).$$

ii) There exist linear contractions

$$\Omega_{(\alpha \bowtie \beta)}(A) \otimes \Omega_{(\gamma \bowtie \delta)}(B) \to \Omega_{(\alpha \bowtie \gamma \bowtie \delta)}(A_b^{*\ast}B), \quad \omega \otimes \omega' \mapsto \omega \bowtie \omega',$$

such that for all $\xi \in \alpha^\omega, \xi' \in \beta^\omega$, $\eta \in \gamma^\omega, \eta' \in \delta^\omega$ and each bijection $i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we have $\omega_{\xi, \xi'} \bowtie \omega_{\eta, \eta'} = \omega_{\xi, \xi'} \bowtie \omega_{\eta, \eta'}$, and $\omega_{\xi, \xi'} \bowtie \omega_{\eta, \eta'} = \omega_{\xi, \xi'}$, where

$$\theta_{(i, n, m)} = [\eta_{\gamma^\omega}]_2 \xi_{\beta^\omega} \in \alpha \bowtie \gamma, \quad \theta_{(i, n, m)} = [\xi_{\beta^\omega}]_2 \eta_{\gamma^\omega} \in \beta \bowtie \delta \quad \text{ for all } n, m \in \mathbb{N}.$$

Proof. The proof of assertion i) is straightforward; we only prove the existence of the first contraction in ii). Let $\xi, \xi', \eta, \eta', \iota, \theta, \theta'$ be as above. Then $\theta \in (\alpha \bowtie \gamma)^\omega$ and $\|\theta\| \leq \|\xi\||\eta\|$. Because

$$\sum_k \theta_k \theta_k = \sum_{m, n} \xi_{\beta^\omega} \eta_{\gamma^\omega} \xi_{\beta^\omega} \eta_{\gamma^\omega} \leq \sum_{m, n} \xi_{\beta^\omega} \eta_{\gamma^\omega} \xi_{\beta^\omega} \eta_{\gamma^\omega} = \|\eta\|^2 \sum_{m, n} \xi_{\beta^\omega} \eta_{\gamma^\omega} \leq \|\xi\|^2 \|\eta\|^2,$$

and similarly $\theta' \in (\beta \bowtie \delta)^\omega$ and $\|\theta'\| \leq \|\xi'\||\eta'\|$. Next, we show that $\omega_{\xi, \xi'} \bowtie \omega_{\eta, \eta'}$, does not depend on $\xi$ and $\xi'$ but only on $\omega_{\xi, \xi'} \bowtie \omega_{\eta, \eta'}$. Let $\xi' \in \mathbb{R}$ and $x \in A_b^{*\ast}B$. Then

$$\omega_{\xi, \xi'}(x) = \sum_{m, n} \xi_{\beta^\omega} \eta_{\gamma^\omega} \xi_{\beta^\omega} \eta_{\gamma^\omega} \xi_{\beta^\omega} \eta_{\gamma^\omega},$$
where the sum converges in norm. Fix any \( n \in \mathbb{N} \). Then we find a sequence \( (k_r) \) in \( \mathbb{N} \) and \( \eta_{r1}'' \cdots, \eta_{rk_r}'' \in \mathbb{R} \) such that the sum \( \sum_{j=1}^{k_r} \eta_{rj}'' \xi_{rj}' \) converges in norm to \( \eta' \) as \( r \) tends to infinity. But then

\[
\sum_m \xi_m \langle \eta_n | x | \xi_m \rangle \eta_n' \xi' = \lim_{r \to \infty} \sum_{m=1}^{k_r} \xi_m \langle \eta_n | x | \xi_m \rangle \eta_n' \xi' = \lim_{r \to \infty} \sum_{m=1}^{k_r} \xi_m \langle \eta_n | x | \xi_m \rangle \eta_n' \xi'.
\]

Note here that \( \langle \eta_n | x | \eta_n' \rangle \xi' \in \mathbb{R} \). Therefore, the sum on the left hand side only depends on \( \omega_{x, r} \in \Omega_{x, b}(A) \) but not on \( \xi, \xi' \), and since \( n \in \mathbb{N} \) was arbitrary, the same is true for \( \omega_{x, r}(x) \). A similar argument shows that \( \omega_{x, r}(x) \) depends on \( \omega_{x, r} \in \Omega_{x, b}(B) \) but not on \( \eta, \eta' \) for each \( \xi \in \mathbb{R} \).

3.3 Concrete Hopf C\(^*\)-bimodules and their convolution algebras

The fiber product construction leads to the following generalization of a Hopf C\(^*\)-algebra and of related concepts.

**Definition 3.8.** Let \( b = (\mathfrak{B}, \mathcal{S}, \mathcal{B}^1) \) be a C\(^*\)-base. A comultiplication on a C\(^*\)-(\( b^1 \), \( b \))-algebra \( A_{\beta, \alpha} \) is a jointly semi-normal morphism \( \Delta \) from \( A_{\beta, \alpha} \) to \( A_{\beta, \alpha} \otimes_{b} A_{\beta, \alpha} \) that is coassociative in the sense that the following diagram commutes:

\[
\begin{array}{c}
A \\
\downarrow \Delta \\
A_{\beta, \alpha} \otimes_{b} A_{\beta, \alpha} \\
\downarrow \Delta \otimes \text{id} \\
A_{\beta, \alpha} \otimes_{b} A_{\beta, \alpha} \\
\end{array}
\]

\[
\begin{array}{c}
\text{id}_{\beta, \Delta} \\
(A_{\beta, \alpha} \otimes_{b} A_{\beta, \alpha})_{(\alpha \circ \Delta)} \\
L(H_{\alpha} \otimes_{b} H_{\alpha} \otimes_{b} H). \\
\end{array}
\]

A (semi-)normal Hopf C\(^*\)-bimodule over \( b \) is a C\(^*\)-(\( b^1 \), \( b \))-algebra with a jointly (semi-)normal comultiplication. When we speak of a Hopf C\(^*\)-bimodule, we always mean a semi-normal one. A morphism of (semi-)normal Hopf C\(^*\)-bimodules \( (A_{\beta, \alpha}^1, \Delta^1) \) to \( (B_{\gamma, \delta}^1, \Delta^1) \) over \( b \) is a jointly (semi-)normal morphism \( \pi \) from \( A_{\beta, \alpha}^1 \) to \( B_{\gamma, \delta}^1 \) satisfying \( \Delta^1 \circ \pi = (\pi \circ \pi) \circ \Delta_{\beta, \alpha} \).

Let \( (A_{\beta, \alpha}^1, \Delta) \) be a Hopf C\(^*\)-bimodule over \( b \). A bounded left Haar weight for \( (A_{\beta, \alpha}^1, \Delta) \) is a completely positive contraction \( \phi : A \to \mathfrak{B} \) satisfying

\[
\phi(ab \phi(b)) = \phi(a)b, \quad \phi((\xi_1 | A(a) | \xi_1')_1) = \xi_1 \phi(\phi(a)) \xi_1'.
\]

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for all \( a \in A, b \in \mathcal{B} \). We call \( \phi \) normal if \( \phi \in \Omega_{\mathcal{M}(\beta)}(A) \), where \( \mathcal{M}(\beta) = \{ T \in L(\mathcal{B}, H) \mid T \mathcal{B}^1 \subseteq \beta, T \beta \subseteq \mathcal{B}^1 \} \). Similarly, a bounded right Haar weight for \( A^b_\alpha(\Delta) \) is a completely positive contraction \( \psi: A \to \mathcal{B}^1 \) satisfying
\[
\psi(a \rho_\alpha(b^1)) = \psi(a)b^1, \quad \psi((\eta_2 \Delta(\alpha))\eta'_2) = \eta_1^* \rho_\alpha(\psi(\alpha)) \eta'_1
\]
for all \( a \in A, b^1 \in \mathcal{B}^1 \), \( \eta, \eta' \in \mathcal{B} \). We call \( \psi \) normal if \( \psi \in \Omega_{\mathcal{M}(\alpha)}(A) \), where \( \mathcal{M}(\alpha) = \{ S \in L(\mathcal{B}, H) \mid S \mathcal{B} \subseteq \alpha, S^\beta \alpha \subseteq \mathcal{B} \} \).

A bounded (left/right) counit for \( A^b_\alpha(\Delta) \) is a jointly semi-normal morphism of \( C^\ast \)-\( (b^1, b) \)-algebras \( \mathcal{V}: A^b_\alpha \to \mathcal{L}(\mathcal{B})^{\mathcal{B}^1, \mathcal{B}^1} \beta \) that makes the (left/right one of the) following two diagrams commute:

\[
\begin{align*}
A^b_\alpha \times \mathcal{B} & \ar[r]^\Delta & A \ar[d] \quad \quad A \ar[d] \ar[r]^\Delta & A^b_\alpha \times \mathcal{B} \\
L(\mathcal{B})^{\mathcal{B}^1, \mathcal{B}^1} \beta & \ar[r]^{\mathcal{V} \circ \id^b} & L(\mathcal{B}, \mathcal{B}) & \ar[r]^{\mathcal{H}(\mathcal{B})} & L(\mathcal{B}) & \ar[r]^{\mathcal{H}(\mathcal{B}, \mathcal{B})} & L(\mathcal{B})^{\mathcal{B}^1, \mathcal{B}^1}
\end{align*}
\]

(14)

**Remark 3.9.** Let \( A^b_\alpha(\Delta) \) be a Hopf \( C^\ast \)-bimodule over \( b \). Evidently, a completely positive contraction \( \phi: A \to \mathcal{B} \) is a normal bounded left Haar weight for \( A^b_\alpha(\Delta) \) if and only if \( \phi \in \Omega_{\mathcal{M}(\beta)}(A) \) and \( (\id^b \phi) \circ \Delta = \rho_\beta \circ \phi \). A similar remark applies to normal bounded right Haar weights.

Let \( A^b_\alpha(\Delta) \) be a normal Hopf \( C^\ast \)-bimodule over \( b \). Combining Propositions 3.5 and 3.7, we obtain for each of the spaces \( \Omega = \Omega_{\alpha}(A), \Omega_\beta(A), \Omega_{\alpha,\beta}(A), \Omega_{\beta,\alpha}(A) \) a map
\[
\Omega \times \Omega \to \Omega, \quad (\omega, \omega') \mapsto \omega \ast \omega' := (\omega \otimes \omega') \circ \Delta. \quad (15)
\]

**Theorem 3.10.** Let \( A^b_\alpha(\Delta) \) be a normal Hopf \( C^\ast \)-bimodule over \( b \). Then \( \Omega_{\alpha}(A), \Omega_\beta(A), \Omega_{\alpha,\beta}(A), \Omega_{\beta,\alpha}(A) \) are Banach algebras with respect to the multiplication (15).

**Proof.** It only remains to show that the multiplication is associative, but this follows from the coassociativity of \( \Delta \). \( \square \)

### 3.4 The legs of a \( C^\ast \)-pseudo-multiplicative unitary

Let \( b = (\mathcal{B}, \mathcal{B}^1) \) be a \( C^\ast \)-base, \((H, \beta, \alpha, \bar{\beta})\) a \( C^\ast \)-\( (b^1, b) \)-module and \( V: H_\beta \otimes_{b^1} H \to H_\alpha \otimes_{b^1} H \) a \( C^\ast \)-pseudo-multiplicative unitary. We associate to \( V \) two algebras and, if \( V \) is well behaved, two Hopf \( C^\ast \)-bimodules as follows. Let
\[
\begin{align*}
\tilde{A}_V := [\beta^2 \alpha_2] \subseteq L(H), \\
A_V := [\alpha_1 \beta^1] \subseteq L(H),
\end{align*}
\]
(16)

where \( [\alpha^2_2, \beta] \subseteq L(H, H_\beta \otimes_{b^1} H) \) and \( [\beta^2, \alpha_1] \subseteq L(H_\alpha \otimes_{b^1} H, H) \) are defined as in Subsection 2.1.
Proposition 3.11. The following relations hold:

\[ \hat{A}_{V^\beta} = A_V^\beta, \quad [\hat{A}_V A_V] = \hat{A}_V, \quad [\hat{A}_V H] = H = [\hat{A}_V \beta] = \hat{A}_V \beta, \]

\[ [\hat{A}_V \rho(\beta)] = [\rho(\beta) \hat{A}_V] = \hat{A}_V = [\hat{A}_V \rho(\beta)] = [\rho(\beta) \hat{A}_V]. \]

\[ A_{V^\beta} = \hat{A}_V, \quad [A_V A_V] = A_V, \quad [A_V H] = H = [A_V \beta] = \hat{A}_V \beta, \]

\[ [A_V \rho(\beta)] = [\rho(\beta) A_V] = A_V = [A_V \rho(\beta)] = [\rho(\beta) A_V]. \]

Proof. First, we have \( \hat{A}_{V^\beta} = \hat{A} \) is a normal Hopf \( C^* \)-algebra (\( \rho(\beta) \hat{A}_V \)) and \( \rho(\beta) H \) because \( V(\beta \rho(\beta)) = \rho(\beta) \hat{A}_V = \hat{A}_V \). Moreover, \( \hat{A}_{V^\beta} = A_{\hat{A}_V} = \hat{A}_V \). \( \Box \)

Consider the \( \ast \)-homomorphisms

\[ \hat{A}_V : \rho(\beta) \hat{A}_V \to L(H_{\beta^\ast \alpha} H), \quad y \mapsto V^\ast (\text{id} \otimes y) V, \]

\[ \Delta_V : \rho(\beta) \hat{A}_V \to L(H_{\beta^\ast \alpha} H), \quad z \mapsto V(z \otimes \text{id}) V^\ast. \]

Proposition 3.12. \( \hat{A}_V \) is a jointly normal morphism of \( C^* \)-\( (b, b^\dagger) \)-algebras \( (\rho(\beta) \hat{A}_V)_{\alpha, \beta}^a \) and \( ((\rho(\beta) \hat{A}_V)_{\alpha, b^\dagger}^a)_{\beta^a}^a \) and \( \Delta_V \) is a jointly normal morphism of \( C^* \)-\( (b, b^\dagger) \)-algebras \( (\rho(\beta) \hat{A}_V)_{\beta^a}^a \) and \( ((\rho(\beta) \hat{A}_V)_{\alpha, b^\dagger}^a)_{\beta^a}^a \). Moreover, \( \hat{A}_{V^\beta} = A_{\Delta_V} = A_{\hat{A}_V}. \)

Proof. We only prove the assertions concerning \( \hat{A}_V \). The relation \( A_{V^\beta} = A_{\Delta_V} \) is easily verified. Next, \( \Delta_V (\rho(\beta) \hat{A}_V) \subseteq (\rho(\beta) \hat{A}_V) \) because \( V(\alpha) = \rho(\beta) \hat{A}_V = \rho(\beta) \hat{A}_V \). To see that \( \hat{A}_V \) is a jointly normal morphism, note that \( V^\ast (\alpha) \subseteq L(H_{\beta^\ast \alpha} H) \) because \( \hat{A}_V \) is a normal Hopf \( C^* \)-bimodule. A sufficient condition, regularity, will be given in subsection 5.1. Coassociativity of \( \Delta_V \) and \( \Delta_V \) follows easily from the commutativity of diagram (8):

Lemma 3.13. If \( \hat{B} \subseteq \rho(\beta) \hat{A}_V \) is a \( C^* \)-algebra, \( \rho(\beta) \hat{A}_V \) and \( \hat{B} \) is a normal Hopf \( C^* \)-bimodule over \( b^\dagger \). Similarly, if \( \rho(\beta) \hat{A}_V \) is a \( C^* \)-algebra, \( \rho(\beta) \hat{A}_V \) and \( \hat{B} \) is a normal Hopf \( C^* \)-bimodule over \( b^\dagger \).
Theorem 3.14. Proof. We only prove the assertion concerning $\hat{B}$; the assertion concerning $B$ follows similarly. Let $B \subseteq p_\beta(\mathcal{B})'$ be a $C^*$-algebra satisfying the assumptions and put $\Delta := \Delta_V$.

By Proposition 3.12, we only need to show that $(\hat{\Delta} + \text{id})(\hat{\Delta}(\hat{b})) = (\text{id} + \hat{\Delta})(\hat{\Delta}(\hat{b}))$ for all $\hat{b} \in \hat{B}$. But this is shown by the following commutative diagram:

Using the maps introduced in subsection 3.2, we construct convolution algebras $\hat{\Omega}_{\beta,\alpha}$ and $\hat{\Omega}_{\alpha,\beta}$ with homomorphisms onto dense subalgebras $A^0_V \subseteq A_V$ and $A^0_V \subseteq A_V$, respectively, as follows. Let

$$\hat{\Omega}_{\beta,\alpha} := \Omega_{\beta,\alpha}(p_\beta(\mathcal{B})'), \quad \hat{\Omega}_{\alpha,\beta} := \Omega_{\alpha,\beta}(p_\beta(\mathcal{B})').$$

Theorem 3.14. i) There exist linear contractions

$$\hat{\Omega}_{\beta,\alpha} : \hat{\Omega}_{\beta,\alpha} \to \Omega_{\beta,\alpha}(p_\beta(\mathcal{B})_\alpha \otimes p_\beta(\mathcal{B})_\beta)'$$

$$\hat{\Omega}_{\alpha,\beta} : \hat{\Omega}_{\alpha,\beta} \to \Omega_{\alpha,\beta}(p_\beta(\mathcal{B})_\beta \otimes p_\beta(\mathcal{B})_\alpha)'$$

such that for all $\xi, \xi' \in \beta^\infty$, $\eta, \eta' \in \alpha^\infty$, $\xi, \xi' \in \hat{\beta}^\infty$ and each bijection $i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we have $\omega_{\xi,\eta} \hat{\omega}_{\xi',\eta'} = \omega_{\xi',\eta'} \omega_{\eta,\eta'}$ and $\omega_{\xi,\eta} \omega_{\xi',\eta'} = \omega_{\eta,\eta'}$, where for all $m, n \in \mathbb{N}$,

$$\theta_{(m,n)} = |\xi'|/2 \xi_m \in \beta \circ \beta, \quad \theta_{(m,n)} = |\eta|/2 \eta_n \in \alpha \circ \alpha,$$

$$\kappa_{(m,n)} = |\xi'|/2 \xi_m \in \alpha \circ \alpha, \quad \kappa_{(m,n)} = |\eta|/2 \eta_n \in \beta \circ \beta.$$

ii) The Banach spaces $\hat{\Omega}_{\beta,\alpha}$ and $\hat{\Omega}_{\alpha,\beta}$ carry the structure of Banach algebras, where the multiplication is given by $\omega \ast \omega' = (\omega \otimes \omega') \circ \Delta_V$ and $\omega \ast \omega' = (\omega \otimes \omega') \circ \Delta_V$, respectively.

iii) There exist contractive algebra homomorphisms $\hat{\pi}_V : \hat{\Omega}_{\beta,\alpha} \to \hat{A}_V$ and $\pi_V : \hat{\Omega}_{\alpha,\beta} \to A_V$ such that for all $\xi \in \beta^\infty$, $\eta \in \alpha^\infty$, $\xi \in \hat{\beta}^\infty$,

$$\hat{\pi}_V(\omega_{\xi,\eta}) = \sum_n (\xi_n |V|\eta_n)_2, \quad \pi_V(\omega_{\eta,\xi}) = \sum_n (\eta_n |V|\xi_n)_1.$$
Proof. i) This is a slight modification of Proposition 3.7 and follows from similar arguments.

ii) The existence of the multiplication in ii) follows from i) and Propositions 3.5 and 3.12, and associativity from coassociativity of $\Delta_V$ and $\Delta_\beta$ (see the proof of Lemma 3.13).

iii) This is a special case of the more general Proposition 4.13 which is proven in subsection 4.3.

If $((A_\beta)_\beta^\alpha, \Delta_\beta)$ and $((A_\alpha)_\alpha^\beta, \Delta_\alpha)$ are Hopf $C^*$-bimodules, they should be thought of as standing in a generalized Pontrjagin duality. This duality is captured by a pairing on the dense subalgebras

$$\hat{A}_\beta^0 := \hat{\pi}_\beta(\hat{\Omega}_{\beta,\alpha}) \subseteq \hat{A}_\beta, \quad \hat{A}_\alpha^0 := \hat{\pi}_\alpha(\hat{\Omega}_{\alpha,\beta}) \subseteq \hat{A}_\alpha.$$

Definition 3.15. We call the algebra $\hat{A}_\beta^0 \subseteq \hat{A}_\beta$, equipped with the quotient norm from the surjection $\pi_\beta$, the Fourier algebra of $V$. Similarly, we call the algebra $\hat{A}_\alpha^0 \subseteq \hat{A}_\alpha$, equipped with the quotient norm from the surjection $\pi_\alpha$, the dual Fourier algebra of $V$.

Proposition 3.16. i) There exists a bilinear map $\cdot \cdot : \hat{A}_\beta^0 \times \hat{A}_\alpha^0 \to \mathcal{L}(\mathcal{H})$ such that $\omega(\pi_\beta(v)) = \pi_\beta(\omega)(\pi_\alpha(v)) = \pi_\alpha(\omega)v = \pi_\beta(\omega)v$ for all $\omega \in \hat{\Omega}_{\beta,\alpha}, v \in \hat{\Omega}_{\alpha,\beta}$.

ii) This map is nondegenerate in the sense that for each $\hat{a} \in \hat{A}_\beta^0$ and $a \in A_\alpha^0$, there exist $\hat{a}' \in \hat{A}_\beta^0$ and $a' \in A_\alpha^0$ such that $(\hat{a}\hat{a}') \neq 0$ and $(\hat{a}'a) \neq 0$.

iii) $(\pi_\beta(\omega)\pi_\alpha(\omega')v) = (\omega \otimes \omega')(\Delta_\beta(a))$ and $(\pi_\beta(v)\pi_\alpha(v')) = (v \otimes v')(\Delta_\alpha(a))$ for all $\omega, \omega' \in \hat{\Omega}_{\beta,\alpha}, a \in A_\alpha^0, v, v' \in \hat{\Omega}_{\alpha,\beta}, a \in \hat{A}_\beta^0$.

Proof. i) If $\omega = \omega_{\beta',\epsilon'}(\eta')$, then $\omega(\pi_\beta(v)) = \sum_{m,n} \eta'(\eta_n V | \eta_m) \eta'_m = \sum_{m,n} \eta'(\eta_n V | \eta'_m) \eta'_m = \omega(\pi_\beta(v))$.

ii) Evident.

iii) For all $\omega, \omega', a$ as above, $(\pi_\beta(\omega)\pi_\alpha(\omega')v) = (\pi_\beta(\omega \ast \omega')v) = (\omega \ast \omega')(\Delta_\beta(a))$. The second equation follows similarly.

As a consequence of part ii) of the preceding result, we obtain the following simple relation between the Fourier algebra $\hat{A}_\beta^0$ and the convolution algebra constructed in Theorem 3.10.

Proposition 3.17. If $((A_\beta)_\beta^\alpha, \Delta_\beta)$ or $((A_\alpha)_\alpha^\beta, \Delta_\alpha)$ is a normal Hopf $C^*$-bimodule, then we have a commutative diagram of Banach algebras and homomorphisms

$$\begin{array}{ccc}
\hat{\Omega}_{\beta,\alpha} & \xrightarrow{\hat{\pi}_\beta} & \hat{A}_\beta^0 \\
\Omega_{\beta,\alpha}(\rho_{\beta}(\mathcal{B})) & \xrightarrow{q} & \hat{\Omega}_{\beta,\alpha}(A_\beta) \\
\end{array} \quad \text{or} \quad \begin{array}{ccc}
\hat{\Omega}_{\alpha,\beta} & \xrightarrow{\hat{\pi}_\alpha} & \hat{A}_\alpha^0 \\
\Omega_{\alpha,\beta}(\rho_{\beta}(\mathcal{B})) & \xrightarrow{q} & \hat{\Omega}_{\alpha,\beta}(A_\alpha) \\
\end{array},$$

respectively, where $q$ is the quotient map and $\hat{\pi}$ or $\pi$ an isometric isomorphism.
3.5 The legs of the unitary of a groupoid

The general preceding constructions are now applied to the $C^*$-pseudo-multiplicative unitary of a locally compact, Hausdorff, second countable groupoid $G$ that was constructed in subsection 2.3. The algebras $A^0_Y$ and $A_Y$ turn out to be the reduced groupoid $C^*$-algebra $C^*_r(G)$ and the function algebra $C_0(G)$, respectively, but unfortunately, we can not determine the Fourier algebras $A^0_Y$ and $A_Y$.

We use the same notation as in subsection 2.3 and let

$$R := L^2(G^n, \mu), \quad \mathcal{B} := C_0(G^n) \subseteq L(R), \quad b := (\mathcal{B}, \mathcal{B}^\perp),$$

$$H := L^2(G, \nu), \quad \alpha = \beta := j(L^2(G, \lambda)), \quad \hat{\beta} := j(L^2(G, \lambda^{-1})),$$

$$V : H_{\hat{b}} \otimes aH \cong L^2(G \times G, \nu_{\hat{a}}) \to L^2(G \times G, \nu_{\hat{b}}) \cong H_{\hat{a}} \otimes bH,$$

$$(\nu \omega)(x, y) = \omega(x, x^{-1}y) \text{ for all } \omega \in C_c(G \times G), \; (x, y) \in G \times G.$$  

Denote by $m : C_0(G) \to L(H)$ the representation given by multiplication operators, and by $L^1(G, \lambda)$ the completion of $C_c(G)$ with respect to the norm given by

$$\|f\| := \sup_{a \in G} \int_{G^n} |f(a)|d\lambda^n(x) \quad \text{for all } f \in C_c(G).$$

Then $L^1(G, \lambda)$ is a Banach algebra with respect to the convolution product

$$(f * g)(y) = \int_{G^{(1)}} g(x)f(x^{-1}y)d\lambda^{(1)}(x) \quad \text{for all } f, g \in L^1(G, \lambda), y \in G,$$

and there exists a norm-decreasing algebra homomorphism $L : L^1(G, \lambda) \to L(H)$ such that

$$(L(f)\xi)(y) = \int_{G^{(1)}} f(x)D^{-1}(x)\xi(x^{-1}y)d\lambda^{(1)}(x) \quad \text{for all } f, \xi \in C_c(G), y \in G.$$  

For all $\xi, \xi' \in L^2(G, \lambda)$ and $\eta \in L^2(G, \lambda)$, $\eta' \in L^2(G, \lambda^{-1})$, let

$$\hat{a}_{\xi, \eta} = (j(\xi)|\nu|j(\eta))_{1} \in A^0_Y \quad \text{and} \quad a_{\eta, \eta'} = (j(\eta)|\nu|j(\eta'))_{1} \in A^0_Y.$$  

Routine arguments show that there exists a unique continuous map

$$L^2(G, \lambda) \times L^2(G, \lambda) \to C_0(G), \; (\xi, \xi') \mapsto \xi * \xi'^*,$$

such that

$$\langle \xi * \xi'^*(x) \rangle = \int_{G^{(1)}} \overline{\xi(y)}\xi'(x^{-1}y)d\lambda^{(1)}(y) \quad \text{for all } \xi, \xi' \in C_c(G), x \in G.$$  

**Lemma 3.18.** Let $\xi, \xi' \in L^2(G, \lambda)$ and $\eta, \eta' \in C_c(G)$. Then $\hat{a}_{\xi, \eta} = m(\xi \cdot \xi'^*)$ and $a_{\eta, \eta'} = L(\eta \eta').$  

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Proof. By continuity, we may assume $\xi, \zeta \in C_c(G)$. Then for all $\xi, \zeta \in C_c(G)$,

\[
\langle \xi \hat{\otimes} \zeta, \zeta' \rangle = \langle \xi \otimes j(\xi) V(\zeta \otimes j(\zeta')) \rangle
= \int_G \int_{G^{(1)}} \xi(x)\xi(y)\zeta(x)x^{-1}y d\lambda^{G}(x) dy = \langle \xi m(\xi \hat{\otimes} \zeta^*), \zeta' \rangle,
\]

\[
\langle \xi \hat{\otimes} \zeta, \zeta' \rangle = \langle j(\eta) \otimes V(j(\eta') \otimes \zeta') \rangle
= \int_G \int_{G^{(1)}} \eta(x)\eta(y)\eta'(x)D^{1/2}(x) \zeta'(x^{-1}y) d\lambda^{G}(x) dy
= \langle \xi L(\eta \hat{\otimes} \zeta'), \zeta' \rangle.
\]

\[\square\]

Remark 3.19. To extend the formula $a_{\eta, \eta'} = L(\eta \hat{\otimes} \eta')$ to all $\eta \in L^2(G, \lambda), \eta' \in L^2(G, \lambda^{-1})$, we would have to extend the representation $L : C_c(G) \rightarrow L(H)$ to some algebra $X$ and the pointwise multiplication $(\eta, \eta') \mapsto \eta \eta'$ to a map $L^2(G, \lambda) \times L^2(G, \lambda^{-1}) \rightarrow X$. Note that pointwise multiplication extends to a continuous map $L^2(G, \lambda) \times L^2(G, \lambda) \rightarrow L^1(G, \lambda)$, but in general this is not what we need. We expect that the map $L : C_c(G) \rightarrow \tilde{A}_V$ does not extend to an isometric isomorphism of Banach algebras $L^1(G, \lambda) \rightarrow \tilde{A}_V$.

The algebra $\tilde{A}_V$ can be considered as a continuous Fourier algebra of the locally compact groupoid $G$. Another Fourier algebra for locally compact groupoids was defined by Paterson in [25] as follows. He constructs a Fourier-Stieltjes algebra $B(G) \subseteq C(G)$ and defines the Fourier algebra $A(G)$ to be the norm-closed subalgebra of $B(G)$ generated by the set $A_f(G) := \{\hat{a}_x \xi \mid x \in L^2(G, \lambda)\}$. The definition of $B(G)$ in [25] immediately implies that $\\|\pi_u(\hat{a}_x \xi)\|_{B(G)} \leq \|\xi\|_1$ for all $\xi \in \alpha^w, \xi' \in \beta^w$ with finitely many non-zero components, whence the following relation holds:

Proposition 3.20. The identity on $A_f(G)$ extends to a norm-decreasing homomorphism of Banach algebras $\tilde{A}_V \rightarrow A(G)$.

\[\square\]

Another Fourier space $\hat{A}(G)$ considered in [25, Note after Proposition 13] is defined as follows. For each $\eta \in L^2(G, \lambda)$ and $u \in G^0$, write $\|\xi_u\| := \|\xi_u\|^{1/2}$. Denote by $M$ the set of all pairs $(\xi, \xi')$ of sequences in $L^2(G, \lambda)$ such that the supremum $\|\xi, \xi'\|_M := \sup_{u \in G^{(1)}} \sum_n \|\xi_u\|_1 \|\xi'_u\|_1$ is finite, and denote by $\hat{A}(G)$ the completion of the linear span of $A_f(G)$ with respect to the norm defined by

\[
\|\hat{a}\|_{\hat{A}(G)} = \inf \Big\{\|\xi, \xi'\|_M \mid \hat{a} = \sum_n \hat{a}_{\xi_n, \xi'_n} \Big\}.
\]

Proposition 3.21. The identity on $\tilde{A}_V \rightarrow \hat{A}(G)$.

\[\square\]

Proof. For all $\xi, \xi' \in L^2(G, \lambda)^w$, we have

\[
\|\xi\|^2 = \sup_{u \in G^0} \sum_n \|\xi_u\|^2, \quad \|\xi'\|^2 = \sup_{v \in G^0} \sum_n \|\xi'_v\|^2,
\]

and therefore $\|\xi, \xi'\|_M = \sup_{u, v \in G^0} \sum_n \|\xi_u\| \|\xi'_v\| \leq \|\xi\| \|\xi'\|$. 

Let us add that a Fourier algebra for measured groupoids was defined and studied by Renault [27], and for measured quantum groupoids by Vallin [37].
Finally, we consider the C*-algebras associated to V. Recall that the reduced groupoid C*-algebra $C_r^*(G)$ is the closed linear span of all operators of the $L^2(G, \lambda)$, where $g \in L^1(G, \lambda)$ [26].

**Theorem 3.22.** Let $V$ be the C*-pseudo-multiplicative unitary of a locally compact groupoid $G$. Then ($\widehat{A}_V|_{\mathcal{H}}^\beta \Lambda$, $\Delta_V$) and ($\Lambda_V|_{\mathcal{H}}^\beta \Lambda$, $\Delta_V$) are Hopf C*-bimodule and

\[
\widehat{A}_V = m(C_0(G)), \quad (\Delta_V(m(f))\omega)(x,y) = f(xy)\omega(x,y),
\]

\[
A_V = C_r^*(G), \quad (\Lambda_V(L(g))\omega')(x',y') = \int_{G} g(z)D^{-1/2}(z)\omega'(z^{-1}x',z^{-1}y')d\lambda'(z)
\]

for all $f \in C_0(G)$, $\omega \in H_0^G \hat{\otimes} H$, $(x,y) \in G \times_r G$ and $g \in C_c(G)$, $\omega' \in H_0^G \hat{\otimes} H$, $(x',y') \in G \times_r G$, where $u' = r(x') = r(y')$.

**Proof.** The first assertion will follow from Example 5.3 and Theorem 5.7 in subsection 5.1. The equations concerning $\Lambda_V$ and $\Delta_V$ follow directly from Lemma 3.18. Let us prove the formulas for $\Delta_V$ and $\Delta_V$. For all $f, \omega, (x,y)$ as above,

\[
(\Delta_V(m(f))\omega)(x,y) = (\Lambda^*(\text{id} \otimes m(f))V\omega)(x,y)
\]

\[
= (\Lambda^*(\text{id} \otimes m(f))L\omega)(x,xy) = f(xy)(V\omega)(x,xy) = f(xy)\omega(x,y),
\]

and for all $g, (x',y'), \omega', u'$ as above,

\[
(\Lambda_V(L(g))\omega')(x',y') = (\Lambda^*(L(g) \otimes \text{id})V^*\omega')(x',y')
\]

\[
= (L(g) \otimes \text{id})V^*\omega'(x',x'^{-1}y')
\]

\[
= \int_{G} g(z)D^{-1/2}(z)(V^*\omega')(z^{-1}x',z'^{-1}y')d\lambda'(z)
\]

\[
= \int_{G} g(z)D^{-1/2}(z)\omega'(z'^{-1}x',z'^{-1}y')d\lambda'(z).
\]

\[
\square
\]

### 4 Representations of a C*-pseudo-multiplicative unitary

Let $G$ be a locally compact group and let $V$ be the multiplicative unitary on the Hilbert space $L^2(G, \lambda)$ given by formula (3). Then one can associate to every unitary representation $\pi$ of $G$ on a Hilbert space $K$ a unitary operator $X$ on $L^2(G, \lambda) \otimes K \cong L^2(G, \lambda; K)$ such that $(Xf)(x) = \pi(x)f(x)$ for all $x \in G$ and $f \in L^2(G, \lambda; K)$, and this operator satisfies the modified pentagon equation

\[
V_{12}X_{13}X_{23} = V_{23}X_{12}.
\]

(17)

For a general multiplicative unitary $V$ on a Hilbert space $H$, Baaj and Skandalis defined a representation on a Hilbert space $K$ to be a unitary $X$ on $H \otimes K$ satisfying equation (17), equipped the class of all such representations with the structure of a C*-tensor category and showed that under the assumption of regularity, this C*-tensor category is the category of representations of a Hopf C*-algebra $(A_{\omega}, \Lambda_{\omega})$ (see [3]). In the case where $V$ is the unitary associated to a group $G$ as above, this category is

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isomorphic to the category of unitary representations of $G$, and $A_{(a)}$ is the full group $C^*$-algebra $C^*(G)$.

We carry over these definitions and constructions to $C^*$-pseudo-multiplicative unitaries and relate them to representations of groupoids. Throughout this section, let $b = (\mathfrak{G}, \mathfrak{B}, \mathfrak{B}^l)$ be a $C^*$-base, $(H, \hat{\beta}, \alpha, \beta)$ a $C^*$-$(b^l, b)$-module and $V : H_{b^l} \otimes \alpha H \rightarrow H_{a \otimes b}H$ a $C^*$-pseudo-multiplicative unitary.

### 4.1 The $C^*$-tensor category of representations

Let $\mathcal{K}_b^a$ be a $C^*$-$(b^l, b)$-module and $X : \mathcal{K}_b^a \otimes \alpha H \rightarrow \mathcal{K}_b^a \otimes \beta H$ an operator satisfying

$$X(\gamma \triangleright \alpha) = \gamma \triangleright \alpha, \quad X(\tilde{\delta} \triangleright \beta) = \tilde{\delta} \triangleright \beta$$  \hspace{1cm} (18)

as subsets of $\mathcal{L}(\mathfrak{G}, \mathcal{K}_b^a \otimes \beta H)$. Then all operators in the following diagram are well defined,

$$
\begin{array}{ccc}
\mathcal{K}_b^a \otimes \alpha H & \xrightarrow{X \otimes \text{id}} & \mathcal{K}_b^a \otimes \beta H \\
\mathcal{K}_b^a \otimes \alpha H & \xrightarrow{X \otimes \text{id}} & \mathcal{K}_b^a \otimes \beta H \\
\end{array}
\begin{array}{c}
\mathcal{K}_b^a \otimes (\alpha \otimes \beta)H \\
(\mathcal{K}_b^a \otimes \alpha H \otimes \beta H) \\
\Sigma_{23}
\end{array}$$

where the canonical isomorphism $\Sigma_{23} : (\mathcal{K}_b^a \otimes \beta H)(\beta \otimes \alpha) \cong \mathcal{K}_b^a \otimes (\alpha \otimes \beta)H$.

We again adopt the leg notation $[3]$ and write

$$X_{12} \text{ for } X \otimes \text{id} \text{ and } X \otimes \text{id}; \quad X_{13} \text{ for } \Sigma_{23}(X \otimes \text{id})(\text{id} \otimes \Sigma).$$

**Definition 4.1.** A representation of $V$ consists of a $C^*$-$(b^l, b)$-module $\mathcal{K}_b^a$ and a unitary $X : \mathcal{K}_b^a \otimes \alpha H \rightarrow \mathcal{K}_b^a \otimes \beta H$ such that equation (18) holds and diagram (19) commutes.

We also call $X$ a representation of $V$ on $\mathcal{K}_b^a$. A (semi-)morphism of representations $(\iota K_{\mathfrak{G}}, X)$ and $(\iota L_{\mathfrak{G}}, Y)$ is an operator $T \in \mathcal{L}(\mathfrak{G}, \mathcal{K}_b^a \otimes \beta H)$ satisfying $Y(T \otimes \text{id}) = (T \otimes \text{id})X$. Evidently, the class of all representations and (semi-)morphisms forms a category; we denote it by $C^*\text{-rep}_{\mathfrak{G}}^a$.

**Examples 4.2.**

i) Consider the canonical isomorphisms

$$\Phi : \mathcal{K}_b^a \otimes \alpha H \rightarrow H, \quad b^l \otimes \zeta \mapsto \rho_a(b^l) \zeta, \quad \Psi : \mathcal{K}_b^a \otimes \beta H \rightarrow H, \quad b \otimes \zeta \mapsto \rho_b(b) \zeta.$$  \hspace{1cm} (20)
The composition $1_{V} := \Psi^{\ast} \Phi$ is a representation on $\mathbb{R}_{\mathbb{C}}$; which we call the \textit{trivial} representation of $V$.

ii) The pair $(\alpha H_{b}, V)$ is a representation which we call the \textit{regular} representation.

iii) Let $(\gamma, K_{i}, X_{i})$, be a family of representations. Then the operator

$$
\mathbb{1}_{i}X_{i} : (\mathbb{1}_{i}K_{i}^{\ast}) \otimes_{b}^{a} H \rightarrow (\mathbb{1}_{i}K_{i}^{\ast}) \otimes_{b}^{a} H
$$

corresponding to $\bigoplus X_{i}$ with respect to the identifications $(\mathbb{1}_{i}K_{i}^{\ast}) \otimes_{b}^{a} H \equiv \bigoplus (K_{i} \gamma \otimes_{b}^{a} H)$ and $(\mathbb{1}_{i}K_{i}^{\ast}) \otimes_{b}^{a} H \equiv \bigoplus (K_{i} \gamma \otimes_{b}^{a} H)$ is a representation on $\bigoplus \mathbb{1}_{i}K_{i}^{\ast}$. We call it the \textit{direct sum} of $(X_{i})$.

iv) Let $\gamma$ be a $C^{\ast}$-base, $L_{a}$ a $C^{\ast}$-$\gamma$-module, $(K, \gamma, \tilde{\delta}, \kappa)$ a $C^{\ast}$-$(b, b', c')$-module, and $X$ a representation on $\gamma K_{b}$. If $X(\kappa \triangleleft \alpha) = \kappa \triangleleft \beta$, then the operator

$$
id \otimes_{c} X : L_{a} \otimes_{c} K_{b} \otimes_{c} H \rightarrow L_{a} \otimes_{c} K_{b} \otimes_{c} H
$$

is a representation on $\gamma \otimes_{\gamma}(L_{a} \otimes_{c} K_{b})_{\gamma \otimes_{\gamma}}$, as one can easily check.

The category of representations admits a tensor product:

**Lemma 4.3.** Let $(\gamma K_{b}, X)$ and $(\xi L_{b'}, Y)$ be representations of $V$. Then the operator

$$
X \boxtimes_{b'} Y : K_{b} \otimes_{b'} L_{b'} \otimes_{b'} H \rightarrow K_{b} \otimes_{b'} L_{b'} \otimes_{b'} H
$$

where $Y_{23} = \text{id} \otimes_{b'} Y$ and where $X_{13}$ acts like $X$ on the first and last factor of the relative tensor product, is a representation of $V$ on $\gamma K_{b} \otimes_{b'} L_{b'}$.

**Proof.** First, the relations (18) for $X$ and $Y$ imply

$$
X_{13} Y_{23}(\gamma \triangleleft \epsilon \triangleleft \alpha) = X_{13}(\gamma \triangleleft \epsilon \triangleleft \alpha) = (\gamma \triangleleft \epsilon \triangleleft \alpha),
$$

$$
X_{13} Y_{23}(\beta \triangleright \phi \triangleright \beta) = X_{13}(\beta \triangleright \phi \triangleright \beta) = (\beta \triangleright \phi \triangleright \beta),
$$

$$
X_{13} Y_{23}(\tilde{\delta} \triangleright \tilde{\phi} \triangleright \beta) = X_{13}(\tilde{\delta} \triangleright \tilde{\phi} \triangleright \beta) = (\gamma \triangleleft \epsilon \triangleright \beta).
$$

If $V$ is an ordinary multiplicative unitary, then $Z := X \boxtimes Y$ satisfies $Z_{12}Z_{13}V_{23} = V_{23}Z_{12}$ because the equations $Y_{12}Y_{13}V_{23} = V_{23}Y_{12}, X_{12}X_{13}V_{23} = V_{23}X_{12}$ imply $X_{13}Y_{12}Y_{13}V_{23} = V_{23}X_{13}X_{12}V_{23}$; here, we used the leg notation [3]. A similar calculation applies to the general case. \hfill \Box

The tensor product turns $C^{\ast}$-$\text{rep}_{\gamma}$ and $C^{\ast}$-$\text{rep}_{\xi}$ into $C^{\ast}$-tensor categories, which are frequently also called $(C^{\ast})$-monoidal categories [8, 14, 20]:

**Theorem 4.4.** The category $C^{\ast}$-$\text{rep}_{\gamma}$ carries the structure of a $C^{\ast}$-tensor category and the category $C^{\ast}$-$\text{rep}_{\xi}$ carries the structure of a tensor category, where both times
the tensor product is given by \((X, Y) \mapsto X \boxtimes Y\) for representations and \((S, T) \mapsto S \otimes T\) for morphisms;

- the associativity isomorphism \(a_{X, Y, Z}: (X \boxtimes Y) \boxtimes Z \rightarrow X \boxtimes (Y \boxtimes Z)\) is the isomorphism \(a_{\alpha, \beta, \delta}\) of equation (6) for all representations \((\alpha, \beta, \gamma)\);

- the unit isomorphisms \(l_\alpha: 1_V \boxtimes X \rightarrow X\) and \(r_\alpha: X \boxtimes 1_V \rightarrow X\) are the isomorphisms \(l_\alpha\) and \(r_\alpha\), respectively, of equation (5) for each representation \((\alpha, \beta, \gamma)\).

Proof. Tedious but straightforward. □

The regular representation tensorially absorbs every other representation:

**Proposition 4.5.** Let \((\gamma K_3, X)\) be a representation of \(V\). Then \(X\) is an isomorphism between the representation \(X \boxtimes V\) and the amplification \(\text{id} \boxtimes V\) on \(\varphi\alpha(K_3 \boxtimes \hat{H}) \varphi\beta\).

Proof. This follows from commutativity of (19). □

We denote by \(\text{End}(1_V)\) the algebra of endomorphisms of the trivial representation \(1_V\). This is a commutative \(C^*\)-algebra, and the category \(C^*\text{-rep}_V\) can be considered as a bundle of \(C^*\)-categories over the spectrum of \(\text{End}_V(1_V)\) [39].

**Proposition 4.6.** \(\text{End}(1_V) = \{b \in M(\mathcal{B}) \cap M(\mathcal{B}^\dag) \subseteq L(\mathcal{R}) \mid \rho_b(1) = \rho_b(1)\}\).

Proof. First, note that \(L(\mathcal{R}_b \mathcal{R}_b^\dag)\) is equal to \(M(\mathcal{B}) \cap M(\mathcal{B}^\dag) \subseteq L(\mathcal{R})\). Let \(\Phi\) and \(\Psi\) be as in (20). Then for each \(x \in L(\mathcal{R}_b \mathcal{R}_b^\dag)\),

\[
x \in \text{End}(1_V) \Leftrightarrow \Psi^* \Phi(x \otimes \text{id}) = (x \otimes \text{id})^* \Psi^* \Phi
\]

\[
\Leftrightarrow \Phi(x \otimes \text{id})^* \Phi = \Psi(x \otimes \text{id})^* \Psi \Leftrightarrow \rho_\alpha(x) = \rho_\beta(x).
\]

□

### 4.2 The legs of representation operators

To every representation, we associate an algebra and a space of generalized matrix elements as follows. Given a representation \(X\) on a \(C^*\text{-}(b, b^\dag)\)-module \(\gamma K_3\), we put

\[
\hat{A}_X := [\langle \beta | X | \alpha \rangle_2] \subseteq L(K) \quad \text{and} \quad A_X := [\langle \gamma | X | \hat{b} \rangle_1] \subseteq L(H),
\]

where \(|\alpha\rangle_2, |\hat{b}\rangle_1, |\beta\rangle_2, |\gamma\rangle_1\) are defined as in subsection 2.1.

**Examples 4.7.**

- i) For the trivial representation \((\gamma K_3, 1_V)\), we have \(\hat{A}_{1_V} = [\beta^* \alpha\rangle\) and \(A_{1_V} = [\rho_\beta(1) \rho_\alpha(1)\rangle\). The space \(\hat{A}_{1_V}\) is related to the \(C^*\)-algebra \(\text{End}(1_V)\) (see Proposition 4.6) as follows: \(\text{End}(1_V) = L(\mathcal{R}_b \mathcal{R}_b^\dag) \cap (\hat{A}_{1_V})\). This relation follows from Proposition 4.6 and the fact that an element \(x \in L(\mathcal{R}_b \mathcal{R}_b^\dag) = M(\mathcal{B}) \cap M(\mathcal{B}^\dag)\) satisfies \(\rho_\alpha(x) = \rho_\beta(x)\) if and only if for all \(\eta \in \beta\) and \(\xi \in \alpha\), the elements \(\eta^* \xi\) and \(x \eta^* \xi\) coincide.
ii) For the regular representation \((\alpha H_b; V)\), the definition above is consistent with definition (16).

iii) If \((X_i)\) is a family of representations and \(X = \bigsqcup X_i\), then \(\tilde{A}_X \subseteq \bigsqcup_i \tilde{A}_{X_i}\) and \(A_X = [\bigcup_i A_{X_i}]\).

iv) If \((\lambda \gamma \hat{L}_\zeta \otimes \varepsilon K)\), \(\hat{\delta} \otimes \id \otimes X\) is the amplification of a representation \((\gamma K_\zeta; X)\) as in Example 4.2 iv), then \(\tilde{A}_{(\id \otimes X)} = \id \otimes \varepsilon \tilde{A}_X\) and \(A_{(\id \otimes X)} = A_X\).

With respect to tensor products, the definition of \(\tilde{A}_X\) and \(A_X\) behaves as follows:

**Lemma 4.8.** Let \((\gamma K_\zeta; X)\) and \((\zeta L_\zeta; Y)\) be representations of \(V\). Then

\[
A_{(X \otimes Y)} = [A_X A_Y], \quad [A_{(X \otimes Y)}] = [A_X A_Y]_2, \quad [A_{(X \otimes Y)}(\hat{\delta} \otimes 1)] = [\hat{\delta} A_X].
\]

The proof involves commutative diagrams of a special kind:

**Notation 4.9.** We shall frequently prove equations for certain spaces of operators using commutative diagrams. In these diagrams, the vertices are labelled by Hilbert spaces, the arrows are labelled by single operators or closed spaces of operators, and the composition is given by the closed linear span of all possible compositions of operators.

**Proof of Lemma 4.8.** The following commutative diagrams show that \(A_{(X \otimes Y)} = [A_X A_Y]_2\):

\[
\begin{array}{cccc}
K_b \otimes L_b & \otimes aH & Y_{b_2} & K_\delta \otimes (e \otimes a)(L_b \otimes b H) & X_{b_1} & (K_b \otimes eL)(y \otimes e \otimes b H) \\
\hat{\delta} & \downarrow & \hat{\delta} & \downarrow & \hat{\delta} & \downarrow \\
\hat{\delta} & \downarrow & \hat{\delta} & \downarrow & \hat{\delta} & \downarrow \\
H & \xrightarrow{A_Y} & H & \xrightarrow{A_X} & H & \xrightarrow{A_X} \\
\end{array}
\]

The relations concerning \(A_{(X \otimes Y)}\) follow similarly. \(\square\)

We now collect some general properties of the spaces introduced above.

**Proposition 4.10.** Let \((\gamma K_\zeta; X)\) be a representation of \(V\).

i) The space \(\tilde{A}_X \subseteq \mathcal{L}(K)\) satisfies

\[
[\tilde{A}_X \tilde{A}_K] = \tilde{A}_X, \quad [\tilde{A}_X \gamma] = [\gamma \tilde{A}_X], \quad [\tilde{A}_X \hat{\delta}] = [\hat{\delta} \tilde{A}_X],
\]

and if \(\tilde{A}_X = \tilde{A}_X\), then \(\tilde{A}_X\) is a \(\mathcal{C}^*\)-(\(b, b^1\))-algebra.

ii) The space \(A_X \subseteq \mathcal{L}(H)\) satisfies

\[
[\bar{A}_X \hat{\beta}] = \hat{\beta}, \quad [A_X \bar{B}] = [B \gamma \hat{\delta}], \quad [A_X \bar{\alpha}] = [\alpha \hat{\delta} \gamma], \quad [A_X A_Y] = A_Y, \quad [A_X \bar{A}] = [\hat{\delta} \gamma A_X], \quad [A_X A_Y] = A_Y, \quad [A_X \bar{A}] = [\alpha A_X], \quad [A_X A_Y] = [A_X \bar{A}].
\]
Proof. i) First, $[\tilde{A}_X A_X] = [(β|2(α|3X_{12}|α|3|α)_2)]$ because the diagram below commutes:

Indeed, cell (C) commutes because for all $ξ ∈ α, η, η′ ∈ β, ζ ∈ K$,

$$|ξ⟩_2⟨η'|_2(ζ ⊗ η) = ρ_{Γ}(η'″η)ζ ⊗ ζ = ρ_{Γ(yαβ)}(η'″η)(ζ ⊗ ζ) = (η'|_3ξ⟩_2(ζ ⊗ η).$$  \hspace{1cm} (24)

cell (P) is diagram (8), and the other cells commute by definition of $\tilde{A}_X$ and because of (7). Next, $[(β|2(α|3X_{12}|α|3|α)_2)] = A_{\chi}$ because the following diagram commutes:

We prove some of the other equations in (22); the remaining ones follow similarly.

$$[\tilde{A}_X K] = [(β|2X|α|2)K] = [(β|2X(K_{δ} ⊗ αH)] = [(β|2(K_{δ}⊗ αH)] = K,$n

$$[\tilde{A}_X \gamma] = [(β|2X|α|2)\gamma] = [(β|2|γ⟩_1|α⟩] = [β γ α] = [β AΓ_V],$$

$$[\tilde{A}_X p_δ(\mathcal{B})] = [(β|2X|α|2)p_δ(\mathcal{B})] = [(β|2X|α|2)] = \tilde{A}_X,$n

$$[p_δ(\mathcal{B})\tilde{A}_X] = [p_δ(\mathcal{B})(β|2X|α|2)] = [(β|2(p_δ(\mathcal{B}) ⊗ id)X|α|2)]$$

$$= [(β|2X(id ⊗ p_δ(\mathcal{B})))|α|2] = [(β|2X(p_δ(\mathcal{B})α)|2] = \tilde{A}_X.$n

ii) First, $[A_X \tilde{B}] = [(γ|1X|δ⟩_1|B⟩] = [(γ|1⟩|γ⟩_1B] = [p_β(\mathcal{B})B] = \tilde{B}$ by (18), and similar calculations show that $[A_X β] = [β γ δ] \gamma$ and $[A_X α] = [α δ γ]$. Next, $[A_X A_V] = A_X \otimes V = A_V$ by Lemma 4.8, Lemma 4.5, and Example 4.7 iv). The equations in the last line of (23) follow from similar calculations as for $A_X$.

Finally, let us prove the equations in the middle line of (23). Since $A_X \subseteq L(H_β) \subseteq $
\[ \rho_B^\op(\mathfrak B)' , \Delta_V(A_X) \] is well defined. Consider the commutative diagram

\[
\begin{array}{c}
\xymatrix{
H_{b'} \otimes_{b''} H \ar[r]^{\rho_1} & H_{b'} \otimes_{b''} H \ar[r]^{\Delta^1} & H_{b'} \otimes_{b''} H \ar[r]^{\rho_1} & H_{b'} \otimes_{b''} H.
}
\end{array}
\]

Since the composition on top is \( \Delta_V(A_X) \) and the composition on the bottom is \( X_{12}X_{13} \), the following diagram commutes and shows that \( |\Delta_V(A_X)| |\beta|_2 = |A_X| |\beta|_2 \):

\[
\begin{array}{c}
\xymatrix{
H \ar[r]^{\rho_1} & K_{b'} \otimes_{b''} H \ar[r]^{\Delta^1} & K_{b'} \otimes_{b''} H \ar[r]^{\rho_1} & H.
}
\end{array}
\]

A similar argument shows that \( |\Delta_V(A_X)| |\alpha|_1 = |\alpha|_1 A_X^\op \).

**Definition 4.11.** We call the \( C^\ast \)-pseudo-multiplicative unitary \( V \) well-behaved if \( \hat{A}_X = \hat{A}_X^\op \) for every representation \( X \) of \( V \) and \( A_Y = A_Y^\op \) for every representation \( Y \) of \( V \op \).

**Proposition 4.12.** If \( V \) is well-behaved, then \( (\hat{A}_V)^{\alpha,\beta} \) and \( (\hat{A}_V)^{\beta,\alpha} \) are concrete Hopf \( C^\ast \)-bimodules.

**Proof.** By Proposition 3.11, the assumption implies that \( (\hat{A}_V)^{\alpha,\beta} \) and \( (\hat{A}_V)^{\beta,\alpha} \) are \( C^\ast \)-algebras, that \( \Delta_V(A_Y) \subseteq A_Y \circ a b A_Y \) and similarly that \( \Delta_V(A_Y) \subseteq A_Y \circ a b A_Y \). Now, the assertion follows from Lemma 3.13. \( \square \)

### 4.3 The universal Banach algebra of representations

Every representation of \( V \) induces a representation of the convolution algebra \( \hat{\Omega}_{b,a} \) introduced in subsection 3.4 as follows.

**Proposition 4.13.**

i) Let \( (\gamma K_{b}, X) \) be a representation of \( V \). Then there exists a contractive algebra homomorphism \( \hat{\pi}_X : \hat{\Omega}_{b,a} \rightarrow \hat{A}_X \) such that

\[
\hat{\pi}_X(\omega_{b,a}) = \sum_n \langle \xi_n | 2X | \xi_n' \rangle_2 \quad \text{for all } \xi \in B^\alpha, \xi' \in A^\alpha.
\]

ii) Let \( (\gamma K_{b}, X) \) and \( (\gamma L_{b}, Y) \) be representations of \( V \) and let \( T \in L(A_X) \). Then \( T \) is a (semi-)morphism from \( (\gamma K_{b}, X) \) to \( (\gamma L_{b}, Y) \) if and only if \( T \) intertwines \( \hat{\pi}_X \) and \( \hat{\pi}_Y \) in the sense that \( T\pi_X(\omega) = \pi_Y(\omega)T \) for all \( \omega \in \hat{\Omega}_{b,a} \).
Proof. i) The sum on the right hand side of (25) depends on \( \omega \xi  \) but not on \( \xi' \), because \( \eta^* (\sum (\omega_{\xi, \xi'})_{n} | X_{n} \rangle \mid \eta') \mid 1 \rangle \) for all \( \eta \in \gamma, \eta' \in \tilde{\delta} \). Thus, \( \tilde{\pi}_X \) is well defined by (25). It is a homomorphism because for all \( \omega, \omega' \in \tilde{\Omega}_{\beta, \alpha}, \eta \in \gamma, \eta' \in \tilde{\delta}, \)

\[
\eta^* \tilde{\pi}_X (\omega) \tilde{\pi}_X (\omega') \eta' = (\omega \otimes \omega') (\langle \eta \mid X_{12}X_{13} | \eta' \rangle )
= (\omega \otimes \omega') (\langle \eta | V_{23}X_{12}V_{23} | \eta' \rangle )
= (\omega \otimes \omega') (V (\langle \eta | X | \eta' \rangle ) 1_{\tilde{b}_{i}^*} \otimes \alpha id) V^*
= (\omega \ast \omega') (\langle \eta | X | \eta' \rangle ) = \eta^* \tilde{\pi}_X (\omega \ast \omega') \eta'.
\]

ii) Straightforward. \( \square \)

For later use, we note the following formula:

**Lemma 4.14.** \( \tilde{\Delta}_V (\tilde{\pi}_V (\omega)) = \tilde{\pi}_{V \otimes V} (\omega) \) for each \( \omega \in \tilde{\Omega}_{\beta, \alpha} \).

**Proof.** For all \( \xi \in \alpha^* \) and \( \xi' \in \beta^* \), we have \( \tilde{\Delta}_V (\tilde{\pi}_V (\omega_{\xi, \xi'})) = \sum (\xi^* | 1_{V_{12}V_{12}} | \xi') = \sum (\xi^* | 1_{V_{23}V_{23}} | \xi') = \tilde{\pi}_{V \otimes V} (\omega_{\xi, \xi'}) \). \( \square \)

Denote by \( \widehat{A}_{(a)} \) the separated completion of \( \tilde{\Omega}_{\beta, \alpha} \) with respect to the seminorm

\[
| \omega | := \sup \{ \| \tilde{\pi}_X (\omega) \| \mid (X \text{ is a representation of } V) \}
\]

and by \( \tilde{\pi}_{(a)} : \tilde{\Omega}_{\beta, \alpha} \to \widehat{A}_{(a)} \) the natural map.

**Proposition 4.15.**

i) There exists a unique algebra structure on \( \widehat{A}_{(a)} \) such that \( \widehat{A}_{(a)} \) is a Banach algebra and \( \tilde{\pi}_{(a)} \) an algebra homomorphism.

ii) For every representation \( X \) of \( V \), there exists a unique algebra homomorphism \( \tilde{\pi}_{X}^{(a)} : \widehat{A}_{(a)} \to \hat{A}_X \) such that \( \tilde{\pi}_{X}^{(a)} \circ \tilde{\pi}_{(a)} = \tilde{\pi}_X \).

iii) If \( V \) is well-behaved, then the Banach algebra \( \widehat{A}_{(a)} \) carries a unique involution turning it into a \( C^* \)-algebra such that \( \tilde{\pi}_{X}^{(a)} \) is a \( * \)-homomorphism for every representation \( X \) of \( V \).

**Proof.** Assertions i) and ii) follow from routine arguments. Let us prove iii). For each \( \omega \in \tilde{\Omega}_{\beta, \alpha} \) and \( \varepsilon > 0 \), choose a representation \( X(\omega, \varepsilon) \) such that \( \| \tilde{\pi}_X (\omega, \varepsilon) \| > | \omega | - \varepsilon \). Let \( X := \sum_{\omega \in \tilde{\Omega}_{\beta, \alpha}} X(\omega, \varepsilon) \), where the sum is taken over all \( \omega \in \tilde{\Omega}_{\beta, \alpha} \) and \( \varepsilon > 0 \). Then evidently \( \tilde{\pi}_X^{(a)} : \widehat{A}_{(a)} \to \hat{A}_X \) is an isometric isomorphism of Banach algebras. We can therefore define an involution on \( \widehat{A}_{(a)} \) such that \( \tilde{\pi}_X^{(a)} \) becomes a \( * \)-isomorphism. Now, let \( Y \) be a representation \( V \). Then \( X \boxplus Y \) is a representation again, and we have a commutative diagram

\[
\begin{array}{ccc}
\hat{A}_X & \xrightarrow{\tilde{\pi}_X^{(a)}} & \hat{A}_X^{\boxplus Y} \\
\downarrow{\hat{\Delta}_Y} & & \downarrow{\tilde{\pi}_Y^{(a)}} \\
\hat{A}_Y & \xrightarrow{\tilde{\pi}_Y^{(a)}} & \hat{A}_Y \\
\end{array}
\]
where $p_X$ and $p_Y$ are the natural maps. Since $\pi^{(u)}_X$ is isometric, so is $\pi^{(u)}_{X \boxtimes Y}$. But $\pi^{(u)}_{X \boxtimes Y}$ also has dense image and therefore is surjective, whence $p_X$ is injective. Since $\pi^{(u)}_X$, $p_X$, $p_Y$ are $*$-homomorphisms, so is $\pi^{(u)}_{X \boxtimes Y}$ and hence also $\pi^{(u)}_Y$.

\[\Box\]

## 4.4 Universal representations and the universal Hopf $C^*$-bimodule

If the unitary $V$ is well-behaved, then the universal Banach algebra $\hat{A}_{(u)}$ constructed above can be equipped with the structure of a semi-normal Hopf $C^*$-bimodule, where the comultiplication corresponds to the tensor product of representations of $V$. The key idea is to identify $\hat{A}_{(u)}$ with the $C^*$-algebra associated to a representation that is universal in the following sense.

**Definition 4.16.** A representation $(\gamma K_\delta, X)$ of $V$ is universal if for every representation $(\lambda L_\phi, Y)$ and every $\xi \in \epsilon, \xi \in L, \eta \in \phi$, there exists a semi-morphism $T$ from $(\gamma K_\delta, X)$ to $(\lambda L_\phi, Y)$ that is a partial isometry and satisfies $\xi \in T\gamma_\phi \xi \in TK, \eta \in T\phi$.

**Remark 4.17.** Evidently, every universal representation is a generator $[20]$ of $C^*\text{-rep}_V^\gamma$ in the categorical sense.

We shall use a cardinality argument to show that $C^*\text{-rep}_V^\gamma$ has a universal representation. Given a topological space $X$ and a cardinal number $c$, let us say that $X$ is $c$-separable if $X$ has a dense subset of cardinality less than or equal to $c$. Let $\omega := |\mathbb{N}|$. Let us also say that a subrepresentation of a representation $(\gamma K_\delta, X)$ of $V$ is a $C^*$-$\{b, b_1\}$-module $\epsilon L_\delta$ such that $L \subseteq K, \epsilon \subseteq \gamma, \hat{\phi} \subseteq \delta$, and $X(L_\delta \otimes_{b} H) = L_\delta \otimes_{b} H$.

**Lemma 4.18.** Let $(\gamma K_\delta, X)$ be a representation, $c, d$ cardinal numbers, $K_0 \subseteq K, \gamma_0 \subseteq \gamma, \hat{\delta}_0 \subseteq \hat{\delta}$ $c$-separable subsets, and assume that the spaces $\epsilon, \delta, \beta_0, \beta_1, \alpha, \beta$ are $d$-separable. Put $e := \omega \sum_{n=0}^d n!$. Then there exists a subrepresentation $\epsilon L_\delta$ of $(\gamma K_\delta, X)$ such that $\gamma_0 \subseteq \epsilon, \hat{\delta}_0 \subseteq \hat{\delta}, K_0 \subseteq L$ and such that $L, \gamma, \hat{\phi}$ are $e(c + 1)$-separable.

**Proof.** Replacing $K_0, \gamma_0, \hat{\delta}_0$ by dense subsets, we may assume that each of these sets has cardinality less than or equal to $c$. Moreover, replacing $\gamma_0$ and $\hat{\delta}_0$ by larger sets, we may assume that $\mathfrak{B} = [\gamma_0, \hat{\delta}_0], \mathfrak{B}_1 = [\gamma_0, \hat{\delta}_0], \gamma_0 \subseteq c + \omega d, |\hat{\delta}_0| \subseteq c + \omega d$.

Now, we can choose inductively $K_n \subseteq K, \hat{\delta}_n \subseteq \hat{\delta}, \gamma_n \subseteq \gamma$ for $n = 1, 2, \ldots$ such that for $n = 0, 1, 2, \ldots$, the following conditions hold:

i) $K_{n+1}$ is large enough so that $K_n + \hat{\delta}_n \mathfrak{R} + \gamma_n \mathfrak{R} \subseteq [K_{n+1}]$, but small enough so that

$$|K_{n+1}| \leq |K_n| + \omega d(|\gamma_n| + |\hat{\delta}_n|) \leq \omega d(|K_n| + |\gamma_n| + |\hat{\delta}_n|);$$

ii) $\gamma_{n+1}$ is large enough so that

$$\gamma_n + \gamma_0 \mathfrak{B} + \rho_{\beta}(\mathfrak{B}_1) \gamma_0 \subseteq [\gamma_{n+1}], \quad K_n \subseteq [\gamma_{n+1}],$$

$$X(\alpha \gamma_{n+1}) \subseteq [\gamma_{n+1}] \mathfrak{R}, \quad X(\hat{\delta}_n) \subseteq [\gamma_{n+1}] \mathfrak{R}.$$
but small enough so that
\[ |γ_{n+1}| ≤ |γ_n|(1 + ωd) + ω|K_n| + ωd|γ_n| + ωd|δ_n| ≤ ωd(|K_n| + |γ_n| + |δ_n|); \]
i
\[ \hat{δ}_{n+1} \text{ is large enough so that} \]
\[ \hat{δ}_n + δ_n \mathcal{B}^1 + p_q(\mathcal{B}^1)\hat{δ}_n ≤ [\hat{δ}_{n+1}], \quad K_n ≤ [\hat{δ}_{n+1}], \quad X\hat{δ}_n) \quad \beta ≤ [β]_2\hat{δ}_{n+1}, \]
but small enough so that
\[ |\hat{δ}_{n+1}| ≤ |\hat{δ}_n|(1 + ωd) + ω|K_n| + ωd|\hat{δ}_n| ≤ ωd(|K_n| + |\hat{δ}_n|). \]

Since \(|K_0| + |γ_0| + |\hat{δ}_0| = 3c + 2ωd\), we can conclude inductively that for all \(n = 0, 1, 2, \ldots\),
\[ |K_{n+1}| + |γ_{n+1}| + |\hat{δ}_{n+1}| ≤ ωd(|K_n| + |γ_n| + |\hat{δ}_n|) ≤ (ωd)^{n+1}(c + ωd). \]
Therefore, the spaces \(L := [\bigcup_n K_n] \), \(ε := [\bigcup_n γ_n] \), \(\hat{φ} := [\bigcup_n \hat{δ}_n] \) are \(e(c + 1)\)-separable.

By construction, \(εL_φ\) is a subrepresentation of \((γK_φ,X)\).

**Proposition 4.19.** There exists a universal representation of \(V\).

**Proof.** Let \(d\) and \(e\) be as in Lemma 4.18. Then there exists a set \(X\) of representations of \(V\) such that every representation \((L_φ, Y)\) of \(V\), where the underlying Hilbert space \(L\) is \(e\)-separable, is isomorphic to some representation in \(X\). Using Lemma 4.18, one easily verifies that the direct sum \(\bigoplus_{X∈X} X\) is a universal representation.

**Theorem 4.20.** Let \(V\) be a well-behaved \(C^*\)-pseudo-multiplicative unitary and let \((γK_φ,X)\) be a universal representation of \(V\).

i) The \(*\)-homomorphism \(\widetilde{φ}^{(a)}_X : \hat{A}_a \rightarrow \hat{A}_X\) is an isometric isomorphism.

ii) If \((L_φ, Y)\) is a representation of \(V\), then there exists a jointly semi-normal morphism \(\tilde{π}_{X,Y}\) of \(C^*\,(b, b^1)\)-algebras \((\hat{A}_X)^Δ, (\hat{A}_Y)^{ε}\hat{φ}\) such that \(\tilde{π}^{(a)}_Y = \tilde{π}_{X,Y} \circ π^{(a)}_X\).

iii) Let \(\hat{Δ}_X := \tilde{π}_{X,X}secX.\) Then \((\hat{A}_X)^Δ, \hat{Δ}_X\) is a semi-normal Hopf \(C^*\)-bimodule.

iv) \(\tilde{π}_{X,Y}\) is a morphism of the semi-normal Hopf \(C^*\)-bimodules \((\hat{A}_X)^Δ, \hat{Δ}_X\) and \((\hat{A}_Y, \hat{Δ}_Y)\).

v) Let \((L_φ, Y)\) be a universal representation of \(V\) and define \(\hat{Δ}_Y\) similarly as \(\hat{Δ}_X\). Then \(\tilde{π}_{X,Y}\) is an isomorphism of the semi-normal Hopf \(C^*\)-bimodules \((\hat{A}_X)^Δ, \hat{Δ}_X\) and \((\hat{A}_Y, \hat{Δ}_Y)\).
Proof. i) Let \( \omega \in \hat{\Omega}_{b, \alpha} \), let \((\omega, L, Y)\) be a representation of \( V \), and let \( \zeta \in L \). Since \( X \) is universal, there exists a semi-morphism \( T \) from \( X \) to \( Y \) that is a partial isometry and satisfies \( \zeta \in TL \). Then by Proposition 4.13, \( \| \hat{\pi}_T(\omega) \zeta \| = \| \hat{\pi}_T(\omega) T^* \zeta \| = \| T \hat{\pi}_X(\omega) T^* \zeta \| \leq \| \hat{\pi}_X(\omega) \| \| \zeta \| \). Since \( Y \) and \( \zeta \) were arbitrary, we can conclude that \( \| \hat{\pi}_X(\omega) \| \leq \| \hat{\pi}_X(\omega) \| \) and hence that \( \hat{\pi}_X(\omega) \) is isometric.

ii) We have to show that \( \hat{\pi}_{X,Y} := \hat{\pi}_Y(\omega) \circ \hat{\pi}_X^{-1}(\omega) \) is a jointly semi-normal morphism of \( C^* \)-\((b,b')\)-algebras. Let \( \xi \in \mathcal{E} \) and \( \eta \in \mathcal{F} \). Since \( X \) is universal, there exists a semi-morphism \( T \) from \( X \) to \( Y \) such that \( \xi \in T \gamma \) and \( \eta \in T \delta \). By Proposition 4.13, \( \hat{\pi}_T(\omega) T = \hat{\pi}_X(\omega) T \) for all \( \omega \in \hat{\Omega}_{b, \alpha} \), and hence \( T \in L_{\pi X} \). The claim follows.

iii) We need to show that \( \hat{\Delta}_X \) is coassociative. We shall prove that \( (\hat{\Delta}_X \ast id) \circ \hat{\Delta}_X = \hat{\pi}_{X,X} \circ \hat{\pi}_{b,b} \). and a similar argument shows that \( (id \ast \hat{\Delta}_X) \circ \hat{\Delta}_X = \hat{\pi}_{X,X} \circ \hat{\pi}_{b,b} \). Let \( S, T \) be semi-morphisms from \( X \) to \( X \). Then \( R := (S \otimes T) \) satisfies the semi-morphism from \( X \) to \( X \times T \), and a generalization of Proposition 4.13 ii) shows that for each \( \omega \in \hat{\Omega}_{b, \alpha} \),

\[
(\hat{\Delta}_X \ast id)(\hat{\Delta}_X(\hat{\pi}_X(\omega))) \cdot R = (S \otimes id) \cdot \hat{\Delta}_X(\hat{\pi}_X(\omega)) \cdot T
\]

\[
= R \cdot \hat{\pi}_X(\omega) = \hat{\pi}_{X,X} \circ \hat{\pi}_{b,b}(\omega) \cdot R
\]

Since \( S \) and \( T \) were arbitrary and \( X \) is universal, we can conclude that \( (\hat{\Delta}_X \ast id) \circ \hat{\Delta}_X(\hat{\pi}_X(\omega)) = \hat{\pi}_{X,X} \circ \hat{\pi}_{b,b}(\omega) \).

iv) We have to show that \( \hat{\pi}_{X,Y} \ast \hat{\pi}_{X,Y} \circ \hat{\Delta}_X = \hat{\Delta}_Y \circ \hat{\pi}_{X,Y} \). Let \( \omega \in \hat{\Omega}_{b, \alpha} \) and \( S \in L_{\pi X}, T \in L_{\pi Y} \). Then \( R := (S \otimes T) \) satisfies \( R(X \times Y) = (V \otimes V)R \), and using Lemma 4.14, we find

\[
(\hat{\pi}_{X,Y} \ast \hat{\pi}_{X,Y})(\hat{\Delta}_X(\hat{\pi}_X(\omega))) \cdot R = R \cdot \hat{\Delta}_X(\hat{\pi}_X(\omega))
\]

\[
= R \cdot \hat{\pi}_{X,X} \circ \hat{\pi}_{b,b}(\omega) = \hat{\pi}_{X,Y} \circ \hat{\pi}_{X,Y} \cdot R = \hat{\Delta}_Y(\hat{\pi}_Y(\omega)) \cdot R.
\]

Since \( S \) and \( T \) were arbitrary, we can conclude \( \hat{\pi}_{X,Y} \ast \hat{\pi}_{X,Y}(\hat{\Delta}_X(\hat{\pi}_X(\omega))) = \hat{\Delta}_Y(\hat{\pi}_Y(\omega)) \).

v) We have to show that \( \hat{\pi}_{X,Y} \ast \hat{\pi}_{X,Y} \circ \hat{\Delta}_X = \hat{\Delta}_Y \circ \hat{\pi}_{X,Y} \). Let \( \omega \in \hat{\Omega}_{b, \alpha} \) and \( S \in L_{\pi X}, T \in L_{\pi Y} \). Then \( R := (S \otimes T) \) satisfies \( R(X \times Y) = (V \otimes V)R \), and using Proposition 4.13, we find

\[
(\hat{\pi}_{X,Y} \ast \hat{\pi}_{X,Y})(\hat{\Delta}_X(\hat{\pi}_X(\omega))) \cdot R = R \cdot \hat{\Delta}_X(\hat{\pi}_X(\omega))
\]

\[
= R \cdot \hat{\pi}_{X,X} \circ \hat{\pi}_{b,b}(\omega) = \hat{\pi}_{X,Y} \circ \hat{\pi}_{X,Y} \cdot R = \hat{\Delta}_Y(\hat{\pi}_Y(\omega)) \cdot R.
\]

Since \( S \) and \( T \) were arbitrary, we can conclude \( \hat{\pi}_{X,Y} \ast \hat{\pi}_{X,Y}(\hat{\Delta}_X(\hat{\pi}_X(\omega))) = \hat{\Delta}_Y(\hat{\pi}_Y(\omega)) \).
4.5 Corepresentations and $W^\ast$-representations

The notion of a representation of a C$^*$-pseudo-multiplicative unitary can be dualized so that one obtains the notion of a corepresentation, and adapted to $W^\ast$-modules instead of C$^*$-modules so that one obtains the notion of a $W^\ast$-representation. We briefly summarize the main definitions and properties of these concepts.

A corepresentation of $V$ consists of a C$^*$-$(\mathcal{B},\mathcal{B}^1)$-module $\mathcal{K}_\beta$ and of a unitary $X: H_\beta \otimes_{\mathcal{B}} K \to H_{\beta} \otimes_{\mathcal{B}} K$ that satisfies $X(\alpha \triangleleft \gamma) = \alpha \triangleright \gamma, X(\beta \triangleright \delta) = \beta \triangleleft \delta, X(\hat{\beta} \triangleright \delta) = \hat{\beta} \triangleleft \delta$ and makes the following diagram commute:

\[
\begin{array}{c}
H_\beta \otimes_{\mathcal{B}} H_\beta \otimes_{\mathcal{B}} K \\
\downarrow_{X_2} \downarrow_{X_3} \downarrow_{X_{23}} \\
H_\beta \otimes_{\mathcal{B}} (H_\alpha \otimes_{\mathcal{B}} K) \\
\end{array}
\]

\[
\begin{array}{c}
H_\beta \otimes_{\mathcal{B}} H_\beta \otimes_{\mathcal{B}} K \\
\downarrow_{V_{12}} \downarrow_{V_{13}} \downarrow_{V_{123}} \\
(H_\beta \otimes_{\mathcal{B}} H_\alpha \otimes_{\mathcal{B}} K) \end{array}
\]

where $V_{12}, X_{13}, X_{23}$ are defined similarly as in subsection 4.1. A (semi-)morphism of corepresentations $(\mathcal{K}_\beta, X)$ and $(\mathcal{L}_\delta, Y)$ is an operator $T \in \mathcal{L}_{\mathcal{B}}((\mathcal{K}_\beta, X), (\mathcal{L}_\delta, Y))$ satisfying $Y(\text{id} \otimes T) = (\text{id} \otimes T)X$. Evidently, the class of all corepresentations $V$ with all (semi-)
morphisms forms a category $\mathcal{C}^\ast$-corep$_V^{(\ast)}$. One easily verifies that there exists an isomorphism of categories $\mathcal{C}^\ast$-corep$_V^{(\ast)} \to \mathcal{C}^\ast$-rep$_V^{(\ast)}$ given by $(\mathcal{K}_\beta, X) \mapsto ((\mathcal{K}_\beta, \Sigma Y^\ast \Sigma)$ and $T \mapsto T$. Thus, all constructions and results on representations carry over to corepresentations. In particular, we can equip $\mathcal{C}^\ast$-corep$_V^{(\ast)}$ with the structure of $\mathcal{C}^\ast$-tensor category and $\mathcal{C}^\ast$-corep$_V^{(\ast)}$ with the structure of a tensor category.

Replacing $\mathcal{B}$ by the $W^\ast$-base $[\mathcal{B}]$ and $\mathcal{C}$-modules by $W^\ast$-modules (see [31]) in definition 4.1, we obtain the notion of a $W^\ast$-representation. If we reformulate this notion using correspondences instead of $W^\ast$-modules, the definition reads as follows. A $W^\ast$-representation of $V$ consists of a Hilbert space $K$ with two commuting non-degenerate and normal representations $\sigma: \mathcal{B} \to \mathcal{L}(K), \sigma: \mathcal{B}^1 \to \mathcal{L}(K)$ and a unitary $X \in \mathcal{L}(K_\alpha \otimes \alpha, K_\beta \otimes \beta)$ that satisfies $X(\sigma(b^1) \otimes \text{id}) = (\text{id} \otimes \rho_{ab}(b^1))X, X(\text{id} \otimes \rho_{ab}(b)) = (\hat{\sigma}(b) \otimes \text{id})X, X(\text{id} \otimes \rho_{ab}(b)) = (\text{id} \otimes \rho_{ab}(b))X$ for all $b^1 \in \mathcal{B}^1, b \in \mathcal{B}$ and that makes the following diagram commute,

\[
\begin{array}{c}
K_\alpha \otimes_{\mathcal{B}} \alpha \xleftarrow{\text{id}} K_\alpha \otimes_{\mathcal{B}} \beta \otimes \alpha \xrightarrow{\mathcal{V}} K_\alpha \otimes_{\mathcal{B}} \beta, \\
\downarrow_{\Sigma_{23}} \downarrow_{X \otimes \text{id}} \downarrow_{X \otimes \text{id}} \\
K_\alpha \otimes_{\mathcal{B}} \alpha \otimes \alpha \xrightarrow{X \otimes \text{id}} (K_\alpha \otimes_{\mathcal{B}} \alpha) \otimes \text{id} \otimes \beta \\
\end{array}
\]

where $\Sigma_{23}$ denotes the isomorphisms that exchange the second and the third factor in the iterated internal tensor products. Here, normality of $\sigma, \hat{\sigma}$ means that they extend to the von Neumann algebras generated by $\mathcal{B}$ and $\mathcal{B}^1$, respectively, in $\mathcal{L}(K)$. A morphism of $W^\ast$-representations $(K, \sigma, \hat{\sigma}, X)$ and $(L, \tau, \hat{\tau}, Y)$ is an operator $T \in \mathcal{L}(K, L)$ that intertwines $\sigma$ and $\tau$ on one side and $\hat{\sigma}$ and $\hat{\tau}$ on the other side, and satisfies
$Y(T \otimes \text{id}) = (T \otimes \text{id})X$. Evidently, the class of all $W^*$-representations of $V$ forms a category $W^\ast\text{-rep}_V$. One easily verifies that there exists a functor $C^\ast\text{-rep}_V^{(x)} \rightarrow W^\ast\text{-rep}_V$ given by $(K, p, \tilde{\rho}, \tilde{\rho}_0, X) \mapsto T$ and $T \mapsto T$. Using a relative tensor product of $W^*$-modules (see [31]), one can equip $W^\ast\text{-rep}_V$ with the structure of a $C^\ast$-tensor category similarly like $C^\ast\text{-rep}_V$ and finds that the functors above preserve the tensor product. Finally, one can consider $W^*$-corepresentations of $V$ which are defined in a straightforward manner.

4.6 Representations of groupoids and of the associated unitaries

Let $G$ be a locally compact, Hausdorff, second countable groupoid with a left Haar system. Then the $C^*$-tensor category of representations of $G$ is equivalent to the $C^*$-tensor category of corepresentations of the $C^*$-pseudo-multiplicative unitary associated to $G$, as will be explained now. We use the notation and results of subsections 2.3 and 3.5,

$$\mathfrak{A} := L^2(G^0, \mu), \quad \mathfrak{B} := \mathfrak{B}^\uparrow := C_0(G^0) \subseteq \mathcal{L}(\mathfrak{A}), \quad b := (\mathfrak{A}, \mathfrak{B}^\uparrow),$$

$$H := L^2(G, \nu), \quad \alpha = \beta := j(L^2(G, \lambda)), \quad \hat{\beta} := j(L^2(G, \lambda^{-1})),$$

$$V: H_b \otimes_\alpha H \cong L^2(G, \mathfrak{B}^\uparrow, \mathfrak{B}^\downarrow) \rightarrow L^2(G, \mathfrak{B}^\uparrow, \mathfrak{B}^\downarrow) \cong H_\alpha \otimes_\beta H;$$

$$V(\omega)(x, y) = \omega(x, y^{-1})$$

for all $\omega \in \mathcal{C}(G \times_G G)$, $(x, y) \in G \times_G G$.

$$C_0(G) \cong \tilde{\Lambda}_\nu \subseteq \mathcal{L}(H), \quad C^*_\nu(G) = A_\nu \subseteq \mathcal{L}(H),$$

and fix further notation. Let $X$ be a locally compact Hausdorff space, $E$ a Hilbert $C^*$-module over $C_0(X)$ and $x \in X$. We denote by $\chi_\nu: C(X) \rightarrow \mathbb{C}$ the evaluation at $x$ and by $E_x := E \otimes_{\chi_\nu} \mathbb{C}$ the fiber of $E$ at $x$; this is the Hilbert space associated to the sesquilinear form $(\eta, \eta') \mapsto (\hat{\eta}, \eta')$ on $E$. Given an element $\xi \in E$ and an operator $T \in \mathcal{L}_{C_0(X)}(E)$, we denote by $\xi_x := \xi \otimes_{\chi_\nu} 1 \in E_x$ and $T_x := T \otimes_{\chi_\nu} \text{id}_\mathbb{C} \in \mathcal{L}(E_x)$ the values of $\xi$ and $T$, respectively, at $x$. Given a locally compact Hausdorff space $Y$ and a continuous map $p: Y \rightarrow X$, the pull-back of $E$ along $p$ is the Hilbert $C^*$-module $p^* E := E \otimes_{p^* C_0(Y)} C_0(Y)$ over $C_0(Y)$, where $p^*: C_0(X) \rightarrow M(C_0(Y))$ denotes the pull-back on functions. This pull-back is functorial, that is, if $Z$ is a locally compact Hausdorff space and $q: Z \rightarrow Y$ is a continuous map, then $(p \circ q)^* E$ is naturally isomorphic to $q^* p^* E$. For $\xi, T$ as above and all $y \in Y$, we have $(p^* \xi)_y = \xi_{p(y)}$ and $(p^* T)_y = T_y$.

The first part of the following definition is a special case of [18, Definition 4.4]:

**Definition 4.21.** A continuous representation of $G$ consists of a Hilbert $C^*$-module $E$ over $C_0(G^0)$ and a unitary $U \in \mathcal{L}_{C_0(G)}(s^* E, r^* E)$ such that $U_0 U_0 = U_{y_0}$ for all $(x, y) \in G \times G$. We denote by $C^\ast\text{-rep}_G$ the category of continuous representations of $G$, where the morphisms between representations $(E, U_E)$ and $(F, U_F)$ are all operator $T \in \mathcal{L}_{C_0(G)}(E, F)$ satisfying $U_F \circ s^* T = r^* T \circ U_E$ in $\mathcal{L}_{C_0(G)}(s^* E, r^* F)$.

The verification of the following result is straightforward:

**Proposition 4.22.**

1. Let $(E, U_E)$ and $(F, U_F)$ be continuous representations of $G$ and represent $C_0(G^0)$ on $F$ by right multiplication operators. Then $(E \otimes F)_x =$...
Let $E$ be a Hilbert $C^*$-module for all $x \in G$, and there exists a continuous representation $U_E \otimes U_F$ of $G$ on $E \otimes F$ such that $(U_E \otimes U_F)_x = (U_E)_x \otimes (U_F)_x$ for all $x \in G$.

ii) If $S_i$ is a morphism of continuous representations $(E_i, U_{i,E})$ and $(F_i, U_{i,F})$ for $i = 1, 2$, then $S_1 \otimes S_2$ is a morphism between $(E_1 \otimes E_2, U_{1,E} \otimes U_{2,E})$ and $(F_1 \otimes F_2, U_{1,F} \otimes U_{2,F})$.

iii) The category $\mathbf{C}^*-\mathbf{rep}_G$ carries the structure of a $C^*$-tensor category such that

- the tensor product is given by the constructions in i) and ii);
- the associativity isomorphism $a_{(E_i, U_{i,E}),(E_j, U_{j,E}),(E_k, U_{k,E})}$ is the canonical isomorphism $(E_i \otimes E_j) \otimes E_k \to E_i \otimes (E_j \otimes E_k)$ for all $(E_i, U_{i,E}),(E_j, U_{j,E}),(E_k, U_{k,E})$;
- the unit consists of the Hilbert $C^*$-module $C_0(G^0)$ and the canonical isomorphism $s' C_0(G^0) \cong C_0(G) \cong \tau^* C_0(G^0)$;
- the isomorphisms $I_{(E,U)}$ and $r_{(E,U)}$ are the canonical isomorphisms $C_0(G^0) \otimes E \cong E \cong E \otimes C_0(G^0)$ for each $(E, U)$.

Define $p_1, p_2 : G \times G \to G$ by $p_1(x, y) = x$, $p_2(x, y) = y$, $m(x, y) = xy$ and $r_1, r_2 : G \times G \to G^0$ by $r_1(x, y) = r(x), r(x, y) = s(x), s_2(x, y) = s(y)$. Then we have a commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{p_1} & G \\
\downarrow{r} & & \downarrow{r} \\
G^0 & \xrightarrow{m} & G \\
\downarrow{r_1} & & \downarrow{r_2} \\
G^0 & \xrightarrow{s} & G \\
\end{array}
\]

(27)

Lemma 4.23. Let $E$ be a Hilbert $C^*$-module over $C_0(G^0)$ and $U \in \mathcal{L}_{C_0(G)}(s^* E, r^* E)$. Then $U_{xy} = U_{xy}$ for all $x, y \in G \times G$ if and only if $m^* U$ is equal to the composition

\[
s^* E \xrightarrow{p_1^* U} t^* E \xrightarrow{p_2^* U} r_1^* E.
\]

Proof. $(p_1^* U)(p_2^* U)_{(x,y)} = U_{xy}$ for all $(x, y) \in G \times G$. We need the following straightforward result which involves the operators defined in (2):

Lemma 4.24. Let $p : Y \to X$ be a continuous map of locally compact Hausdorff spaces, let $L$ be a Hilbert space with a nondegenerate injective $*$-homomorphism $C_0(Y) \hookrightarrow \mathcal{L}(L)$, and let $\gamma$ be a Hilbert $C^*$-module over $C_0(X)$.

i) There exists an isomorphism $\Phi_{L,\gamma}^f : L \otimes f^* \gamma \to L f^* \otimes \gamma$ of Hilbert spaces given by $\zeta \otimes (\xi \otimes f^* g) \mapsto g\xi f^* \otimes \zeta$.

ii) There exists an isomorphism $\Psi_{L,\gamma}^f : f^* \gamma \to [r_L^f (\gamma) C_0(Y)] \subseteq \mathcal{L}(L, L f^* \otimes \gamma)$ of Hilbert $C^*$-modules over $C_0(Y)$ given by $\zeta \otimes f^* g \mapsto r_L^f (\xi) g$.

iii) For all $U \in \mathcal{L}_{C_0(Y)}(f^* \gamma)$ and $\omega \in f^* \gamma$, we have $\Phi_{L,\gamma}^f (i \partial L \otimes U)(\Phi_{L,\gamma}^f)^* \Psi_{L,\gamma}^f (\omega) = \Psi_{L,\gamma}^f (U \omega)$ in $\mathcal{L}(L, L f^* \otimes \gamma)$.
To each representation \( \gamma \) of \( G \), we functorially associate a corepresentation of \( V \):

**Proposition 4.25.**

i) Let \((E, U)\) be a continuous representation of \( G \). Put \( K := \mathbb{R} \otimes E \) and identify \( E \) with the subspace \( \gamma := \{ r(\xi) \mid \xi \in E \} \subseteq L(\mathbb{R}, K) \) via \( \xi \mapsto r(\xi) \). Then \( s_{K_\delta} \) is a \( \mathbb{C}^*(\beta, b^1) \)-module, we have canonical identifications

\[
H_\gamma \otimes E \cong H_{\rho_\gamma} \otimes \gamma \cong H_{\rho_\gamma} \otimes \delta \cong H_{\rho_\gamma} \otimes b K,
\]

and \( s_{K_\delta} \) together with the unitary \( X := \Phi_H \gamma (id_H \otimes U) \Phi_H \gamma \) form a corepresentation \( F(E, U) := (s_{K_\delta}, X) \) of \( V \).

ii) Let \( T \) be a morphism of continuous representations \((E, U_E), (F, U_F)\) of \( G \). Then \( FT := id_H \otimes T \) is a morphism of the corepresentations \( F(E, U), F(F, U_F) \).

iii) The assignments \((E, U) \mapsto F(E, U)\) and \( T \mapsto FT \) form a functor \( F : \mathbb{C}^*\text{-}\text{rep}_G \rightarrow \mathbb{C}^*\text{-}\text{corep}_G \).

**Proof.**

i) The assertion on \( s_{K_\delta} \) is easily checked. In \( L(H, H_{\alpha} \otimes K) \), we have

\[
\Phi_H \gamma (id_H \otimes U)(\Phi_H \gamma)^*[[\gamma] \triangleright C_0(G)] = \Phi_H \gamma (id_H \otimes U)r_H(s^*\gamma) = \Phi_H \gamma (U s^*\gamma) = \Phi_H \gamma (r^*\gamma) = [\gamma] \triangleright C_0(G).
\]

Note here that \( r_H(\cdot) \) denotes the operator defined in (2), while \( r \) denotes the range map of \( G \). Using the relations \( \gamma = \delta \) and \( \alpha = \beta = [C_0(G)] \), we can conclude

\[
X[\gamma] \triangleright \alpha = [\gamma] \triangleright \alpha = [\alpha] \triangleright \gamma, \quad X[\gamma] \triangleright \beta = [\beta] \triangleright \gamma, \quad X[\beta] \triangleright \delta = X[\gamma] \triangleright \beta = [\delta] \triangleright \beta.
\]

To finish the proof, we have to show that diagram (26) commutes. We apply Lemma 4.24 to the maps \( p = r_1, r_2, s_2 : G_c \times G \rightarrow C_0 \) in diagram (27), the space \( L = H_{\rho_\gamma} \otimes \alpha H \) and the representation \( C_0(G_c \times G) \rightarrow L(L) \) given by multiplication operators, use the relations \( r_1^* = p_1(\alpha \circ \gamma) \) and \( s_2^* = p_2(\beta) \), and find that \( X_1 X_2 3 = \text{equal to the composition} \)

\[
L \otimes \gamma \xrightarrow{\Phi_H \gamma} L \otimes \gamma \xrightarrow{id_L \otimes p_2 U} L \otimes \gamma \xrightarrow{id_L \otimes p_2 U} L \otimes \gamma \xrightarrow{\Phi_H \gamma} L \otimes \gamma.
\]

which coincides by Lemma 4.23 with \( \Phi_H \gamma (id_L \otimes m^* U)(\Phi^*_H \gamma)^* \). Since \( V_1^* p_2^* (f)V_1 = \hat{\Lambda}_1(f) = m^* f \) for all \( f \in C_0(G) \), this composition is equal to \( V_1^* X_3 V_2^* \).

ii) Straightforward.

Conversely, we functorially associate to every corepresentation \( V \) a representation of \( G \). In the formulation of this construction, we apply Lemma 4.24 to \( Y = \hat{G} \) and \( L = H \).

**Proposition 4.26.**

i) Let \( X \) be a corepresentation of \( V \) on a \( \mathbb{C}^*(\beta, b^1) \)-module \( s_{K_\delta} \). Then \( \gamma := \delta, X[\gamma] \triangleright C_0(G) = [\gamma] \triangleright C_0(G) \), and \( \gamma \) together with the unitary \( U := \Psi_{H \gamma} X(\Psi_{H \gamma})^*: s^* \gamma \rightarrow r^* \gamma \) form a continuous representation \( \mathbb{G}(s_{K_\delta}, X) := (\gamma, U) \) of \( G \).
Lemma 4.24 iii) implies that the functors \( G : \mathcal{C}(\mathcal{K}_B, X) \rightarrow \mathcal{C}(\mathcal{K}_B, X) \) form a functor \( \mathbf{G}^* - \mathrm{rep} \rightarrow \mathbf{C}^* - \mathrm{rep}_\mathbf{G} \).

**Proof.** i) Since \( \alpha = \beta \) and \( [\langle \alpha \rangle]_1 \mathcal{I} = X \langle [\gamma]_1 \delta \rangle = \langle [\gamma]_1 \delta \rangle \) as subsets of \( \mathcal{L}(\mathcal{H}, H_{\mathcal{K}(\mathcal{K} \otimes \mathcal{K})}) \), we can conclude \( \gamma = [\mathcal{P}(\mathcal{B}) \gamma] \) and \( [\langle \alpha \rangle_1 \gamma]_1 \mathcal{I} = [\langle \alpha \rangle_1 \gamma]_1 \mathcal{I} = \mathcal{P}(\mathcal{B}) \delta = \delta \). The relation \( \mathcal{X}(\mathcal{H}_2, \mathcal{K} \otimes \mathcal{K}) \) will follow from Example 5.3 ii) and Proposition 5.8. Finally, \( X = \Phi_\mathcal{H}, \mathcal{I}(\mathcal{H}, \Phi_\mathcal{H}) \) and reversing the arguments in the proof of Proposition 4.25, we conclude from \( \mathcal{X}(\mathcal{K}_B, \mathcal{K} \otimes \mathcal{K}) \) that \( \mathcal{P}_1 U \circ \mathcal{P}_2 U = m' U \). By Lemma 4.23, \( U \) is a representation on \( \gamma \).

ii), iii) Straightforward. \( \square \)

We define an equivalence between \( \mathcal{C}^* - \mathrm{tensor} \) categories to be an equivalence of the underlying \( \mathcal{C}^* \) -categories and tensor categories [20].

**Theorem 4.27.** The functors \( \mathbf{G}^* - \mathrm{rep} \rightarrow \mathbf{G}^* - \mathrm{rep}_\mathbf{G} \) extend to an equivalence of \( \mathcal{C}^* - \mathrm{tensor} \) categories.

**Proof.** Lemma 4.24 iii) implies that the functors \( \mathbf{F}, \mathbf{G} \) form equivalences of categories. The verification of the fact that they preserve the monoidal structure is tedious but straightforward. \( \square \)

**Remark 4.28.** Let us note that a similar equivalence holds between the categories of measurable representations of \( G \) and \( W^* - \mathrm{corep} \) of \( V \).

### 5 Regular, proper and étale \( \mathcal{C}^* \)-pseudo-multiplicative unitaries

In this section, we study particular classes of \( \mathcal{C}^* - \mathrm{pseudo-multiplicative} \) unitaries. As before, let \( \mathfrak{h} = (\mathcal{H}, \mathcal{B}, \mathcal{B}^1) \) be a \( \mathcal{C}^* \) -base, let \( (H, \mathcal{B}, \alpha, \beta) \) be a \( \mathcal{C}^* - (\mathfrak{A}, \mathfrak{B}, \mathfrak{B}^1) \)-module, and let \( V : H^\mathfrak{A}_\mathfrak{B} \otimes \mathfrak{A} \mathcal{H} \rightarrow H^\mathfrak{A}_\mathfrak{B} \mathcal{H} \) be a \( \mathcal{C}^* - \mathrm{pseudo-multiplicative} \) unitary.

#### 5.1 Regularity

In [3], Baaj and Skandalis showed that the pairs \( (\mathcal{A}_\mathcal{V}, \mathcal{A}_\mathcal{V}) \) and \( (\mathcal{A}_\mathcal{V}, \Delta \mathcal{V}) \) associated to a multiplicative unitary \( V \) on a Hilbert space \( H \) form Hopf \( \mathcal{C}^* \) -algebras if the unitary satisfies the regularity condition \( [\mathcal{H}(\mathcal{V})/\mathcal{H}]_1 = \mathcal{K}(\mathcal{H}) \). This condition was generalized by Baaj in [1, 2] and extended to pseudo-multiplicative unitaries by Enock [10].

We now formulate a generalized regularity condition for \( \mathcal{C}^* - \mathrm{pseudo-multiplicative} \) unitaries and show that the pairs \( (\mathcal{A}_\mathcal{V}^\alpha, \mathcal{A}_\mathcal{V}^\alpha) \) and \( (\mathcal{A}_\mathcal{V}^\beta, \Delta \mathcal{V}^\beta) \) associated to such
a unitary $V$ in subsection 3.4 are concrete Hopf $C^*$-bimodules if $V$ is regular. This regularity condition involves the space

$$C_V := [(\alpha_1|V|\alpha_2) \subseteq L(H).$$

**Proposition 5.1.** We have

$$[C_V C_V] = C_V, \quad C_V \alpha_\alpha = C_V, \quad [C_V \alpha_\beta] = \alpha, \quad [C_V \beta_\beta] = \beta, \quad [C_V \beta_\beta] = \beta.$$  

**Proof.** The proof is completely analogous to the proof of Proposition 3.11; for example, the first equation follows from the commutativity of the following two diagrams:

![Diagram](image)

**Definition 5.2.** A $C^*$-pseudo-multiplicative unitary $(b, H, \hat{\beta}, \alpha, \beta, V)$ is semi-regular if $C_V \supseteq [\alpha \alpha]$, and regular if $C_V = [\alpha \alpha]$.

**Examples 5.3.**

i) By Proposition 5.1, $V$ is (semi-)regular if and only if $V^{op}$ is (semi-)regular.

ii) The $C^*$-pseudo-multiplicative unitary associated to a locally compact Hausdorff groupoid $G$ as in Theorem 2.7 is regular. To prove this assertion, we use the notation introduced in subsection 2.3 and calculate that for each $\xi, \xi' \in C_c(G)$, $\zeta \in C_c(G) \subseteq L^2(G, v)$, $y \in G$,

$$\langle (j_1(\xi')|1V|j_1(\xi))2\xi \rangle(y) = \int_{G^{(y)}} \overline{\xi(x)}\xi(x^{-1})y d\lambda^{(y)}(x),$$

$$\langle j_1(\xi')j_1(\xi)^*\xi \rangle(y) = \xi(y) \int_{G^{(y)}} \overline{\xi(x)}\xi(x) d\lambda^{(y)}(x).$$

Using standard approximation arguments, we find $[(\alpha_1|V|\alpha_2)] = [S(C_c(G, \times, G))] = [\alpha \alpha]$, where for each $\omega \in C_c(G, \times, G)$, the operator $S(\omega)$ is given by

$$S(\omega)\xi(y) = \int_{G^{(y)}} \omega(x,y)\xi(x) d\lambda^{(y)}(x) \quad \text{for all} \quad \zeta \in C_c(G), \quad y \in G.$$
iii) In [35], we introduce compact \( C^\ast \)-quantum groupoids and construct for each such quantum groupoid a \( C^\ast \)-pseudo-multiplicative unitary that turns out to be regular.

We shall now deduce several properties of semi-regular and regular \( C^\ast \)-pseudo-multiplicative unitaries, using commutative diagrams as explained in Notation 4.9.

**Proposition 5.4.** If \( V \) is semi-regular, then \( C_V \) is a \( C^\ast \)-algebra.

**Proof.** Assume that \( V \) is regular. Then the following two diagrams commute, whence \([C_V C^\ast_V] = \langle (\alpha)_1 (\alpha)_2 V_{23} (\alpha)_1 (\alpha)_2 \rangle = C_V^\ast\):

![Diagrams](image)

Now, assume that \( V \) is semi-regular. Then cell (R) in the first diagram need not commute, but still \( \langle (\alpha)_1 (\alpha)_2 \rangle \subseteq \langle [\alpha]_{2} V_{23} [\alpha]_3 \rangle \) and hence \([C_V C^\ast_V] \subseteq \langle [\alpha]_1 [\alpha]_1 V_{23} [\alpha]_1 [\alpha]_2 \rangle = C_V^\ast\). A similar argument shows that also \([C^\ast_V C_V] \subseteq C_V^\ast\), and from Proposition 3.11 and [1, Lemme 3.3], it follows that \( C_V \) is a \( C^\ast \)-algebra.

**Proposition 5.5.** Assume that \( C_V = C_V^\ast\).

i) Let \((\gamma, X)_{\tilde{K}}\) be a representation of \( V \). Then \((\hat{A}_X)_{\hat{K}}\tilde{\beta}\) is a \( C^\ast-(b, b^\dagger)\)-algebra.

ii) \((\hat{A}^\alpha_{V})_{\hat{H}}\tilde{\beta}\) is a \( C^\ast-(b, b^\dagger)\)-algebra and \((A_V)_{\hat{H}}\tilde{\alpha}\) a \( C^\ast-(b^\dagger, b)\)-algebra.

The proof uses the following central lemma:
Lemma 5.6. Let \((\gamma K_\delta, X)\) be a representation of \(V\). Then \([X(1 \otimes C_V)X^*|\beta]\) is \([|\beta]\) in \((\gamma K_\delta, X)\).

Proof. The following diagram commutes and shows that we have \([X(1 \otimes C_V)X^*|\beta]\) is \([|\beta]\) in \((\gamma K_\delta, X)\).

\[
\begin{align*}
K & \quad \xrightarrow{|\beta|_2} \quad K_{\gamma} \otimes b_H \\
K_{\gamma} \otimes b_H & \quad \xrightarrow{|\alpha|_2} \quad K_{\gamma} \otimes b_H \\
\hat{\alpha} & \quad \xrightarrow{|\alpha|_2} \quad \hat{\alpha} \otimes (H_a \otimes b_H) \\
\hat{\alpha} \otimes (H_a \otimes b_H) & \quad \xrightarrow{|\beta|_2} \quad \hat{\alpha} \otimes (H_a \otimes b_H) \\
K_{\gamma} \otimes b_H & \quad \xrightarrow{|\beta|_2} \quad K_{\gamma} \otimes b_H \\
\end{align*}
\]

Indeed, cell (P) commutes by (19), and the remaining cells because of (18) or by inspection.

Proof of Proposition 5.5. i) By Proposition 4.10, it suffices to show that \(\hat{\alpha} X = \hat{\alpha} X\). But by Proposition 4.10 and Lemma 5.6, \(\hat{\alpha} X = (p_{\alpha}(B_\delta))\hat{\alpha} X = |\beta|_2 X(1 \otimes C_V)X^*|\beta|_2\).

ii) Statement i) applied to \((\gamma K_\delta, X) = (\alpha H_b, V)\) yields the first assertion. The second one follows after replacing \(V\) by \(V^{op}\), where we use Propositions 3.11 and 5.1.

The main result of this subsection is the following:

Theorem 5.7. If \(V\) is semi-regular, then \((\hat{\alpha} V)_{\delta, b}, \hat{\alpha} V)\) and \((\alpha V)_{\delta, a, \delta V}\) are normal Hopf \(C^*\)-bimodules.

Proof. We prove the assertion concerning \((\hat{\alpha} V)_{\delta, b}, \hat{\alpha} V)\); for \((\alpha V)_{\delta, a, \delta V),\) the arguments are similar. By Proposition 5.5, \((\hat{\alpha} V)_{\delta, b}\) is a \(C^*\)-\((b, b^1)\)-algebra, and by Proposition 4.10, applied to \(V^{op} = V^\vee, \) we have \(\Delta_{\delta, b}(\hat{\alpha} V) \subseteq (\hat{\alpha} V)_{\delta, b} \oplus (\hat{\alpha} V)\). Now, the claim follows from Lemma 3.13.

We collect several auxiliary results on regular \(C^*\)-pseudo-multiplicative unitaries.

Proposition 5.8. Assume that \(V\) is regular.

i) Let \((\gamma K_\delta, X)\) be a representation of \(V\). Then \([X(1 \otimes C_V)X^*|\beta]\) is \([|\beta]\) in \((\gamma K_\delta, X)\) and \([X(1 \otimes C_V)X^*|\beta]\) is \([|\beta]\) in \((\gamma K_\delta, X)\).

ii) \([V(1 \otimes C_V)X^*|\beta]\) is \([|\beta]\) in \((\gamma K_\delta, X)\) and \([V(1 \otimes C_V)X^*|\beta]\) is \([|\beta]\) in \((\gamma K_\delta, X)\).
Proof: Using Lemma 5.6 and the relation $\hat{A}_X = \hat{A}_X^*$ (Proposition 5.5), we find that $[X|\alpha\rangle_2\hat{A}_X] = [X|\alpha\rangle_2(\alpha_2X^*|\beta\rangle_2) = [X(1\otimes C_V)X^*|\beta\rangle_2] = [\beta\rangle_2\hat{A}_X].$

Replacing $(\gamma,b\delta,X)$ by $(a\beta^*,V),$ we obtain the first equation in ii), and replacing $V$ by $V^{op}$ and using Proposition 3.11, we obtain $[\Sigma V^*\Sigma|\alpha\rangle_2\hat{A}_V^*] = [\hat{\beta}\rangle_2\hat{A}_V^*], which yields $[V^*|\beta\rangle_1A_V] = [\alpha\rangle_1A_V].$

Finally, let us prove the equation $[X|\tilde{\delta}\rangle_1|\alpha\rangle_1A_V] = [\alpha\rangle_2|\gamma\rangle_1A_V].$ The following commutative diagram shows that $[X|\tilde{\delta}\rangle_1|\alpha\rangle_1A_V] = [\alpha\rangle_2|\gamma\rangle_1A_V]:$

Moreover, also the following diagram commutes,

and hence $[X|\tilde{\delta}\rangle_1|\alpha\rangle_1A_V] = [(\alpha\rangle_2X_1|\tilde{\delta}\rangle_1|\alpha\rangle_1A_V] = [(\alpha\rangle_2|\gamma\rangle_1A_V] = [\gamma\rangle_1A_V].$

The last result in this subsection involves the algebras $\tilde{A}_V = [\beta^*\gamma\alpha]$ and $\tilde{A}_{1_{V^{op}}} = [\beta^*\gamma\alpha]$ associated to the trivial representations of $V$ and $V^{op},$ respectively.

**Proposition 5.9.** If $V$ is regular, then $[\beta\hat{A}_V] = [\alpha\hat{A}_V]$ and $[\beta\hat{A}_{1_{V^{op}}} = [\alpha\hat{A}_{1_{V^{op}}}].$

**Proof.** The following diagram commutes

and shows that $[\beta\hat{A}_V] = [\beta\hat{A}_V] = [\beta\hat{A}_V] = [\alpha\hat{A}_V] = [\beta\hat{A}_V] = [\alpha\hat{A}_V].$ The second equation follows by replacing $V$ with $V^{op}.$
5.2 Proper and étale $C^*$-pseudo-multiplicative unitaries

In [3], Baaj and Skandalis characterized multiplicative unitaries that correspond to compact or discrete quantum groups by the existence of fixed or cofixed vectors, respectively, and showed that from such vectors, one can construct a Haar state and a counit on the associated legs. We adapt some of their constructions to $C^*$-pseudo-multiplicative unitaries as follows.

Given a $C^*$-$b^{(1)}$-module $K$, let $M(\gamma) = \{ T \in L(\mathfrak{F}, K) \mid T \mathfrak{B}^{(1)} \subseteq \gamma, T^*\gamma \subseteq \mathfrak{B}^{(1)} \}$.

**Definition 5.10.** A fixed element for $V$ is an operator $\eta \in M(\hat{\mathfrak{B}}) \cap M(\alpha) \subseteq L(\mathfrak{F}, H)$ satisfying $V|\eta_1 = |\eta_1$. A cofixed element for $V$ is an operator $\xi \in M(\alpha) \cap M(\hat{\mathfrak{B}}) \subseteq L(\mathfrak{F}, H)$ satisfying $V|\xi_2 = |\xi_2$. We denote the set of all fixed/cofixed elements for $V$ by $\Fix(V)\cofix(V)$.

**Example 5.11.** Let us consider the $C^*$-pseudo-multiplicative unitary associated to a locally compact, Hausdorff, second countable groupoid $G$ in subsection 2.3. We identify $M(L^2(G, \lambda))$ in the natural way with the completion of the space

$$\left\{ f \in C(G) \mid r: \text{supp} f \to G \text{ is proper}, \sup_{u \in G^0} \int_{G^1} |f(x)|^2 d\lambda^u(x) \text{ is finite} \right\}$$

with respect to the norm $f \mapsto \sup_{u \in G^0} \left( \int_{G^1} |f(x)|^2 d\lambda^u(x) \right)^{1/2}$. Similarly as in [34, Lemma 7.11], one easily verifies that

i) $\eta_0 \in M(L^2(G, \lambda))$ is a fixed element for $V$ if and only if for each $u \in G^0$, $\eta_0|_{G^1(u)} = 0$ almost everywhere with respect to $\lambda^u$;

ii) $\xi_0 \in M(L^2(G, \lambda))$ is a cofixed element for $V$ if and only if $\xi_0(x) = \xi_0(s(x))$ for all $x \in G$.

Let us collect some easy properties of fixed and cofixed elements.

**Remarks 5.12.**

i) $\Fix(V) = \cofix(V)\op$ and $\cofix(V) = \Fix(V)\op$.

ii) $\Fix(V)\ast\Fix(V)$ and $\cofix(V)\ast\cofix(V)$ are contained in $M(\mathfrak{B}) \cap M(\mathfrak{B}\op)$.

iii) The relations $\Fix(V) \subseteq M(\hat{\mathfrak{B}}) \cap M(\alpha)$ imply $\rho_{\hat{\mathfrak{B}}} (\mathfrak{B}\ast) \Fix(V) = (\Fix(V)\mathfrak{B}^{(1)} \subseteq \hat{\mathfrak{B}}$ and $\rho_{\mathfrak{B}}(\mathfrak{B}) (\Fix(V) = (\Fix(V)\ast \mathfrak{B} \subseteq \alpha).$ Likewise, we have $\rho_{\mathfrak{B}}\ast (\mathfrak{B}\ast) \Fix(V) \subseteq \mathfrak{B}$.

**Lemma 5.13.** Let $\xi, \xi' \in \cofix(V)$ and $\eta, \eta' \in \Fix(V)$. Then

$$\langle \xi|_2 V|\xi' \rangle_2 = \rho_{\alpha}(\xi, \xi') = \rho_{\beta}(\xi, \xi') \quad \langle \eta|_1 V|\eta' \rangle_1 = \rho_{\beta}(\eta, \eta') = \rho_{\alpha}(\eta, \eta').$$

**Proof.** Let $\xi \in H$. Then $\langle \xi|_2 V|\xi' \rangle_2 = \langle \xi|_2 \xi' \rangle_2 = \rho_{\alpha}(\xi, \xi') \xi$ and $\langle \xi|_2 V|\xi' \rangle_2 = \langle \xi|_2 \xi' \rangle_2 = \rho_{\beta}(\xi, \xi') \xi$. The second equation follows similarly.

**Proposition 5.14.**

i) $\rho_{\hat{\mathfrak{B}}}(M(\mathfrak{B})) \cofix(V) \subseteq \cofix(V) \ast \cofix(V)$ and $\rho_{\mathfrak{B}}(\mathfrak{B}) \Fix(V) \subseteq \Fix(V)$.

ii) $[\cofix(V)\ast\cofix(V)] = \cofix(V)$ and $[\Fix(V)\ast\Fix(V)] = \Fix(V)$.

iii) $[\cofix(V)\ast\cofix(V)]$ and $[\Fix(V)\ast\Fix(V)]$ are $C^*$-subalgebras of $M(\mathfrak{B}) \cap M(\mathfrak{B}\op)$; in particular, they are commutative.
Proof. We only prove the assertions concerning $\text{Cofix}(V)$; the other assertions follow similarly.

i) Let $T \in M(\mathcal{B})$ and $\xi \in \text{Cofix}(V)$. Then $\rho_\beta(T)\xi \subseteq M(\beta) \cap M(\alpha)$ because $\rho_\beta(\mathcal{B})\beta \subseteq \beta$ and $\rho_\beta(\mathcal{B})\alpha \subseteq \alpha$. The relation $V(\tilde{\beta} \triangleright \tilde{\beta}) = \alpha \triangleright \tilde{\beta}$ furthermore implies

$$V|\rho_\beta(T)\xi|^2 = V\rho(\tilde{\beta} \triangleright \tilde{\beta})(T)|\tilde{\beta} \triangleright \tilde{\beta}|^2 = \rho(\tilde{\omega} \triangleright \tilde{\omega})(T)V|\tilde{\omega} \triangleright \tilde{\omega}|^2 = |\rho_\beta(T)\xi|^2.$$

ii) Using i) and the relation $\text{Cofix}(V) \subseteq M(\tilde{\beta})$, we find that

$$[\text{Cofix}(V)\text{Cofix}(V)^* \text{Cofix}(V)] \subseteq [\text{Cofix}(V)M(\mathcal{B}^\dagger)]$$

$$= [\rho_\beta(M(\mathcal{B}^\dagger)) \text{Cofix}(V)] \subseteq \text{Cofix}(V).$$

Therefore, $[\text{Cofix}(V)^* \text{Cofix}(V)]$ is a $C^*$-algebra and $\text{Cofix}(V)$ is a Hilbert $C^*$-module over $[\text{Cofix}(V)^* \text{Cofix}(V)]$. Now, [17, p. 5] implies that the inclusion above is an equality.

iii) This follows from ii) and Remark 5.12 ii).

Definition 5.15. We call the $C^*$-pseudo-multiplicative unitary $V$ étale if $\eta^*\eta = \text{id}_\mathcal{A}$ for some $\eta \in \text{Fix}(V)$, proper if $\xi^*\xi = \text{id}_\mathcal{A}$ for some $\xi \in \text{Cofix}(V)$, and compact if it is proper and $\mathcal{B}, \mathcal{B}^\dagger$ are unital.

Example 5.16. The $C^*$-pseudo-multiplicative unitary associated to a locally compact, Hausdorff, second countable groupoid $G$ (Theorem 2.7) is étale/proper/compact if and only if $G$ is étale/proper/compact. This follows from similar arguments as in [34, Theorem 7.12].

Remarks 5.17. i) By Remark 5.12, $V$ is étale/proper if and only if $V^{op}$ is proper/étale.

ii) If $V$ is proper, then $\text{id}_\mathcal{B} \in \widehat{A}_V$; if $V$ is étale, then $\text{id}_\mathcal{B} \in \hat{A}_V$. This follows directly from Lemma 5.13.

The first main result of this subsection shows how one can construct a counit on $((\widehat{A}_V)^{\alpha,\beta}_H, \widehat{\Delta}_V)$ from a fixed element for $V$.

Theorem 5.18. Let $V$ be an étale $C^*$-pseudo-multiplicative unitary.

i) There exists a unique contractive homomorphism $\hat{\varepsilon}: \widehat{A}_V \to \hat{A}_1$ such that $\hat{\pi}_1 = \hat{\varepsilon} \circ \hat{\pi}_V : \Omega_{\beta,\alpha} \to \hat{A}_1$.

ii) Assume that $V$ is regular. Then $\hat{\varepsilon}$ is a jointly normal morphism from $(\widehat{A}_V)^{\alpha,\beta}_H$ to $([\hat{A}])^\beta_{\hat{\pi}}$, and a bounded counit for $(\widehat{A}_V)^{\alpha,\beta}_H, \widehat{\Delta}_V)$.

Proof. Choose an $\eta_0 \in \text{Fix}(V)$ with $\eta_0^*\eta_0 = \text{id}_\mathcal{A}$ and define $\hat{\varepsilon}: \widehat{A}_V \to L(\mathcal{A})$ by $\hat{a} \mapsto \eta_0^*\eta_0$. Then $\hat{\varepsilon}$ is contractive. For all $\xi \in \alpha$, $\eta \in \beta, \xi \in \mathcal{A}$,

$$\langle \eta \xi^*V |\xi|^2 \eta_0^*\xi \rangle = \langle \eta \xi^*V(\eta_0 \xi \otimes \xi \xi) \rangle = \eta_0^*(\eta^*\xi)\xi^*,$$

and hence $\hat{\pi}(\omega)\eta_0 = \eta_0 \hat{\pi}_1(\omega)$ for all $\omega \in \Omega_{\beta,\alpha}$. In particular, $\hat{\varepsilon}(\hat{\pi}_V(\omega)) = \eta_0^*\hat{\pi}_V(\omega)\eta_0 = \eta_0^*\eta_0\hat{\pi}_1(\omega) = \hat{\pi}_1(\omega)$.

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Proof. For all $\hat{\alpha} \in \mathcal{L}^2(\beta \hat{\beta}, \beta \hat{\beta} H^{\beta})$, and $[\eta_0^* \alpha] \supseteq [\eta_0^* \eta_0 \beta] = \mathcal{B}$ and $[\eta_0^* \hat{\beta}] \supseteq [\eta_0^* \eta_0 \mathcal{B}^1] = \mathcal{B}^1$. It remains to show that diagram (14) commutes. Clearly, $(\tilde{e} + \text{id}) (x) = \langle \eta_0^{|s_x} \eta_0 \rangle$ and $(\text{id} + \tilde{e}) (x) = \langle \eta_0^{|s_x} \eta_0 \rangle$ for all $x \in (\hat{A}_V)_{\hat{\beta}} \hat{\beta} \hat{\alpha}(\hat{A}_V)$. Now, the left square in diagram (14) commutes because for all $\hat{\alpha} \in \hat{A}_V$, 

\[
\langle \eta_0^{|s_{\hat{\alpha}}} \hat{A}_V (\hat{\alpha}) \eta_0 \rangle = \langle \eta_0^{|s_{\hat{\alpha}}} \hat{A}_V (\hat{\alpha}) \eta_0 \rangle = \langle \eta_0^{|s_{\hat{\alpha}}} (1 \otimes \hat{a}) \eta_0 \rangle = \langle \eta_0^{|s_{\hat{\alpha}}} (1 \otimes \hat{a}) \eta_0 \rangle = \rho_B (\eta_0^* \eta_0) \hat{\alpha} = \hat{\alpha}.
\]

To see that the left square in diagram (14) commutes, let $\eta \in \beta, \hat{\xi} \in \alpha$ and consider the following diagram:

\[
\begin{array}{cccccc}
H & \overset{|(\xi)\rangle_2}{\longrightarrow} & H_{\hat{\beta} \hat{\alpha}} \alpha \ H & \overset{V}{\longrightarrow} & H_{\hat{\beta} \hat{\alpha}} \alpha \ H & \overset{\text{id}}{\longrightarrow} & H_{\hat{\beta} \hat{\alpha}} \alpha \ H \overset{(|\eta|)_{\langle 2 \rangle}}{\longrightarrow} & H \\
\downarrow{|\eta|_{\langle 2 \rangle}} & & \downarrow{|\eta|_{\langle 2 \rangle}} & & \downarrow{|\eta|_{\langle 2 \rangle}} & & \downarrow{|\eta|_{\langle 2 \rangle}} & \downarrow{|\eta|_{\langle 2 \rangle}} \\
H_{\hat{\beta} \hat{\alpha}} \alpha \ H & \overset{\text{id}}{\longrightarrow} & H_{\hat{\beta} \hat{\alpha}} \alpha \ H & \overset{\text{id}}{\longrightarrow} & H_{\hat{\beta} \hat{\alpha}} \alpha \ H & \overset{\text{id}}{\longrightarrow} & H_{\hat{\beta} \hat{\alpha}} \alpha \ H & \overset{\text{id}}{\longrightarrow} & H_{\hat{\beta} \hat{\alpha}} \alpha \ H \\
\end{array}
\]

The lower cell commutes by Lemma 4.14, cell (*) commutes because $\mathcal{V}_{\mathcal{O}}(\eta_0) = (1, \mathcal{V}_{\mathcal{O}}(\eta_0))$, and the other cells commute as well. Since $\eta \in \beta$ and $\hat{\xi} \in \alpha$ were arbitrary, the claim follows. \(\square\)

As an example, we consider the unitary associated to a groupoid (subsection 2.3).

**Proposition 5.19.** Let $G$ be a locally compact, Hausdorff, second countable groupoid and let $\mathcal{A} : H_{\hat{\beta} \hat{\alpha}} \alpha \ H \rightarrow H_{\hat{\beta} \hat{\alpha}} \alpha \ H$ be the associated $C^*$-pseudo-multiplicative unitary.

i) Let $G$ be étale. Then $V$ is étale, $\hat{A}_V \cong C_0(G)$, $\hat{A}_1 \cong C_0(G^0)$, and $\tilde{e} : \hat{A}_V \rightarrow \hat{A}_1$ is given by the restriction of functions on $G$ to functions on $G^0$.

ii) Let $G$ be proper. Then $\mathcal{V}_{\mathcal{O}}$ is étale, $A_V = A_{\mathcal{V}_{\mathcal{O}}} = C^*_r (G)$, and for each $f \in C_c (G)$, the operator $\tilde{e} (L(f)) \in \mathcal{L}(L^2 (G^0, \mu))$ is given by

\[
(\tilde{e} (L(f)) \xi) (u) = \mathcal{F} \int_{G^0} f (x) D^{-1/2} (x) \xi (s(x)) \, d\lambda^u (x) \quad \text{for all } \xi \in L^2 (G^0, \mu), \ u \in G.
\]

**Proof.** For all $\xi, \xi' \in C_c (G), \xi' \in L^2 (G^0, \mu)$ and $u \in G^0$, we have by Lemma 3.18

\[
(\tilde{e} (m(\xi \ast \xi'))) (u) = (\tilde{e} (\hat{a} \hat{g} \hat{m}')) (u) = (j(\xi)^* j(\xi') \xi) (u) = \int_{G^0} \xi (x) \xi' (x) \, d\lambda^u (x) = (\xi \ast \xi') (u),
\]

\[
(\tilde{e} (L(\xi))) (u) = (\tilde{e} (a \hat{g} \hat{m}')) (u) = (j(\xi)^* j(\xi') \xi) (u) = \int_{G^0} \xi (x) D^{-1/2} (x) \xi (s(x)) \, d\lambda^u (x). \quad \square
\]

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The second main result of this subsection shows how one can construct a Haar weight on \((\hat{\mathcal{A}}_V)_H^{\alpha,\beta}, \hat{\Delta}_V\) from a cofixed element for \(V\).

**Theorem 5.20.** Let \(V\) be a proper regular \(C^*\)-pseudo-multiplicative unitary. Then there exists a normal bounded left Haar weight \(\phi\) for \((\hat{\mathcal{A}}_V)_H^{\alpha,\beta}, \hat{\Delta}_V\).

**Proof.** Choose \(\xi_0 \in \text{Cofix}(V)\) with \(\xi_0^* \xi_0 = \text{id}_\beta\). By Proposition 3.11 and Remark 5.12 i), \(\xi_0^* \hat{A}_V \xi_0 = [\xi_0^* \rho_\alpha(\mathcal{B})^\dagger \hat{A}_V \rho_\alpha(\mathcal{B})^\dagger \xi_0] = [\beta^* \hat{A}_V \beta] \subseteq \mathcal{B}^\dagger\). Hence, we can define a completely positive map \(\phi\): \(\hat{A}_V \rightarrow \mathcal{B}^\dagger\) by \(\hat{a} \mapsto \xi_0^* \hat{a} \xi_0\), and \(\phi \in \Omega_M(\hat{\Delta}_V)\). For all \(\hat{a} \in \hat{A}_V\), \((\text{id} \comp \phi)(\hat{\Delta}_V(\hat{a})) = (\xi_0|_2 V^* (\text{id} \comp \hat{a}) V(\xi_0)|_2 = (\xi_0|_2 (\text{id} \comp \hat{a}) \xi_0)|_2 = \rho_\alpha(\xi_0^* \hat{a} \xi_0)\). \(\square\)

As an example, we again consider the unitary associated to a groupoid.

**Proposition 5.21.** Let \(G\) be a locally compact, Hausdorff, second countable groupoid and let \(V\): \(H_{\hat{b}_b^G} \otimes H \rightarrow H_{\hat{a}_a^G} \otimes H\) be the associated \(C^*\)-pseudo-multiplicative unitary.

i) Let \(G\) be proper. Then \(V\) is proper, \(\hat{\Delta}_V \cong C_\ell(G)\), and the map \(\phi\): \(\hat{A}_V \rightarrow C_\ell(G^0)\) given by \((\phi(f))(\alpha) = \int_{G^0} f(x) \, d\lambda^\alpha(x)\) is a normal bounded left Haar weight for \((\hat{\mathcal{A}}_V)_H^{\alpha,\beta}, \hat{\Delta}_V\).

ii) Let \(G\) be étale. Then \(V^{op}\) is proper and there exists a normal bounded left and right Haar weight \(\phi\) for \((\hat{\mathcal{A}}_V)_H^{\alpha,\beta}, \hat{\Delta}_V\) given by \(L(f) \mapsto f|_{C^\ell(G)}\) for all \(f \in C^\ell_c(G)\).

**Proof.** This follows from Theorem 5.20 and similar calculations as in 5.19. \(\square\)

**References**


