Abstract

We study coactions of concrete Hopf C*-bimodules in the framework of (weak) C*-pseudo-Kac systems, define reduced crossed products and dual coactions, and prove an analogue of Baaj-Skandalis duality.

1 Introduction

In a seminal article [1], Baaj and Skandalis developed a duality theory for coactions of Hopf C*-algebras on C*-algebras that extends the Takesaki-Takai duality of actions of locally compact abelian groups to all locally compact groups and, more generally, to all regular locally compact quantum groups. More precisely, Baaj and Skandalis introduce the notion of a Kac system which consists of a regular multiplicative unitary and an additional symmetry, consider coactions of the two Hopf C*-algebras (the “legs”) of the multiplicative unitary, define for every coaction of each leg a reduced crossed product that carries a coaction of the other leg, and show that two applications of this construction yield a stabilization of the original coaction.

In this article, we extend the duality theory of Baaj and Skandalis to coactions of concrete Hopf C*-bimodules, applying the methods and concepts introduced in [10] to the constructions in [1]. In particular, our theory covers coactions of the Hopf C*-bimodules associated to a locally compact groupoid. An article on examples is in preparation.

Let us mention that a similar duality theory like the one presented here was developed in the PhD thesis of the author [9]. Our new approach allows us to drop a rather restrictive condition (decomposability) needed in [9], and to work in the framework of C*-algebras instead of the somewhat exotic C*-families. Moreover, the approach presented here greatly simplifies the complex assumptions needed [9].

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Organization

This article is organized as follows:

First, we fix notation and terminology, and summarize some background on (Hilbert) C*-modules.

In Section 2, we recall the definition of C*-pseudo-multiplicative unitaries and concrete Hopf C*-bimodules given in [10].

In Section 3, we introduce weak C*-pseudo-Kac systems which provide the framework for the construction of reduced crossed products.
In Section 4, we consider coactions of concrete Hopf $C^*$-bimodules and construct reduced crossed products and duals for such coactions.

In Section 5, we introduce $C^*$-pseudo-Kac systems and establish an analogue of Baaj-Skandalis duality for coactions of concrete Hopf $C^*$-bimodules.

In Section 6, we associate to each locally compact groupoid a $C^*$-pseudo-Kac system.

**Preliminaries**

Given a subset $Y$ of a normed space $X$, we denote by $[Y] \subseteq X$ the closed linear span of $Y$.

Given a Hilbert space $H$ and a subset $X \subseteq \mathcal{L}(H)$, we denote by $X'$ the commutant of $X$. Given Hilbert spaces $H$, $K$, a $C^*$-subalgebra $A \subseteq \mathcal{L}(H)$, and a $*$-homomorphism $\pi: A \to \mathcal{L}(K)$, we put

$$\mathcal{L}'(H,K) := \{ T \in \mathcal{L}(H,K) \mid T a = \pi(a) T \text{ for all } a \in A \};$$

thus, for example, $A' = \mathcal{L}^{\text{id}_A}(H)$.

We shall make extensive use of (right) $C^*$-modules, also known as Hilbert $C^*$-modules or Hilbert modules. A standard reference is [4].

All sesquilinear maps like inner products of Hilbert spaces or $C^*$-modules are assumed to be conjugate-linear in the first component and linear in the second one.

Let $A$ and $B$ be $C^*$-algebras. Given $C^*$-modules $E$ and $F$ over $B$, we denote the space of all adjointable operators $E \to F$ by $\mathcal{L}_B(E,F)$.

Let $E$ and $F$ be $C^*$-modules over $A$ and $B$, respectively, and let $\pi: A \to \mathcal{L}_B(F)$ be a $*$-homomorphism. Then one can form the internal tensor product $E \otimes \pi F$, which is a $C^*$-module over $B$ [4, Chapter 4]. This $C^*$-module is the closed linear span of elements $\eta \otimes_A \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \otimes_A \xi \mid \eta' \otimes_A \xi' \rangle = \langle \xi \mid \pi(\eta') \xi' \rangle$ and $\langle \eta \otimes_A \xi \mid b \rangle = \eta \otimes_A \xi b$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. We denote the internal tensor product by "$\otimes$"; thus, for example, $E \otimes_F F = E \otimes_A F$. If the representation $\pi$ or both $\pi$ and $A$ are understood, we write "$\otimes_A$" or "$\otimes_B$", respectively, instead of "$\otimes$".

Given $E$, $F$ and $\pi$ as above, we define a flipped internal tensor product $F \otimes \pi E$ as follows.

We equip the algebraic tensor product $F \otimes E$ with the structure maps $\langle \xi \otimes \eta \mid \xi' \otimes \eta' \rangle := \langle \xi \mid \pi(\eta') \xi' \rangle$ and $\langle \eta \otimes b \rangle := \xi b \otimes \eta$, and by factoring out the null-space of the semi-norm $\xi \mapsto \| \xi \|^{1/2}$ and taking completion, we obtain a $C^*$-$B$-module $F \otimes \pi E$. This is the closed linear span of elements $\xi \otimes \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \xi \otimes \eta \mid b \rangle := \langle \xi \mid \pi(\eta) b \rangle$ for all $\eta \in E$, $\xi \in F$, and $b \in B$. As above, we write "$\otimes_A^*$" or simply "$\otimes^*$" instead of "$\otimes$" if the representation $\pi$ or both $\pi$ and $A$ are understood, respectively.

Evidently, the usual and the flipped internal tensor product are related by a unitary map $\Sigma: F \otimes \pi E \to \pi E \otimes F$, $\eta \otimes \xi \mapsto \xi \otimes \eta$.

We shall frequently consider the following kind of $C^*$-modules. Let $H$ and $K$ be Hilbert spaces. We call a subset $\Gamma \subseteq \mathcal{L}(H,K)$ a concrete $C^*$-module if $[\Gamma^* \Gamma] = \Gamma$. If $\Gamma$ is a concrete $C^*$-module, then evidently $\Gamma^*$ is a concrete $C^*$-module as well, the space $\overline{\Gamma} := [\Gamma^* \Gamma]$ is a $C^*$-algebra, and $\Gamma$ is a full right $C^*$-module over $\overline{\Gamma}$ with respect to the inner product given by $\langle \xi | \zeta' \rangle = \zeta^* \zeta'$ for all $\zeta, \zeta' \in \Gamma$.

2 $C^*$-pseudo-multiplicative unitaries and concrete Hopf $C^*$-bimodules

We recall several constructions and definitions from [10] which are fundamental to the duality theory developed in the following sections.
**C*-bases and C*-factorizations**  C*-bases and C*-factorizations are simple but convenient concepts used in the definition of the C*-relative tensor product; for details, see [10, Section 2.1].

A C*-base is a triple $(\mathcal{B}, \mathfrak{S}, \mathcal{B}^1)$, shortly written as $\mathfrak{S}$, consisting of a Hilbert space $\mathfrak{S}$ and two commuting nondegenerate C*-algebras $\mathcal{B}, \mathcal{B}^1 \subseteq \mathcal{L}(\mathfrak{S})$. We shall mainly be interested in the following example of C*-bases: If $\mu$ is a proper KMS-weight on a C*-algebra $\mathcal{B}$, then the triple $(\pi_\mu(\mathcal{B}), H_\mu, J_\mu \pi_\mu(\mathcal{B}) J_\mu)$ is a C*-base, where $H_\mu$ denotes the GNS-space, $\pi_\mu: \mathcal{B} \to \mathcal{L}(H_\mu)$ the GNS-representation, and $J_\mu: H_\mu \to H_\mu$ the modular conjugation associated to $\mu$.

A C*-factorization of a Hilbert space $H$ with respect to a C*-base $\mathfrak{S}$, is a closed subspace $\alpha \subseteq \mathcal{L}(\mathfrak{S}, H)$ satisfying $[\alpha^* \alpha] = \mathcal{B}$, $[\alpha \mathcal{B}] = \alpha$, and $[\alpha \beta] = H$. We denote the set of all C*-factorizations of a Hilbert space $H$ with respect to a C*-base $\mathfrak{S}$, by $\text{C*-fact}(H; \mathfrak{S})$.

Let $\alpha$ be a C*-factorization of a Hilbert space $H$ with respect to a C*-base $\mathfrak{S}$. Then $\alpha$ is a concrete C*-module and a full right C*-module over $\mathcal{B}$ with respect to the inner product $\langle \xi | \xi' \rangle := \xi^* \xi'$. We shall frequently identify $\alpha \otimes \mathfrak{S}$ with $H$ via the unitary

$$\alpha \otimes \mathfrak{S} \xrightarrow{\sim} H, \quad \xi \otimes \zeta \mapsto \xi \zeta.$$  

(1)

There exists a nondegenerate and faithful representation $\rho_\alpha: \mathcal{B}^1 \to \mathcal{L}(\alpha \otimes \mathfrak{S}) \cong \mathcal{L}(H)$ such that for all $b^1 \in \mathcal{B}^1$, $\xi, \alpha, \zeta \in \mathfrak{S}$,

$$\rho_\alpha(b^1)(\xi \otimes \zeta) = \xi \otimes b^1 \zeta \quad \text{or, equivalently,} \quad \rho_\alpha(b^1)\xi \zeta = \xi b^1 \zeta.$$  

Let $K$ be a Hilbert space. Then each unitary $U: H \to K$ induces a map

$$U_\alpha: \text{C*-fact}(H; \mathfrak{S}) \to \text{C*-fact}(K; \mathfrak{S}), \quad \alpha \mapsto U \alpha,$$  

and $\rho_{U_\alpha}(b)(\xi) = U \rho_\alpha(b)(U^* \xi)$ for all $\alpha \in \text{C*-fact}(H; \mathfrak{S})$ and $b \in \mathcal{B}^1$ because

$$\rho_{U_\alpha}(b)(U \xi) = U \rho_\alpha(b)(U^* \xi) \text{ for all } \xi \in \alpha, \zeta \in \mathfrak{S}.$$  

(2)

Let $\beta$ be a C*-factorization of $K$ with respect to $\mathfrak{S}$. We put

$$\mathcal{L}(H_\alpha, K_\beta) := \{ T \in \mathcal{L}(H, K) \mid T \alpha \subseteq \beta, T^* \beta \subseteq \alpha \}.$$  

Evidently, $\mathcal{L}(H_\alpha, K_\beta)^* = \mathcal{L}(K_\beta, H_\alpha)$. Let $T \in \mathcal{L}(H_\alpha, K_\beta)$. Then the map $T_\alpha: \alpha \to \beta$ given by $\xi \mapsto T \xi$ is an adjointable operator of $\alpha$-modules, $(T_\alpha)^* = (T^*)_\beta$, and $T \rho_\alpha(b) = \rho_\beta(b) T$ for all $b \in \mathcal{B}^1$.

Let $\alpha$ be a C*-factorization of a Hilbert space $H$ with respect to a C*-base $\mathfrak{S}$, and let $\mathcal{E} \mathcal{K}_\mathfrak{C}$ be another C*-base. We call a C*-factorization $\beta \in \text{C*-fact}(H; \mathcal{E} \mathcal{K}_\mathfrak{C})$ compatible with $\alpha$, written $\alpha \perp \beta$, if $[\rho_\alpha(\mathcal{B}^1) | \beta \mathcal{B}_\mathfrak{C}]$ is $\text{C*-fact}(H; \mathcal{E} \mathcal{K}_\mathfrak{C})$ compatible, and $[\rho_\beta(\mathcal{C}^1) | \alpha \mathcal{B}_\mathfrak{S}]$ is $\text{C*-fact}(H; \mathcal{E} \mathcal{K}_\mathfrak{C})$ compatible. In that case, $\rho_\alpha(\mathcal{B}^1)$ and $\rho_\beta(\mathcal{C}^1)$ commute. We put $\text{C*-fact}(H_\alpha, \mathcal{E} \mathcal{K}_\mathfrak{C}) := \{ \beta \in \text{C*-fact}(H; \mathcal{E} \mathcal{K}_\mathfrak{C}) \mid \alpha \perp \beta \}$.

**The C*-relative tensor product**  The C*-relative tensor product of Hilbert spaces is a symmetrized version of the internal tensor product of C*-modules and a C*-algebraic analogue of the relative tensor product of Hilbert spaces over a von Neumann algebra. We briefly summarize the definition and the main properties; for details, see [10, Section 2.2].

Let $H$ and $K$ be Hilbert spaces, $\mathfrak{S}$, a C*-base, and $\alpha \in \text{C*-fact}(H; \mathfrak{S})$ and $\beta \in \text{C*-fact}(K; \mathfrak{S})$. The C*-relative tensor product of $H$ and $K$ with respect to $\alpha$ and $\beta$ is the internal tensor product

$$H_\alpha \otimes_\beta K := \alpha \otimes \mathfrak{S} \otimes \beta.$$  

We frequently identify this Hilbert space with $\alpha \otimes_\beta K$ and $H_\rho_\alpha \otimes_\beta$ via the isomorphisms

$$\alpha \otimes_\beta K \cong H_\alpha \otimes_\beta K \cong H_\rho_\alpha \otimes_\beta, \quad \xi \otimes \eta \zeta \equiv \xi \otimes \zeta \otimes \eta \equiv \xi \zeta \otimes \eta.$$  

(3)
Using these isomorphisms, we define for each $\xi \in \alpha$ and $\eta \in \beta$ two pairs of adjoint operators

\[ |\xi|_1 : K \rightarrow H_{\alpha \otimes b}K, \quad \eta \mapsto \xi \otimes \eta, \quad \langle |\xi|_1 : \xi \otimes \eta \mapsto \rho_0(\langle \xi | \eta \rangle), \langle \xi | \eta \rangle. \]
\[ |\eta|_2 : H \rightarrow H_{\alpha \otimes b}K, \quad \eta \mapsto \xi \otimes \eta, \quad \langle |\eta|_2 : \xi \otimes \eta \mapsto \rho_0(\langle \xi | \eta \rangle), \langle \xi | \eta \rangle. \]

We put $|\alpha|_1 := \{ |\xi|_1 | \xi \in \alpha \}$ and similarly define $|\alpha|_1, |\beta|_2, |\gamma|_2$.

Let $L, M$ be Hilbert spaces, $\gamma \in C^\ast$-fact($L; \mathfrak{B}_M$), $\delta \in C^\ast$-fact($M; \mathfrak{B}_M$), and $S \in \mathcal{L}(H, L)$, $T \in \mathcal{L}(K, M)$. Using the isomorphisms (3), we define an operator $S \otimes T \in \mathcal{L}(H_{\alpha \otimes b}K, L_{\alpha \otimes b}M)$ in the following cases:

1) if $S \in \mathcal{L}(H_{\alpha}, L_{\alpha})$ and $T\rho_0(b) = \rho_0(b)T$ for all $b \in \mathfrak{B}$, then $S \otimes T := S_0 \otimes T$;
2) if $T \in \mathcal{L}(K_{\beta}, M_{\beta})$ and $S\rho_0(b) = \rho_0(b)S$ for all $b \in \mathfrak{B_1}$, then $S \otimes T := S \otimes T_\beta$.

If $S \in \mathcal{L}(H_{\alpha}, L_{\alpha})$ and $T \in \mathcal{L}(K_{\beta}, M_{\beta})$, then both times $S \otimes T = S_0 \otimes T_\beta$. Put $S_0 := S \otimes \text{id}_b : H_{\alpha \otimes b}K \rightarrow L_{\gamma \otimes b}K$ and $T_\beta := \text{id}_b \otimes T : H_{\alpha \otimes b}K \rightarrow H_{\alpha \otimes b}M$.

Given $C^\ast$-algebras $C, D$ and $\ast$-homomorphisms $\rho : C \rightarrow \rho_0(\mathfrak{B}_1) \subseteq \mathcal{L}(H), \sigma : D \rightarrow \rho_0(\mathfrak{B}_2) \subseteq \mathcal{L}(K)$, we define $\ast$-homomorphisms $\rho_1 : C \rightarrow \mathcal{L}(H_{\alpha \otimes b}K), c \mapsto \rho(c)|_1$, and $\sigma_2 : D \rightarrow \mathcal{L}(H_{\alpha \otimes b}K), d \mapsto \sigma(d)|_2$.

Let $\gamma \in \mathfrak{B}_1$ and $\delta \in \mathfrak{B}_2$, be $C^\ast$-bases. Then there exist compatibility-preserving maps

\[ C^\ast$-fact($H_{\alpha}; \gamma \in \mathfrak{B}_1$) $\rightarrow \mathcal{C}^\ast$-fact($H_{\alpha \otimes b}K; \rho_0(\mathfrak{B}_1) \delta$), \]
\[ C^\ast$-fact($K_{\beta}; \delta \in \mathfrak{B}_2$) $\rightarrow \mathcal{C}^\ast$-fact($H_{\alpha \otimes b}K; \rho_0(\mathfrak{B}_2) \gamma$), \]

and for all $\gamma \in C^\ast$-fact($H_{\alpha}; \gamma \in \mathfrak{B}_1$) and $\delta \in C^\ast$-fact($K_{\beta}; \delta \in \mathfrak{B}_2$),

\[ \rho_\gamma(\gamma) = \rho_\gamma|_1, \quad \rho_\delta(\delta) = \rho_\delta|_2, \quad \gamma \ast \beta = \beta \ast \gamma \ast \delta. \]

The $C^\ast$-relative tensor product is symmetric, functorial, and associative in a natural sense [10, Section 2.2]. Moreover, if the $C^\ast$-base $\mathfrak{B}_\mathfrak{B}$ arises from a proper KMS-weight $\mu$ on a $C^\ast$-algebra $B$, then $H_{\alpha \otimes b}K$ can be identified with a von Neumann-algebraic relative tensor product, also known as Connes’ fusion; see [10, Section 2.3].

The spatial fiber product of $C^\ast$-algebras and concrete Hopf $C^\ast$-bimodules

The spatial fiber product of $C^\ast$-algebras is a $C^\ast$-algebraic analogue of the fiber product of von Neumann algebras [8] and of the relative tensor product of $C_0(X)$-algebras [2]. We briefly summarize the definition and main properties; for details, see [10, Section 3].

Throughout this paragraph, let $\mathfrak{B}_\mathfrak{B}$ be a $C^\ast$-base.

A (nondegenerate) concrete $C^\ast$-base $\mathfrak{B}_\mathfrak{B}$-algebra $(H, A, \alpha)$ consists of a Hilbert space $H$, a (nondegenerate) $C^\ast$-algebra $A \subseteq \mathcal{L}(H)$, and a $C^\ast$-factorization $\alpha \in C^\ast$-fact($H; \mathfrak{B}_\mathfrak{B}$) such that $\rho_0(\mathfrak{B}_1)A \subseteq A$. If $(H, A, \alpha)$ is a nondegenerate concrete $C^\ast$-base $\mathfrak{B}_\mathfrak{B}$-algebra, then $A' \subseteq \rho_0(\mathfrak{B}_2)'$.

Given a Hilbert space $H$ and a $C^\ast$-algebra $A \subseteq \mathcal{L}(H)$, we put

\[ C^\ast$-fact($A; \mathfrak{B}_\mathfrak{B}$) $:= \{ \beta \in C^\ast$-fact($H; \mathfrak{B}_\mathfrak{B}$) | (H, A, $\beta$) is a concrete $C^\ast$-base $\mathfrak{B}_\mathfrak{B}$-algebra\}. \]

Let $\alpha \in C^\ast$-fact($A; \mathfrak{B}_\mathfrak{B}$) and let $\epsilon \in \mathfrak{B}_\mathfrak{B}$ be a $C^\ast$-base. We put $C^\ast$-fact($A_\alpha; \epsilon \mathfrak{B}_\mathfrak{B}$) $:= \{ \beta \in C^\ast$-fact($A; \mathfrak{B}_\mathfrak{B}$) | $\beta \ast \alpha$ \} and if $\beta \in C^\ast$-fact($A_\alpha; \epsilon \mathfrak{B}_\mathfrak{B}$) (and $A$ is nondegenerate), we call $(H, A, \alpha, \beta)$ a (nondegenerate) concrete $C^\ast$-base $\mathfrak{B}_\mathfrak{B}$-algebra.
Let \((H, A, \alpha)\) be a concrete \(C^*_{-\Bbb R}\)-algebra and \((K, B, \beta)\) a concrete \(C^*_{-\Bbb R}\)-algebra. The fiber product of \((H, A, \alpha)\) and \((K, B, \beta)\) is the \(C^*\)-algebra
\[
A_n \otimes_{\beta} B := \{ T \in \mathcal{L}(H_n \otimes K_n) \mid T|\alpha|_{[1]} \subseteq \{ |\alpha|_{[1]} B \}
\quad \text{and} \quad T|\beta|_{[2]} \subseteq \{ |\beta|_{[2]} A \}\}.
\]

We do not know whether \(A_n \otimes_{\beta} B\) is nondegenerate if \(A\) and \(B\) are nondegenerate. If \(\varepsilon \in \mathcal{R}_{\Bbb C}\), is a \(C^*\)-base, then \(\gamma \beta \in C^*\)-fact \((A_n \otimes_{\beta} B; \varepsilon \mathcal{R}_{\Bbb C})\) for each \(\gamma \in C^*\)-fact \((A_n; \varepsilon \mathcal{R}_{\Bbb C})\) and \(\alpha \beta \delta \in C^*\)-fact \((A_n \otimes_{\beta} B; \varepsilon \mathcal{R}_{\Bbb C})\) for each \(\delta \in C^*\)-fact \((B_n; \varepsilon \mathcal{R}_{\Bbb C})\).

We do not expect the fiber product to be associative. Given \(C^*\)-bases \(\mathcal{R}_{\Bbb C}\), and \(\varepsilon \mathcal{R}_{\Bbb C}\), a concrete \(C^*_{-\Bbb R}\)-algebra \((H, A, \alpha)\), a concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebra \((K, B, \beta, \gamma)\), and a concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebra \((L, C, \delta)\), we put
\[
A_n \otimes_{\beta} B \otimes_{\delta} C := \left( (A_n \otimes_{\beta} B) \otimes_{\gamma} C \right) \cap \left( A_n \otimes_{\beta} (B_n \otimes_{\delta} C) \right);
\]
here, we canonically identify \((H_n \otimes_{\beta} K_n)_{\alpha \gamma} \otimes \delta L\) with \(H_n \otimes_{\alpha \beta} (K_n \otimes_{\gamma} L)\).

**Definition 2.1.** Let \(\mathcal{R}_{\Bbb C}\) be a \(C^*\)-base and \((H, A, \alpha)\), \((K, B, \beta)\) concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebras. A morphism from \((H, A, \alpha)\) to \((K, B, \beta)\) is a \(*\)-homomorphism \(\pi: A \to B\) such that \(\beta = [L^*_{\pi}(H_n, K_n)_{\alpha}]\), where \(L^*_{\pi}(H_n, K_n)_{\alpha} := \{ V \in \mathcal{L}(H_n, K_n) \mid \forall a \in A : \pi(a)V = V\alpha \}\).

We denote the set of all morphisms from \((H, A, \alpha)\) to \((K, B, \beta)\) by \(\text{Mor}(A_n, B_n)\). In [10, Definition 3.11], we imposed an additional condition on morphisms which is automatically satisfied:

**Lemma 2.2.** Let \(\mathcal{R}_{\Bbb C}\) be a \(C^*\)-base and let \(\pi\) be a morphism of concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebras \((H, A, \alpha)\) and \((K, B, \beta)\). Then \(\pi(a\rho_n(b^i)) = \pi(a)\rho_n(b^i)\) for all \(a \in A\) and \(b^i \in \mathcal{B}^i\).

**Proof.** Let \(a \in A\) and \(b^i \in \mathcal{B}^i\). Then for all \(V \in L^*(H_n, K_n)_{\xi, \alpha}, \xi \in \mathcal{R}_{\Bbb C}\),
\[
\pi(a)\rho_n(b^i)V\xi = \pi(a)V\xi b^i = V\rho_n(b^i)\xi = \pi(a\rho_n(b^i))V\xi.
\]
Since \([L^*_{\pi}(H_n, K_n)_{\xi}] = [\mathcal{B}^i]\) is a \(\alpha\)-basis, the claim follows. ☐

The image of a nondegenerate concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebra under a morphism is always nondegenerate [10, Remark 3.12].

Let \(\phi\) be a morphism of nondegenerate concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebras \((H, A, \alpha)\) and \((L, \gamma, C)\), and let \(\psi\) be a morphism of nondegenerate concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebras \((K, B, \beta)\) and \((M, \delta, D)\). Then there exists a unique \(*\)-homomorphism \(\phi \ast \psi: A_n \otimes_{\beta} B \to C_n \otimes_{\delta} D\) such that \((\phi \ast \psi)(T) \cdot (X \otimes Y) = (X \otimes Y) \cdot T\) whenever \(T \in A_n \otimes_{\beta} B\) and one of the following conditions holds: i) \(X \in L^*(H_n, L_n)\) and \(Y \in L^*(K_n, K_n)\) or ii) \(X \in L^*(H_n, L_n)\) and \(Y \in L^*(K, D)\). Moreover, let \(\varepsilon \in \mathcal{R}_{\Bbb C}\), be a \(C^*\)-base and assume that \(A_n \otimes_{\beta} B \subseteq \mathcal{L}(H_n \otimes_{\xi} K_n)\) is nondegenerate. If \(\alpha' \in C^*\)-fact \((A_n; \varepsilon \mathcal{R}_{\Bbb C})\), \(\gamma' \in C^*\)-fact \((C_n; \varepsilon \mathcal{R}_{\Bbb C})\), \(\phi \in \text{Mor}(A_n, C_n)\), then \(\alpha' \psi \in \text{Mor}\left((A_n \otimes_{\beta} B)_{\langle \alpha', \beta \rangle}, (C_n \otimes_{\delta} D)_{\langle \gamma', \delta \rangle}\right)\). Similarly, if \(\beta' \in C^*\)-fact \((B_n; \varepsilon \mathcal{R}_{\Bbb C})\), \(\delta' \in C^*\)-fact \((D_n; \varepsilon \mathcal{R}_{\Bbb C})\), \(\psi \in \text{Mor}(B_n, D_n)\), then \(\phi \ast \psi \in \text{Mor}\left((A_n \otimes_{\beta} B)_{\langle \alpha, \beta \rangle}, (C_n \otimes_{\delta} D)_{\langle \gamma', \delta \rangle}\right)\).

**Lemma 2.3.** Let \(H\) be a Hilbert space, \(\alpha \in C^*\)-fact \((H; \mathcal{R}_{\Bbb C})\), and let \(\rho\) be a morphism of concrete \(C^*_{-\Bbb R}\mathcal{R}_{\Bbb C}\)-algebras \((K, C, \gamma)\) and \((L, D, \delta)\). Then \([\alpha]_{\langle 2 \rangle}C\langle \alpha \rangle_{\langle 1 \rangle} \subseteq \mathcal{L}(K_n \otimes_{\alpha} H)\) and \([\alpha]_{\langle 2 \rangle}D\langle \alpha \rangle_{\langle 1 \rangle} \subseteq \mathcal{L}(L_n \otimes_{\alpha} H)\) are \(C^*\)-algebras, and there exists a \(*\)-homomorphism
\[
\text{Ind}_{\alpha}(\rho) : [\alpha]_{\langle 2 \rangle}C\langle \alpha \rangle_{\langle 1 \rangle} \to [\alpha]_{\langle 2 \rangle}D\langle \alpha \rangle_{\langle 1 \rangle}\]
such that for all \(c \in C\) and \(\xi, \xi' \in \alpha,\]
\[
\text{Ind}_{\alpha}(\rho)(\xi)_{\langle 2 \rangle}C\langle \xi \rangle_{\langle 1 \rangle} = |\xi\rangle_{\langle 2 \rangle}\langle c|\langle \xi' \rangle_{\langle 1 \rangle}.
\]
Proof. The space $[\|\alpha\|_2^2 \mathcal{C}(\alpha)_2]_2$ is a $C^*$-algebra because
\[
[\|\alpha\|_2^2 \mathcal{C}(\alpha)_2]_2 (\|\alpha\|_2^2 \mathcal{C}(\alpha)_2)_2^* \subseteq [\|\alpha\|_2^2 \mathcal{C}(\alpha)_2]_2 \subseteq [\|\alpha\|_2^2 \mathcal{C}(\alpha)_2].
\]
Likewise, $[\|\alpha\|_2^2 \mathcal{D}(\alpha)_2]_2$ is a $C^*$-algebra. The existence of $\text{Ind}_\alpha(\rho)$ follows as in the first part of the proof of [10, Proposition 3.13].

A concrete Hopf $C^*$-bimodule is a tuple consisting of a $C^*$-base $\mathfrak{g} \mathfrak{B}_B$, a nondegenerate concrete $C^*$-$\mathfrak{g} \mathfrak{B}_B$-$\mathfrak{g} \mathfrak{B}_B$-algebra $(H, A, \alpha, \beta)$, and a $*$-homomorphism $\Delta : A \to A_\beta \ast A$ subject to the following conditions:

i) $\Delta \in \text{Mor} (A_\alpha, (A_\beta \ast A)_\alpha)$ and $\text{Mor} (A_\beta, (A_\alpha \ast A)_\beta)$, and

ii) the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{\Delta} & A_\alpha \ast A \\
\downarrow & & \downarrow \text{id} \ast \Delta \\
A_\beta \ast A & \xrightarrow{\Delta \ast \text{id}} & (A_\alpha \ast A)_\alpha \ast (A_\beta \ast A)
\end{array}
\]

If $(\mathfrak{g} \mathfrak{B}_B, H, A, \alpha, \beta, \Delta)$ is a concrete Hopf $C^*$-bimodule, then the spaces $(\Delta \ast \text{id})((\Delta(A))$ and $(\text{id} \ast \Delta)((\Delta(A))$ are contained in $A_\beta \ast A$.

If $(\mathfrak{g} \mathfrak{B}_B, H, A, \alpha, \beta, \Delta)$ is a concrete Hopf $C^*$-bimodule and $\mathfrak{g} \mathfrak{B}_B$ and $H$ are understood, we shall denote this concrete Hopf $C^*$-bimodule briefly by $(A_\alpha, \Delta)$.

**C*-pseudo-multiplicative unitaries and the associated legs.** The notion of a $C^*$-pseudo-multiplicative unitary extends the notion of a multiplicative unitary [1], of a continuous field of multiplicative unitaries [2], and of a pseudo-multiplicative unitary on $C^*$-modules [5, 11], and is closely related to pseudo-multiplicative unitaries on Hilbert spaces [12]; see [10, Section 4.1]. The precise definition is as follows.

Let $H$ be a Hilbert space, $\mathfrak{g} \mathfrak{B}_B$, a $C^*$-base, $\alpha \in C^*$-fact($H; \mathfrak{g} \mathfrak{B}_B$), $\beta \in C^*$-fact($H; \mathfrak{g} \mathfrak{B}_B$), $\beta \in C^*$-fact($H; \mathfrak{g} \mathfrak{B}_B$) such that $\alpha, \beta, \hat{\beta}$ are pairwise compatible. Let $V : H_\beta \otimes_\beta H \to H_\alpha \otimes_\alpha H$ be a unitary such that

\[
V_\alpha(\alpha \ast \alpha) = \alpha \ast \alpha, \quad V_\beta(\beta \ast \beta) = \hat{\beta} \ast \beta, \quad V_\epsilon(\hat{\beta} \ast \hat{\beta}) = \alpha \ast \beta, \quad V_\epsilon(\beta \ast \alpha) = \beta \ast \beta.
\]

(4)

Then all operators in the following diagram are well-defined [10, Lemma 4.1],

\[
\begin{array}{cccc}
H_\beta \otimes_\beta H & \xrightarrow{V \otimes \text{id}} & H_\alpha \otimes_\alpha H & \xrightarrow{\text{id} \otimes V} & H_\alpha \otimes_\beta H, \\
\downarrow \text{id} \otimes \otimes_\beta & & \downarrow \text{id} \otimes \otimes_\alpha & & \downarrow \text{id} \otimes \otimes_\beta \\
H_\beta \otimes \otimes_\beta (H_\alpha \otimes_\beta H) & \xrightarrow{V \otimes \text{id}} & (H_\beta \otimes_\beta H) \otimes \otimes_\beta H \\
\downarrow \Sigma_{[23]} & & \downarrow \Sigma_{[23]} & & \downarrow \Sigma_{[23]}
\end{array}
\]

where $\Sigma_{[23]}$ denotes the isomorphism

\[
(H_\alpha \otimes_\beta H) \otimes_\beta \otimes_\alpha H \cong (H_\alpha \otimes_\beta \beta) \otimes_\alpha \otimes_\alpha H \cong \beta \cong (H_\beta \otimes_\beta H) \otimes \alpha_\beta \otimes_\beta H,
\]

(\xi \otimes \eta) \otimes \eta \mapsto (\xi \otimes \eta) \otimes \xi.

6
We put $V_{1[2]} := V \otimes \text{id}$, $V_{2[1]} := \text{id} \otimes V$, $V_{1[3]} := (\text{id} \otimes \Sigma)V_{1[2]} \Sigma_{1[3]}$; we call $V$ a $C^*$-pseudo-multiplicative unitary if Diagram (5) commutes, that is, if $V_{1[2]}V_{2[1]}V_{1[3]} = V_{2[1]}V_{1[2]}$. In that case, also $V^{op} := \Sigma V^{*} \Sigma : H_{\beta} \otimes_a H \rightarrow H_{\alpha} \otimes_b H$ is a $C^*$-pseudo-multiplicative unitary, called the opposite of $V$.

Let $V : H_{\beta} \otimes_b H \rightarrow H_{\alpha} \otimes_a H$ be a $C^*$-pseudo-multiplicative unitary. Then the spaces

$\hat{A} := \hat{A}(V) := \{[(\beta)_{1[3]} V (\alpha)_{1[2]}] \subseteq \mathcal{L}(H)\}$, \quad $A := A(V) := \{(\alpha)_{1[3]} V (\beta)_{1[1]} \subseteq \mathcal{L}(H)\}$

satisfy

$[\hat{A} \hat{A}] = [\hat{A} \hat{A}(\mathfrak{B})] = [\rho_{\beta}(\mathfrak{B}) \hat{A}] = [\hat{A} \rho_{\alpha}(\mathfrak{B}^*)] = [\rho_{\alpha}(\mathfrak{B}^* \hat{A})] = \hat{A} \subseteq \mathcal{L}(H_{\beta})$,

$[A A] = [A \rho_{\beta}(\mathfrak{B})] = [\rho_{\beta}(\mathfrak{B}) A] = [\rho_{\alpha}(\mathfrak{B}^* A)] = \rho_{\alpha}(\mathfrak{B}^* A) \subseteq \mathcal{L}(H_{\beta})$.

Define maps

$\Delta_V : \hat{A} \rightarrow \mathcal{L}(H_{\beta} \otimes_a H)$, $y \mapsto V^{*}(1 \otimes y)V$, \quad $\Delta_V : A \rightarrow \mathcal{L}(H_{\alpha} \otimes_a H), \quad z \mapsto V(z \otimes 1)V^{*}$.

We call $V$ well-behaved if the tuples $(\mathfrak{g}_H, \mathfrak{h}_H, \mathfrak{a}_H, \hat{A}, \hat{B}, \hat{C})$ and $(\mathfrak{g}_H, \mathfrak{h}_H, A, \alpha, \beta, \Delta_V)$ are concrete Hopf $C^*$-bimodules. We call $V$ regular if the subspace $\{(\alpha)_{1[3]} V (\beta)_{1[1]} \subseteq \mathcal{L}(H)\}$ is equal to $[\alpha \alpha^{*}]$. If $V$ is regular, then it is well-behaved [10, Theorem 4.14].

3 Weak $C^*$-pseudo-Kac systems

Reduced crossed products for coactions of Hopf $C^*$-algebras can conveniently be constructed in the framework of Kac systems [1] or, more generally, of weak Kac systems [13]. To adapt the construction to coactions of concrete Hopf $C^*$-bimodules, we generalize the notion of a weak Kac system as follows.

Let $H$ be a Hilbert space. Recall that for each $C^*$-base $\mathfrak{e} \mathcal{R}_e$, and each $C^*$-factorization $\gamma \in C^*$-fact($H; \mathfrak{e} \mathcal{R}_e$), $\delta \in C^*$-fact($H; \mathfrak{e} \mathcal{R}_e$), there exists a flip map

$\Sigma : H_{\beta} \otimes_a H = \gamma \otimes \mathcal{R} \otimes \delta \mapsto \delta \otimes \mathcal{R} \otimes \gamma = H_{\alpha} \otimes_a H$, \quad $\eta \otimes \zeta \otimes \xi \mapsto \xi \otimes \zeta \otimes \eta$.

Let $\mathfrak{g}_H$, be a $C^*$-base, $\alpha, \hat{\alpha} \in C^*$-fact($H; \mathfrak{g}_H$), $\beta, \hat{\beta} \in C^*$-fact($H; \mathfrak{g}_H$), and let $U : H \rightarrow H$ be a symmetry, that is, a self-adjoint unitary. Assume that $U \alpha = \tilde{\alpha}$ and $U \beta = \tilde{\beta}$; then also $U \hat{\alpha} = \alpha$ and $U \hat{\beta} = \beta$. For each $T \in \mathcal{L}(H_{\beta} \otimes_a H, H_{\alpha} \otimes_b H)$, put

$\hat{T} := \Sigma(1 \otimes_U)\Sigma(1 \otimes_U)\Sigma : H_{\beta} \otimes_a H \xrightarrow{U \otimes \Sigma} H_{\beta} \otimes_a H \xrightarrow{\Sigma(U \otimes_U)\Sigma(U \otimes_U)\Sigma} H_{\alpha} \otimes_b H$,

$\tilde{T} := \Sigma(U \otimes_U)\Sigma(U \otimes_U)\Sigma : H_{\beta} \otimes_a H \xrightarrow{U \otimes \Sigma} H_{\beta} \otimes_a H \xrightarrow{\Sigma(U \otimes_U)\Sigma(U \otimes_U)\Sigma} H_{\alpha} \otimes_b H$.

Switching between the $C^*$-bases $\mathfrak{g}_H$, and $\mathfrak{g}_H$, and relabeling the $C^*$-factorizations $\alpha, \hat{\alpha}, \beta, \hat{\beta}$ suitably, we can iterate the maps $T \mapsto \hat{T}$ and $T \mapsto \tilde{T}$. The two relations $\Sigma(1 \otimes_U)\Sigma(1 \otimes_U)\Sigma = \Sigma(U \otimes_U)\Sigma(U \otimes_U)$ and $\Sigma(1 \otimes_U)\Sigma(U \otimes_U) = \text{id}$ imply

$\tilde{T} = \text{Ad}_U(U \otimes_U)(T) = \hat{T}$, \quad $\tilde{T} = \hat{T}$, \quad $\tilde{T} = \hat{T}$ \quad for all $T \in \mathcal{L}(H_{\beta} \otimes_a H, H_{\alpha} \otimes_b H)$.

(6)

Definition 3.1. A balanced $C^*$-pseudo-multiplicative unitary $(\alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)$ consists of

- $C^*$-factorizations $\alpha, \hat{\alpha} \in C^*$-fact($H; \mathfrak{g}_H$) and $\beta, \hat{\beta} \in C^*$-fact($H; \mathfrak{g}_H$),
- a symmetry $U : H \rightarrow H$, and
Lemma 3.4. Let \( \mathcal{L} \) be a \( C^* \)-pseudo-multiplicative unitary \( V : H_\beta \otimes B \rightarrow H_\alpha \otimes B \)
satisfying the following conditions:

i) \( \alpha, \beta, \beta \) are pairwise compatible,

ii) \( U_\alpha = \tilde{\alpha} \) and \( U_\beta = \tilde{\beta} \),

iii) \( \tilde{V} \) and \( \tilde{V} \) are \( C^* \)-pseudo-multiplicative unitaries.

Remarks 3.2. i) Since \( \tilde{V} = (U \otimes U) \tilde{V}(U \otimes U) \), the unitary \( \tilde{V} \) is \( C^* \)-pseudo-multiplicative if and only if \( \tilde{V} \) is \( C^* \)-pseudo-multiplicative.

ii) If \( (\alpha, \hat{\alpha}, \beta, \beta, U, V) \) is a balanced \( C^* \)-pseudo-multiplicative unitary, then also the tuples
\( (\beta, \beta, \alpha, \hat{\alpha}, U, V) \), \( (\alpha, \hat{\alpha}, \beta, \beta, U, Ad_{U \otimes U}(V)) \), and \( (\beta, \beta, \hat{\alpha}, \alpha, U, \tilde{V}) \) are balanced \( C^* \)-pseudo-multiplicative unitaries. This follows easily from (6). Moreover, in that case
\( (\alpha, \hat{\alpha}, \tilde{\beta}, \beta, U, V^{op}) \) is a balanced \( C^* \)-pseudo-multiplicative unitary as well, as can be seen from the relation

\[
(\tilde{V}^{op}) = \Sigma U_{12}(\Sigma V^{*}\Sigma) U_{12} = \Sigma(\Sigma U_{11} V U_{11}^{*}\Sigma)^{op} = (\tilde{V})^{op}.
\] (7)

iii) If \( (\alpha, \hat{\alpha}, \beta, \beta, U, V) \) is a balanced \( C^* \)-pseudo-multiplicative unitary, the relations (4) for the unitaries \( \tilde{V} : H_\alpha \otimes \beta H \rightarrow H_\hat{\alpha} \otimes \beta H \) and \( \tilde{V} : H_\alpha \otimes \beta H \rightarrow H_\beta \otimes \hat{\alpha} H \) (read as follows:

\[
\tilde{V}_\alpha(\beta \circ \hat{\alpha}) = \tilde{\beta} \circ \hat{\alpha},
\tilde{V}_\alpha(\hat{\alpha} \circ \beta) = \tilde{\hat{\alpha}} \circ \beta,
\tilde{V}_\alpha(\alpha \circ \beta) = \beta \circ \hat{\alpha},
\tilde{V}_\alpha(\beta \circ \alpha) = \beta \circ \alpha,
\tilde{V}_\alpha(\alpha \circ \alpha) = \alpha \circ \alpha.
\]

These relations furthermore imply

\[
\tilde{V}_\alpha(\beta \circ \hat{\alpha}) = \beta \circ \hat{\alpha},
\tilde{V}_\alpha(\hat{\alpha} \circ \beta) = \hat{\beta} \circ \beta,
\tilde{V}_\alpha(\alpha \circ \beta) = \hat{\beta} \circ \beta,
\tilde{V}_\alpha(\beta \circ \alpha) = \beta \circ \alpha,
\tilde{V}_\alpha(\alpha \circ \alpha) = \alpha \circ \alpha.
\]

iv) If \( (\alpha, \hat{\alpha}, \beta, \beta, U, V) \) is a balanced \( C^* \)-pseudo-multiplicative unitary, then \( \tilde{A}(V), A(V) \in \mathcal{L}(H_\alpha) \) because

\[
[\tilde{A}(V) \tilde{\alpha}] = [\langle \beta |_{12} V | \alpha \rangle_{12} \tilde{\alpha}] = [\langle \beta |_{12} | \beta \rangle_{12} \tilde{\alpha}] = [\rho_\alpha(\mathcal{B}^1) \tilde{\alpha}] = \tilde{\alpha}
\] (8)

and similarly \( [A(V) \tilde{\alpha}] = [\langle \alpha |_{11} V | \beta \rangle_{11} \tilde{\alpha}] = [\langle \alpha |_{11} | \alpha \rangle_{11} \tilde{\alpha}] = [\rho_\alpha(\mathcal{B}) \tilde{\alpha}] = \tilde{\alpha} \).

The auxiliary unitaries \( \tilde{V} \) and \( \tilde{V} \) defined above allow us to treat the right and the left leg of \( V \), respectively, as the left or the right leg of some \( C^* \)-pseudo-multiplicative unitary:

Proposition 3.3. Let \( (\alpha, \hat{\alpha}, \beta, \beta, U, V) \) be a balanced \( C^* \)-pseudo-multiplicative unitary. Then

\[
\tilde{A}(\tilde{V}) = Ad_{U}(A(V)), \quad \hat{\Delta}_\alpha = \hat{\Delta}_\alpha \circ \hat{\Delta}_\alpha \circ \hat{\Delta}_\alpha \circ \hat{\Delta}_\alpha = \hat{\Delta}_\alpha, \quad \tilde{A}(V) = \tilde{A}(V), \quad \Delta_\alpha = \Delta_\alpha.
\]

In particular, \( \tilde{V} \) and \( \tilde{V} \) are well-behaved if \( V \) is well-behaved.

For the proof, we need the following lemma:

Lemma 3.4. The following diagrams commute:

\[
\begin{array}{ccc}
(H_{\beta} \otimes B)_{\beta} \otimes \alpha \otimes \beta & \xrightarrow{V_{13}} & (H_{\alpha} \otimes B)_{\alpha} \otimes \beta \otimes \beta H \xrightarrow{V_{13}} (H_{\beta} \otimes B)_{\beta} \otimes \alpha \otimes \beta H \\
\end{array}
\] (9)
and
\[ H_\beta \otimes_{\alpha n} (H_\alpha \otimes H) \xrightarrow{\varphi[12]} H_\beta \otimes_{\beta n} (H_\beta \otimes \alpha H) \xrightarrow{\varphi[13]} H_\alpha \otimes_{\beta n} (H_\beta \otimes \alpha H). \] (10)

**Proof.** Let us prove that diagram (9) commutes. Put \( W := \Sigma V \Sigma \). We insert the relation \( \tilde{\nu} = U_{[1]} W U_{[1]} \) into the pentagon equation \( \tilde{\nu}_{[12]} \tilde{\nu}_{[13]} \tilde{\nu}_{[23]} = \tilde{\nu}_{[123]} \tilde{\nu}_{[13]} \) and obtain
\[ U_{[1]} W_{[12]} U_{[1]} \cdot U_{[1]} W_{[13]} U_{[1]} \cdot U_{[2]} = \tilde{\nu}_{[123]} \cdot U_{[1]} W_{[12]} U_{[1]}. \]
Since \( U_{[1]} \) commutes with \( \tilde{\nu}_{[123]} \), we can cancel \( U_{[1]} \) everywhere in the equation above and find \( W_{[12]} W_{[13]} \tilde{\nu}_{[123]} = \tilde{\nu}_{[123]} W_{[12]} \). Now, we conjugate both sides of this equation by the automorphism \( \Sigma_{[23]} \Sigma_{[12]} \), which amounts to renumbering the legs of the operators according to the permutation \( (1, 2, 3) \leftrightarrow (2, 3, 1) \), and find \( W_{[12]} \Sigma_{[23]} W_{[123]} = W_{[12]} \Sigma_{[12]} \). If we retrace the derivation of this equation in diagrammatic form, we obtain diagram (9).

A similar argument shows that diagram (10) commutes.

**Proof of Proposition 3.3.** Inserting the relations
\[ \tilde{\nu} = \Sigma U_{[2]} V U_{[2]} \Sigma; \quad H_\alpha \otimes H \rightarrow H_\beta \otimes H, \quad \tilde{\nu} = \Sigma U_{[1]} V U_{[1]} \Sigma; \quad H_\alpha \otimes H \rightarrow H_\beta \otimes H, \]
into the definition of \( A(\tilde{\nu}) \) and \( \tilde{A}(\tilde{\nu}) \), respectively, we find
\[ A(\tilde{\nu}) = [\tilde{\alpha} [12] \Sigma U_{[2]} V U_{[2]} \Sigma \tilde{\alpha} [12]] = [\tilde{\beta} [12] V \Sigma U_{[2]} V \Sigma] = [\tilde{\alpha} [12] V \Sigma U_{[2]} V \Sigma] = A(\tilde{\nu}). \]

To prove \( \Delta_\nu = \Delta \nu \), consider an element \( \tilde{\alpha} = \langle \xi' [123] V \xi [12] \rangle \in \tilde{A}(V) \), where \( \xi' \in \beta, \xi' \in \alpha \). By definition, \( \Delta_\nu(\tilde{\alpha}) = \tilde{\nu}(\tilde{\alpha} \otimes 1) \tilde{\nu}^*. \) The commutative diagram
\[ \begin{array}{ccc}
H_\beta \otimes H & \xrightarrow{\varphi[12]} & H_\beta \otimes_{\beta n} H \\
\downarrow \varphi[13] & & \downarrow \varphi[13] \\
H_\beta \otimes_{\beta n} H & \xrightarrow{\varphi[12]} & H_\beta \otimes_{\beta n} H \\
\end{array} \]
diagram (9) and the pentagon diagram (5) imply
\[ \tilde{\nu}(\otimes 1) \tilde{\nu}^* = \langle \xi' [13] \tilde{\nu}_{[1]2} \tilde{\nu}_{[12]} \rangle \tilde{\nu}^* \]
\[ = \langle \xi' [13] \tilde{\nu}_{[1]2} \tilde{\nu}_{[12]} \rangle \tilde{\nu}^* = \langle \xi' [13] \tilde{\nu}_{[1]2} \tilde{\nu}_{[12]} \rangle \xi [13]. \]
The following expression shows that the expression above is equal to \( \tilde{\nu}^*(1 \otimes \tilde{\alpha}) V = \Delta_\nu(\tilde{\alpha}): \)
\[ \begin{array}{ccc}
H_\beta \otimes_{\beta n} H & \xrightarrow{\varphi[12]} & H_\beta \otimes_{\beta n} H \\
\downarrow \varphi[13] & & \downarrow \varphi[13] \\
H_\beta \otimes_{\beta n} H & \xrightarrow{\varphi[12]} & H_\beta \otimes_{\beta n} H \\
\end{array} \]
Since elements of the form like $\tilde{a}$ are dense in $\widehat{A}(V)$, we can conclude $\Delta = \tilde{\Delta}$. A similar argument shows that $\tilde{\Delta} = \Delta$; here, we have to use relation (10).

The remaining equations in Proposition 3.3 follow from the relation $\tilde{V} = \text{Ad}_{(U \otimes V)}(\tilde{V})$ and those equations that we have proved already. □

The definition of a weak $C^*$-pseudo-Kac system involves the following conditions:

**Lemma 3.5.** Let $(\alpha, \tilde{a}, \beta, \tilde{b}, U, V)$ be a balanced $C^*$-pseudo-multiplicative unitary.

i) The following conditions are equivalent:

(a) The following diagram commutes:

\[ H_\alpha \otimes_0 H_\beta \otimes_0 H \xrightarrow{\tilde{V}_{123}} H_\beta \otimes_0 H_\beta \otimes_0 H \]

\[ \xrightarrow{\tilde{V}_{123}} H_\alpha \otimes_0 H_\alpha \otimes_0 H. \]

(b) $(1 \otimes \tilde{a})\tilde{V} = \tilde{V}(1 \otimes \tilde{a})$ in $\mathcal{L}(H_\alpha \otimes_0 H, H_\beta \otimes_0 H)$ for each $\tilde{a} \in \widehat{A}(V)$.

(c) $(\text{Ad}_{C}(\tilde{a}) \otimes 1)V = V(\text{Ad}_{C}(\tilde{a}) \otimes 1)$ in $\mathcal{L}(H_\beta \otimes_0 H, H_\alpha \otimes_0 H)$ for each $\tilde{a} \in \widehat{A}(V)$.

(d) $\hat{A}(V)$ and $\text{Ad}_{C}(\hat{A}(V))$ commute.

ii) The following conditions are equivalent:

(a) The following diagram commutes:

\[ H_\beta \otimes_0 H_\alpha \beta \otimes_0 H \xrightarrow{\tilde{V}_{123}} H_\alpha \otimes_0 H_\alpha \beta \otimes_0 H \]

\[ \xrightarrow{\tilde{V}_{123}} H_\beta \otimes_0 H_\beta \otimes_0 H. \]

(b) $(a \otimes 1)\tilde{V} = \tilde{V}(a \otimes 1)$ in $\mathcal{L}(H_\beta \otimes_0 H, H_\beta \otimes_0 H)$ for each $a \in A(V)$.

(c) $(1 \otimes \text{Ad}_{C}(a))V = V(1 \otimes \text{Ad}_{C}(a))$ in $\mathcal{L}(H_\beta \otimes_0 H, H_\alpha \otimes_0 H)$ for each $a \in A(V)$.

(d) $A(V)$ and $\text{Ad}_{C}(A(V))$ commute.

*Proof.* In i), conditions (a) and (b) are equivalent because

\[ \begin{align*}
(a) & \iff \forall \xi \in \alpha, \xi' \in \beta : \langle \xi'_{13}| \tilde{V}_{123} | \xi_{13} \rangle = \langle \xi'_{13}| \tilde{V}_{123} | \xi_{13} \rangle \\
& \iff \forall \xi \in \alpha, \xi' \in \beta : (1 \otimes \tilde{a})\tilde{V} = \tilde{V}(1 \otimes \tilde{a}), \text{ where } \tilde{a} = \langle \xi'_{13}| \tilde{V}| \xi_{13} \rangle \implies (b).
\end{align*} \]

The remaining equivalences follow similarly. □

**Definition 3.6.** We call a balanced $C^*$-pseudo-multiplicative unitary $(\alpha, \tilde{a}, \beta, \tilde{b}, U, V)$ a weak $C^*$-pseudo-Kac system if $V$ is well-behaved and if the equivalent conditions in Lemma ?? hold.

The notion of a weak $C^*$-pseudo-Kac system is symmetric in the following sense:

**Proposition 3.7.** Let $(\alpha, \tilde{a}, \beta, \tilde{b}, U, V)$ be a weak $C^*$-pseudo-Kac system. Then also the following tuples are weak $C^*$-pseudo-Kac systems:

\[ (\beta, \beta, \alpha, \tilde{a}, U, V), \quad (\beta, \tilde{b}, \tilde{a}, \alpha, U, V), \quad (\alpha, \tilde{a}, \beta, \tilde{b}, U, V^{op}). \]

*Proof.* The $C^*$-pseudo-multiplicative unitaries $\tilde{V}, \tilde{V}, V^{op}$ are well-behaved by Proposition 3.3 and [10, Lemma 4.4], respectively, and the tuples (??) are balanced $C^*$-pseudo-multiplicative unitaries by Remark 3.2 ii). Using Proposition 3.3 and [10, Lemma 4.4], one easily checks that these tuples satisfy the conditions i)(d) and ii)(d) of Lemma ?? □

**Definition 3.8.** Given a weak $C^*$-pseudo-Kac system $(\alpha, \tilde{a}, \beta, \tilde{b}, U, V)$, we call the tuples (??) its predual, its dual, and its opposite, respectively.
4 Coactions and reduced crossed products

Coactions of concrete Hopf $C^*$-bimodules

**Definition 4.1.** Let $(\mathfrak{B}_B, H, A, \alpha, \beta, \Delta)$ be a concrete Hopf $C^*$-bimodule and $(K, C, \gamma)$ a nondegenerate concrete $C^*$-$\mathfrak{B}_B$-algebra. Then a coaction of $(\mathfrak{B}_B, H, A, \alpha, \beta, \Delta)$ on $(K, C, \gamma)$ is a morphism $\delta_C$ from $(K, C, \gamma)$ to $(K \otimes_B H, C \otimes_B A, \gamma \circ \alpha)$ that makes the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\delta_C} & C \otimes_B \gamma \\
\downarrow & & \downarrow \\
C \otimes_B A & \xrightarrow{id \otimes \Delta} & C \otimes_B (A \otimes_B A) \\
\end{array}
$$

We also refer to the tuple $(K, C, \gamma, \delta_C)$ as a coaction. We call such a coaction fine if $\delta_C$ is injective and $[\delta_C(C) | \beta, [1] = [\beta]_{[\beta]} C]$ as subsets of $\mathcal{L}(K, K \otimes_B H)$.

A covariant morphism between coactions $(K, C, \gamma, \delta_C)$ and $(L, D, \delta, \delta_{12})$ is a morphism $\rho$ from $(K, C; \gamma)$ to $(L, D, \delta)$ that makes the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\rho} & D \\
\downarrow & & \downarrow \\
C \otimes_B A & \xrightarrow{\rho \otimes id_A} & D \otimes_B A. \\
\end{array}
$$

**Remarks 4.2.**

i) Note that by [10, Remark 3.12], the $C^*$-algebra $\delta_C(C)$ and hence also the $C^*$-algebra $C \otimes_B A$ is nondegenerate.

ii) Evidently, the class of all coactions of a fixed concrete Hopf-$C^*$-bimodule forms a category with respect to covariant morphisms.

iii) For every concrete Hopf $C^*$-bimodule $(\mathfrak{B}_B, H, A, \alpha, \Delta)$, the triple $(H, A, \alpha, \Delta)$ is a coaction.

We shall study coactions of concrete Hopf $C^*$-bimodules in a separate article.

Reduced crossed products for coactions of $(\mathfrak{B}_B, H, A, \alpha, \beta, \Delta_V)$ Till the end of this section, we fix a weak $C^*$-pseudo-Kac system $(\alpha, \tilde{\alpha}, \tilde{\beta}, \tilde{U}, V)$. To shorten the notation, we put

$$
\tilde{A} := \tilde{A}(V), \quad \tilde{\Delta} := \tilde{\Delta}_Y, \quad A := A(V), \quad \Delta := \Delta_V,
$$

and write $(\alpha, \tilde{\alpha}, \tilde{\Delta})$ and $(\beta A_\alpha, \Delta)$ for the concrete Hopf $C^*$-bimodules $(\mathfrak{B}_B, H, \tilde{\alpha}, \tilde{\beta}, \tilde{\Delta})$ and $(\mathfrak{B}_B, H, A, \alpha, \beta, \Delta)$, respectively.

First, we define reduced crossed products for coactions of $(\mathfrak{B}_B, A, \Delta)$:

**Definition 4.3.** Let $(K, C, \gamma, \delta_C)$ be a coaction of $(\mathfrak{B}_B, A, \Delta)$. The associated reduced crossed product is the $C^*$-subalgebra $C \rtimes_{\delta_C} \tilde{A} \subseteq \mathcal{L}(K \otimes_B H)$ generated by

$$
\delta(C)(1 \otimes \tilde{A}) \subseteq \mathcal{L}(K \otimes_B H).
$$

If $\delta_C$ is understood, we shortly write $C \rtimes \tilde{A}$ for $C \rtimes_{\delta_C} \tilde{A}$.

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Remark 4.4. In the situation above, \( C \rtimes \hat{A} \subseteq C \otimes_{\beta} L(H) \), as can be seen from the inclusions \( \delta_C(C) \subseteq C \otimes_{\beta} A \) and

\[
(C \otimes_{\beta} A)(1 \otimes \hat{A})[\gamma_1 \gamma_2] = (C \otimes_{\beta} A)[\gamma_1 \gamma_2] \hat{A} \subseteq [\gamma_1 \gamma_2] A \hat{A},
\]

\[
(C \otimes_{\beta} A)(1 \otimes \hat{A})[\beta_1 \beta_2] \subseteq [C \otimes_{\beta} A][\beta_1 \beta_2] \subseteq [\beta_1 \beta_2] C;
\]

here, we used the inclusion \( \hat{A} \beta \subseteq \beta \) [10, Lemma 4.5].

Frequently, it is useful to know that the set (11) is linearly dense in \( C \rtimes \hat{A} \):

**Proposition 4.5.** Let \( (K, C, \gamma, \delta_C) \) be a coaction of \((\beta A, \Delta)\). Then

\[ C \rtimes \hat{A} = [\delta_C(C)(1 \otimes \hat{A})] \]

**Proof.** We only need to prove \( [(1 \otimes \hat{A})\delta_C(C)] \subseteq [\delta_C(C)(1 \otimes \hat{A})] \). By definition of \( \hat{A} \),

\[ [(1 \otimes \hat{A})\delta_C(C)] = \left[ [\beta][13](1 \otimes V)[\alpha][13] \delta_C(C) \right] = \left[ [\beta][13](1 \otimes V)[\delta_C(C) \otimes 1][\alpha][13] \right]. \]

Note that \( \delta_C(C) \otimes 1 \subseteq L(K, \hat{A}; \beta \hat{A}; \beta_1 H) \) is well-defined because \( \delta_C(C) \subseteq C \otimes_{\beta} A \) commutes with \( 1 \otimes \rho_{\beta}(\mathfrak{A}) \) by [10, Lemma 3.8]. Put \( \delta_C^{(2)} := (\text{id} \ast \Delta) \circ \Delta \). By definition of \( \Delta \),

\[ [\beta][13](1 \otimes V)[\delta_C(C) \otimes 1][\alpha][13] = [\beta][13](\delta_C^{(2)}(C))(1 \otimes V)[\alpha][13]. \]

Since \( \delta_C^{(2)}(C) = (\text{id} \ast \Delta)(\delta_C(C)) = (\delta_C \ast \text{id})(\delta_C(C)) \subseteq \delta_C(C) \gamma_{\alpha} \otimes_{\beta} A \),

\[ [\beta][13](\delta_C^{(2)}(C))(1 \otimes V)[\alpha][13] \subseteq [\delta_C(C)[\beta][13](1 \otimes V)[\alpha][13]] = [\delta_C(C)(1 \otimes \hat{A})]. \]

**Corollary 4.6.** Let \( (K, C, \gamma, \delta_C) \) be a coaction of \((\beta A, \Delta)\). Then \( \delta_C(C) \) and \( 1 \otimes \hat{A} \) are nondegenerate \( C^* \)-subalgebras of \( M(C \rtimes \hat{A}) \), and the maps \( c \mapsto \delta_C(c) \) and \( \hat{a} \mapsto 1 \otimes \hat{a} \) extend to \( * \)-homomorphisms \( M(C) \rightarrow M(C \rtimes \hat{A}) \) and \( M(\hat{A}) \rightarrow M(C \rtimes \hat{A}) \), respectively.

The reduced crossed product \( C \rtimes \hat{A} \) carries a dual coaction \( \hat{\delta}_C \) of \((\alpha \hat{A}, \beta; \hat{\Delta})\):

**Theorem 4.7.** Let \( (K, C, \gamma, \delta_C) \) be a coaction of \((\beta A, \Delta)\). Then there exists a coaction

\[ (K \otimes_{\beta} H, C \rtimes \hat{A}, \gamma \circ \beta, \hat{\delta}_C) \]  \hspace{1cm} (13)

of the concrete Hopf \( C^* \)-bimodule \((\alpha \hat{A}, \beta; \hat{\Delta})\) such that for all \( c \in C \) and \( \hat{a} \in \hat{A} \),

\[ \hat{\delta}_C(\delta_C(c)(1 \otimes \hat{a})) = (\delta_C(c) \otimes 1)(1 \otimes \hat{\Delta}(\hat{a})). \]  \hspace{1cm} (14)

If the coaction \((H, A, \beta; \hat{\Delta})\) is fine, then also the coaction (12) is fine.

**Proof.** The triple \( (K \otimes_{\beta} H, C \rtimes \hat{A}, \gamma \circ \beta) \) is a concrete \( C^* \)-\( C^* \)-bimodule because

\[ \rho_{(\gamma \circ \beta)}(\mathfrak{B})(C \rtimes \hat{A}) \subseteq [(1 \otimes \rho_{\beta}(\mathfrak{B}) \hat{A}) \delta_C(C)] = C \rtimes \hat{A} \]

by Proposition 4.5. Moreover, \( C \rtimes \hat{A} \) is nondegenerate because \( \delta_C(C) \) and \( 1 \otimes \hat{A} \) are nondegenerate (see Remark 4.2 i)).

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Consider the $*$-homomorphism
\[
\tilde{\delta}_C : C \otimes \bar{A} \to \mathcal{L}(K_{\beta} \otimes \delta \beta \otimes \delta \gamma, H), \quad x \mapsto (1 \otimes \bar{V})(x \otimes 1)(1 \otimes \bar{V}^*).
\]
Since $\bar{V}(1 \otimes \bar{V}^*) = a \otimes 1$ for all $a \in A$ (Lemma ??) and $\delta_C(C) \subseteq C \otimes \beta \beta$, we have
\[
(1 \otimes \bar{V})(\delta_C(c) \otimes 1)(1 \otimes \bar{V}^*) = \delta_C(c) \otimes 1 \quad \text{for all } c \in C.
\]
Moreover, by Proposition 3.3, $\bar{V}(a \otimes 1)\bar{V}^* = \hat{\Delta}(a)$ for all $a \in \bar{A}$. Consequently, Equation (13) holds for all $c \in C$ and $\hat{a} \in \hat{A}$.

Let us show that $\tilde{\delta}_C(C \otimes \bar{A}) \subseteq (C \otimes \bar{A}) \gamma \beta \gamma \alpha \beta \gamma$. By Proposition 4.5 and Equation (13),
\[
\tilde{\delta}_C(C \otimes \bar{A}) = [(\delta_C(C) \otimes 1)(1 \otimes \hat{\Delta}(\bar{A}))].
\]
Since $\hat{\Delta}(\bar{A})|\alpha|_{12} \subseteq [\alpha]_{12} \bar{A}$,
\[
\tilde{\delta}_C(C \otimes \bar{A})|\alpha|_{12} \subseteq [(\delta_C(C) \otimes 1)|\alpha|_{12}(1 \otimes \bar{A})]
= [\alpha|_{12}|\delta_C(C)(1 \otimes \bar{A})] = [\alpha|_{12}(C \otimes \bar{A})]. \tag{15}
\]
On the other hand, since $A \beta \bar{A} \subseteq \beta \beta \{10, \text{Lemma } 4.5\}$,
\[
\delta_C(C)|\gamma \beta \bar{A} \subseteq [(C \otimes \beta \beta \gamma \gamma) |\gamma]_{12} \beta] \subseteq [\gamma]_{12} \beta \gamma = |\gamma|_{12} \beta = |\gamma \beta \bar{A}|,
\]
and therefore,
\[
\tilde{\delta}_C(C \otimes \bar{A})|\gamma \beta \bar{A}|_{12} = [(1 \otimes \hat{\Delta}(\bar{A}))|\gamma \beta \bar{A}|_{12}]
\subseteq [(1 \otimes \hat{\Delta}(\bar{A}))|\gamma \beta \bar{A}|_{12}]
= [\gamma]_{11} \hat{\Delta}(\bar{A})|\beta \bar{A}|_{12} = [\gamma]_{12} \beta \gamma = |\gamma \beta \bar{A}|_{12} \bar{A}.
\]
Let us show that the $*$-homomorphism $\tilde{\delta}_C$ is a morphism of concrete $C^\ast_{\text{sa}} \delta \gamma \beta$-algebras. By definition of $\tilde{\delta}_C$,
\[
\tilde{\delta}_C(x) = (1 \otimes \bar{V})(x \otimes 1)|\alpha|_{12} = (1 \otimes \bar{V})(x \otimes 1)|\alpha|_{12} x
\]
for each $x \in C \otimes \bar{A}$ and $\xi \in \hat{\beta}$, and therefore,
\[
\gamma \beta \bar{A} \simeq \tilde{\delta}_C(x)(1 \otimes \bar{V})(x \otimes 1)|\alpha|_{12} = (1 \otimes \bar{V})(x \otimes 1)|\alpha|_{12} x
\]
\[
\tilde{\delta}_C \circ \text{id} = \tilde{\delta}_C \circ \text{id} \circ \hat{\Delta} : C \otimes \bar{A} \to \mathcal{L}(K_{\beta} \otimes \delta \beta \otimes \delta \gamma, H)
\]
are both given by $\delta_C(c) (1 \otimes \hat{a}) \to (\delta_C(c) \otimes 1)(1 \otimes \hat{\Delta}(\hat{a}))$ for all $c \in C$, $\hat{a} \in \hat{A}$, where $\hat{\Delta}(\hat{a}) := (\hat{\Delta} \circ \text{id}) \circ \hat{\Delta} : \hat{A} \to \mathcal{L}(H \otimes \delta \beta \otimes \delta \gamma, H)$. Summarizing, we find that (12) is a coaction of $(a \hat{A}, \hat{\Delta})$ as claimed. By construction, $\tilde{\delta}_C$ is injective, and Equation (??) shows that this coaction is fine if the coaction $(H, A, \hat{\beta}, \hat{\Delta})$ is fine. \hfill \bx
Definition 4.8. Let \((K, C, \gamma, \delta_C)\) be a coaction of \((\beta A_n, \Delta)\). We call \((K, \otimes_\beta H, C \rtimes_\alpha \hat{A}, \gamma \circ \hat{\delta}, \hat{\delta}_C)\) the associated dual coaction of \((\alpha \hat{A}_\beta, \hat{\Delta})\).

The reduced crossed product construction is functorial in the following sense:

**Proposition 4.9.** Let \(\rho\) be a covariant morphism between a coaction \((K, C, \gamma, \delta_C)\) and a coaction \((L, D, \delta, \delta_D)\) of \((\beta A_n, \Delta)\). Then there exists a unique covariant morphism \(\rho \rtimes \text{id}\) between the associated dual coactions such that for all \(c \in C\) and \(\alpha, \beta\),

\[
(\rho \rtimes \text{id})(\delta_C(c)(1 \otimes \alpha)) = \delta_D(\rho(c))(1 \otimes \beta).
\]  

(16)

**Proof.** By Remark 4.4, \(C \rtimes_\alpha \hat{A} \subseteq C \otimes_\beta \mathcal{L}(H)\) and \(D \rtimes_\beta \hat{A} \subseteq D \otimes_\beta \mathcal{L}(H)\), and by [10, Theorem 3.15], there exists a morphism

\[
\rho \rtimes \text{id} \in \text{Mor}\big((C \otimes_\beta \mathcal{L}(H))_{\gamma \circ \hat{\delta}}, (D \otimes_\beta \mathcal{L}(H))_{\delta \circ \hat{\delta}}\big).
\]

Denote by \(\rho \rtimes \text{id}: C \rtimes_\alpha \hat{A} \to D \rtimes_\beta \hat{A}\) the restriction of \(\rho \rtimes \text{id}\). Then

\[
(\rho \rtimes \text{id})(\delta_C(c)(1 \otimes \alpha)) = (\rho \rtimes \text{id})(\delta_C(c)) \cdot (\rho \rtimes \text{id})(1 \otimes \alpha) = \delta_D(\rho(c))(1 \otimes \beta)
\]

for all \(c \in C\) and \(\alpha \in \hat{A}\). In particular, \(\rho \rtimes \text{id} \in \text{Mor}\big((C \rtimes_\alpha \hat{A})_{\gamma \circ \hat{\delta}}, (D \rtimes_\beta \hat{A})_{\delta \circ \hat{\delta}}\big)\). Finally, Equations (13) and (14) imply that \(\rho \rtimes \text{id}\) is covariant with respect to \(\hat{\delta}_C\) and \(\hat{\delta}_D\).

**Corollary 4.10.** The assignments \((K, C, \gamma, \delta_C) \mapsto (K, \otimes_\beta H, C \rtimes_\alpha \hat{A}, \gamma \circ \hat{\delta}, \hat{\delta}_C)\) and \(\rho \mapsto \rho \rtimes \text{id}\) define a functor from the category of coactions of \((\beta A_n, \Delta)\) to the category of coactions of \((\alpha \hat{A}_\beta, \hat{\Delta})\).

**Reduced crossed products for coactions of \((\beta \delta)\) We extend the reduced crossed product construction to coactions of the concrete Hopf \(C^*-\)bimodule \((\alpha \hat{A}_\beta, \hat{\Delta})\) as follows:

**Definition 4.11.** Let \((K, C, \gamma, \delta_C)\) be a coaction of \((\alpha \hat{A}_\beta, \hat{\Delta})\). The associated reduced crossed product is the \(C^*-\)subalgebra \(C \rtimes_{\delta_C, \gamma} A \subseteq \mathcal{L}(K, \otimes_\alpha H)\) generated by

\[
\delta(C)(1 \otimes \text{Ad}_\gamma(A)) \subseteq \mathcal{L}(K, \otimes_\alpha H).
\]

(17)

If \(\delta_C\) is understood, we shortly write \(C \rtimes_\alpha A\) for \(C \rtimes_{\delta_C, \gamma} A\).

This construction has the same formal properties like the reduced crossed product for coactions of \((\beta A_n, \Delta)\); for the proofs, one simply replaces the weak \(C^*-\)pseudococycle \((\alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)\) by its predual, applies Proposition 3.3, and uses the results of the preceding paragraph.

**Proposition 4.12.** Let \((K, C, \gamma, \delta_C)\) be a coaction of \((\alpha \hat{A}_\beta, \hat{\Delta})\). Then \(C \rtimes_\alpha A = [\delta_C(C)(1 \otimes \text{Ad}_\gamma(A))]\).

**Theorem 4.13.** Let \((K, C, \gamma, \delta_C)\) be a coaction of \((\alpha \hat{A}_\beta, \hat{\Delta})\). There exists a coaction \((K, \otimes_\beta H, C \rtimes_\alpha A, \gamma \circ \hat{\delta}, \hat{\delta}_C)\) of \((\beta A_n, \Delta)\) such that

\[
\hat{\delta}_C(\delta_C(c)(1 \otimes \text{Ad}_\gamma(a))) = (\delta_C(c) \otimes 1)(1 \otimes \text{Ad}(\varphi(a)) \Delta(a))
\]

for all \(c \in C\) and \(a \in A\). If the coaction \((H, A, \alpha, \Delta)\) is fine, then also \((K, \otimes_\beta H, C \rtimes_\alpha A, \gamma \circ \hat{\delta}, \hat{\delta}_C)\) is fine.
Definition 4.14. Let \((K, C, \gamma, \delta_C)\) be a coaction of \((\hat{A}, \hat{\Delta})\). Then we call \((K, \Theta \hat{A}, H, C \times_r A, \gamma \circ \hat{\alpha}, \hat{\delta}_C)\) the associated dual coaction of \((\hat{A}, \Delta)\).

Proposition 4.15. Let \(\rho\) be a covariant morphism between a coaction \((K, C, \gamma, \delta_C)\) and a coaction \((L, D, \delta_D)\) of \((\hat{A}, \hat{\Delta})\). Then there exists a unique covariant morphism \(\rho \times_r \text{id}\) between the associated dual coactions such that

\[
(\rho \times_r \text{id})(\delta_C(c)(1 \otimes \text{Ad}_r(a))) = \delta_D(\rho(c))(1 \otimes \text{Ad}_r(a))
\]

for all \(c \in C\) and \(a \in A\).

Corollary 4.16. The assignments \((K, C, \gamma, \delta_C) \mapsto (K, \Theta \hat{A}, H, C \times_r A, \gamma \circ \hat{\alpha}, \hat{\delta}_C)\) and \(\rho \mapsto \rho \times_r \text{id}\) define a functor from the category of coactions of \((\hat{A}, \hat{\Delta})\) to the category of coactions of \(C\).

5 \(C^*\)-pseudo-Kac systems

To obtain an analogue of Baaj-Skandalis duality [1], we need to refine the notion of a weak \(C^*\)-pseudo-Kac system as follows.

As before, we fix the \(C^*\)-base \(\mathcal{H}\), and the Hilbert space \(\mathcal{H}\).

Definition 5.1. We call \(\alpha, \beta, \delta\), \(\mathcal{U}, \mathcal{V}\) a \(C^*\)-pseudo-Kac-system if

i) the \(C^*\)-pseudo-multiplicative unitaries \(\mathcal{V}, \hat{\mathcal{V}}, \hat{\mathcal{V}}\) are regular, and

ii) \((\Sigma(1 \otimes \mathcal{U})\mathcal{V})^3 = 1\) in \(\mathcal{L}(\mathcal{H} \otimes \mathcal{H})\).

Remark 5.2. In leg notation, the equation \(\left(\Sigma(1 \otimes \mathcal{U})\mathcal{V}\right)^3 = 1\) takes the form \(\left(\Sigma \left[U_{12}\right] V\right)^3 = 1\).

Conjugating by \(\Sigma\) or \(\mathcal{V}\), we find that this condition is equivalent to the relation \(\left(U_{12}\mathcal{V}\Sigma\right)^3 = 1\) and to the relation \(\left(V \Sigma U_{12}\right)^3 = 1\).

Remark 5.3. Definition 5.1 greatly simplifies the definition of a pseudo-Kac system on \(C^*\)-modules given in [9] and still covers our main examples. The two concepts are related as follows.

Let \((\alpha, \hat{\alpha}, \beta, \hat{\beta}, \mathcal{U}, \mathcal{V})\) be a \(C^*\)-pseudo-Kac system and assume that \(\mathcal{B} \cong \mathcal{B}^\alpha\). Then the \(C^*\)-modules \(\alpha, \hat{\alpha}, \beta, \hat{\beta}\), the representations \(\rho_\alpha, \rho_\hat{\alpha}, \rho_\beta, \rho_\hat{\beta}\), the unitaries \(U_\alpha : \alpha \to \hat{\alpha}, U_\beta : \beta \to \hat{\beta}, U_{\hat{\alpha}} : \hat{\alpha} \to \alpha, U_{\hat{\beta}} : \hat{\beta} \to \beta\) together with the family of unitaries

\[
\begin{align*}
V_{\alpha \beta} : & \alpha \otimes \beta \to \alpha \otimes \beta, \\
V_{\hat{\alpha} \hat{\beta}} : & \hat{\alpha} \otimes \hat{\beta} \to \hat{\beta} \otimes \hat{\alpha}, \\
V_{\beta \hat{\alpha}} : & \beta \otimes \hat{\alpha} \to \beta \otimes \hat{\alpha}, \\
V_{\hat{\beta} \alpha} : & \hat{\beta} \otimes \alpha \to \hat{\beta} \otimes \alpha,
\end{align*}
\]

form a pseudo-Kac system in the sense of [9, Definition 2.53].

We shall use the following reformulation of condition ii) in Definition 5.1:

Lemma 5.4. Let \((\alpha, \hat{\alpha}, \beta, \hat{\beta}, \mathcal{U}, \mathcal{V})\) be a balanced \(C^*\)-pseudo-multiplicative unitary. Then

\[
(\Sigma \left[U_{12}\right]V)^3 = 1 \text{ if and only if } \hat{V} \mathcal{V} \hat{V} = U_{12}\Sigma.
\]

Proof. Rearranging the factors in the product

\[
U_{12}\left(\Sigma \left[U_{12}\right]V\right)^3 U_{12}\Sigma = \Sigma U_{12} V U_{12} \Sigma \cdot V \cdot \Sigma U_{12} V U_{12} \Sigma = \hat{V} \cdot V \cdot \hat{V}.
\]

Thus, \(\left(\Sigma U_{12}\right)^3 = 1\) if and only if \(U_{12}\Sigma = U_{12} U_{12} U_{12}\Sigma\) is equal to \(\hat{V} \mathcal{V} \hat{V}\).

The notion of a \(C^*\)-pseudo-Kac system is symmetric in the following sense:
Proposition 5.5. Let \((\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, U, V)\) be a \(\ast\)-pseudo-Kac system. Then also the tuples \((\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, U, V)\) are \(\ast\)-pseudo-Kac systems.

Proof. Equations (6), (7) and [10, Remark 4.11] imply that the tuples \((\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, U, V)\) satisfy condition i) of Definition 5.1. To check that they also satisfy condition ii), we use Remark 5.2:

\[
(\Sigma U_{12} V)^3 = (V \Sigma U_{12})^3 = 1, \quad (\tilde{V} \Sigma U_{12})^3 = (\Sigma U_{12} V)^3 = 1, \\
(U_2 V^* \Sigma)^3 = (U_{23} V U^*)^3 = ((V \Sigma U_{12})^3)^* = 1.
\]

The following two propositions are essential for our duality theorem:

Proposition 5.6. Let \((\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, U, V)\) be a \(\ast\)-pseudo-Kac system. Then it is a weak \(\ast\)-pseudo-Kac system.

Proof. Since \(V\) is regular, it is well-behaved. We show that condition i)(a) in Lemma ?? holds. By Lemma 3.4, \(V_{123} = V_{13} V_{23} V_{12}\). We multiply this equation by \(V_{12}\) on the left and by \(\Sigma_{123}\) on the right, use the pentagon equation for \(V\), and obtain

\[
V_{123} V_{12} \Sigma_{123} V_{13} V_{23} = V_{13} V_{123} V_{23} V_{13} V_{123} = V_{123} V_{13} V_{23} V_{123} V_{13} = V_{123} V_{13} V_{23} V_{123} V_{13}.
\]

By Lemma ??, we can replace \(V_{123} V_{12} \Sigma_{123} V_{13} V_{23}\) by \(V_{123}^\ast U_{11}\), and find that \(V_{123}^\ast U_{11} V_{23} = V_{23} V_{123}^\ast U_{11}\). Since \(V_{123}\) is unitary and \(U_{11} V_{23} = V_{23} U_{11}\), we can conclude \(V_{123}^\ast V_{23} = V_{23}^\ast V_{123}\), that is, condition i)(a) in Lemma ?? holds. A similar argument shows that condition ii)(a) in Lemma ?? holds.

Proposition 5.7. Let \((\alpha, \tilde{\alpha}, \beta, \tilde{\beta}, U, V)\) be a \(\ast\)-pseudo-Kac system. Then \([A(V) \tilde{\alpha}](V)] = [\tilde{\alpha} \alpha^*]\).

We need the following lemma:

Lemma 5.8. Let \(V: H_\beta \otimes H \rightarrow H_\alpha \otimes H\) be a regular \(\ast\)-pseudo-multiplicative unitary. Then \([V \alpha]_{[2]} \tilde{\epsilon}(V) = [\beta]_{[2]} \tilde{\epsilon}(V)\).

Proof. The first commutative diagram in [10, Proof of Proposition 4.12] shows

\[
[V \alpha]_{[2]} \tilde{\epsilon}(V) = [V \alpha]_{[2]} \tilde{\epsilon}(V)^* = [\alpha]_{[2]} V_{12}^\ast [\beta]_{[3]} [\beta]_{[2]},
\]

and the following commutative diagram shows that this equals \([\alpha]_{[2]} \tilde{\epsilon}(V)^* = [\beta]_{[2]} \tilde{\epsilon}(V)\):

\[
\begin{array}{ccc}
H & \xrightarrow{V_{12}} & H_{\alpha} \otimes H_{\beta} \\
H_{[\alpha]} & \xrightarrow{V_{12}^\ast} & (H_{[\beta]} \otimes H_{[\beta]})_{[\alpha]_{[2]} \otimes [\beta]_{[2]}} \\
\end{array}
\]

Proof of Proposition 5.5. Since \(V\) is regular and \(V^* = \Sigma U_{12} V \Sigma U_{12} V \Sigma U_{12}\),

\[
[\tilde{\alpha} \beta^*] = [U \alpha \beta^* U] = [U \alpha]_{[2]} V^* [\alpha]_{[1]} U] = [U \alpha]_{[2]} V \Sigma U_{12} V \Sigma U_{12} [\alpha]_{[1]} U] = [\alpha]_{[1]} V \Sigma U_{12} V [\alpha]_{[2]} V = [\alpha]_{[1]} V \Sigma U_{12} V [\alpha]_{[2]} V = [\alpha]_{[1]} V \Sigma U_{12} V [\alpha]_{[2]} V.
\]

We multiply on the right by \(\tilde{\alpha} := \tilde{\alpha}(V)\), use Equation (8) and Lemma 5.6, and find

\[
[\tilde{\alpha} \alpha^*] = [\tilde{\alpha} \alpha^* \tilde{\alpha}] = [\alpha]_{[1]} V \Sigma U_{12} [\alpha]_{[2]} [\tilde{\alpha}] = [\alpha]_{[1]} V \Sigma U_{12} [\beta]_{[2]} [\tilde{\alpha}] = [\alpha]_{[1]} V \Sigma U_{12} [\beta]_{[2]} \tilde{\alpha} = [\alpha]_{[1]} V [\tilde{\beta}]_{[1]} \tilde{\alpha} = [\alpha]_{[1]} V [\tilde{\alpha}](V) \tilde{\alpha}(V) = [\alpha]_{[1]} V [\tilde{\alpha}](V) \tilde{\alpha}(V).
\]
The duality theorem Let \((\alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)\) be a C*-pseudo-Kac system, put

\[\hat{A} := \hat{A}(V), \quad \hat{\Delta} := \hat{\Delta}_V, \quad A := A(V), \quad \Delta := \Delta_V,\]

and write \((\alpha, \hat{\alpha}, \Delta)\) and \((\beta, \hat{\beta}, \Delta)\) for the concrete Hopf C*-bimodules \((\hat{\beta} \triangleright_{\beta, Y}, H, \hat{\alpha}, \beta, \hat{\beta}, \Delta)\) and \((\hat{\alpha} \triangleright_{\hat{\alpha}, Y}, H, A, \alpha, \beta, \Delta)\), respectively.

Before we can state the duality theorem, we need some preliminaries. Let \((K, C, \gamma, \delta_C)\) be a coaction of \((\beta, \alpha, \Delta)\). By Lemma 2.3, \([\beta]_{[2]} C(\beta)_{13} \subseteq L(K, \otimes_{\delta_H} H_{\beta})\) is a C*-algebra and there exists a \(*\)-homomorphism

\[\text{Ind}_a(\delta_C) : [\beta]_{[2]} C(\beta)_{13} \to [\beta]_{[2]} \delta_C(C)(\beta)_{13} \subseteq L(K, \otimes_{\delta_H} H_{\beta}).\]

Likewise, if \((K, C, \gamma, \delta_C)\) is a coaction of \((\alpha, \hat{\alpha}, \hat{\Delta})\), then \([\alpha]_{[2]} C(\alpha)_{13} \subseteq L(K, \otimes_{\delta_H} H_{\alpha})\) is a C*-algebra and there exists a \(*\)-homomorphism

\[\text{Ind}_a(\delta_C) : [\alpha]_{[2]} C(\alpha)_{13} \to [\alpha]_{[2]} \delta_C(C)(\alpha)_{13} \subseteq L(K, \otimes_{\delta_H} H_{\alpha}).\]

**Theorem 5.9.**

1. Let \((K, C, \gamma, \delta_C)\) be a fine coaction of \((\beta, \alpha, \Delta)\). There exists a natural isomorphism

\[C \rtimes_{\gamma} \hat{A} \rtimes_{\alpha} A \cong [\beta]_{[2]} C(\beta)_{13} \subseteq L(K, \otimes_{\delta_H} H)\]

which identifies the bidual coaction \(\delta_{C}\) on \(C \rtimes_{\gamma} \hat{A} \rtimes_{\alpha} A\) with the map \(\text{Ad}_{(\delta_{C})^{-1}} \circ \text{Ind}_a(\delta_C)\).

2. Let \((K, C, \gamma, \delta_C)\) be a fine coaction of \((\alpha, \hat{\alpha}, \hat{\Delta})\). There exists a natural isomorphism

\[C \rtimes_{\gamma} A \rtimes_{\alpha} \hat{A} \cong [\alpha]_{[2]} C(\alpha)_{13} \subseteq L(K, \otimes_{\delta_H} H)\]

which identifies the bidual coaction \(\delta_{C}\) on \(C \rtimes_{\gamma} A \rtimes_{\alpha} \hat{A}\) with the map \(\text{Ad}_{(\delta_{C})^{-1}} \circ \text{Ind}_a(\delta_C)\).

**Proof.**

1. By Proposition 4.8, and Proposition 4.5, the iterated crossed product \(C \rtimes_{\gamma} \hat{A} \rtimes_{\alpha} A\) is equal to

\[[\delta_{C}(C) \otimes 1] (1 \otimes \hat{\Delta}(\hat{A})) (1 \otimes 1 \otimes \text{Ad}_C(A)) \subseteq L(K, \otimes_{\delta_H} H_{\beta, \alpha}).\]

By definition of \(\Delta\) and \(\hat{\Delta}\) and by Lemma 2.3, conjugation by \(1 \otimes V\) maps this C*-algebra isomorphically onto

\[[\delta_{C}^{(2)}(C) \otimes 1] (1 \otimes 1 \otimes \hat{\Delta}(\hat{A})) (1 \otimes 1 \otimes \text{Ad}_C(A)) \subseteq L(K, \otimes_{\delta_H} H_{\beta, \alpha}).\]

where \(\delta_{C}^{(2)} = (\text{id} \circ \Delta) \circ \delta_{C} = (\delta_{C} \circ \text{id}) \circ \delta_{C} \circ \text{id}.\) Since \(\delta_{C}\) is injective, the formula

\[\delta_{C}^{(2)}(c)(1 \otimes 1 \otimes T) \mapsto \delta_{C}(c)(1 \otimes T), \quad \text{where } c \in C, \ T \in L(H),\]

defines an isomorphism of this C*-algebra onto

\[[\delta_{C}(C) \otimes 1] (1 \otimes \hat{\Delta} \cdot \text{Ad}_C(A)) \subseteq L(K, \otimes_{\delta_H} H).\]

By Proposition 3.3 and Proposition 5.5, applied to the C*-pseudo-Kac system \((\hat{\beta}, \beta, \alpha, \hat{\alpha}, U, \hat{V})\), we have \([\hat{A} \text{Ad}_C(A)] = [A(\hat{V}) \hat{A}(\hat{V})] = [\beta_{\beta_{\alpha}}].\) We insert this relation into \((?)\), use the fact that \(\delta_{C}\) is a fine coaction, and find

\[C \rtimes_{\gamma} \hat{A} \rtimes_{\alpha} A \cong [\delta(C) \beta]_{[2]}(\beta)_{13} \subseteq L(K, \otimes_{\delta_H} H).\]
This isomorphism identifies an element of the form

$$(\delta_C(c) \otimes 1)(1 \otimes \hat{\Delta}(\hat{a})) (1 \otimes 1 \otimes \text{Ad}_C(a)) \in C \rtimes_s \hat{A} \rtimes_s A$$

with

$$\delta_C(c)(1 \otimes \hat{a} \cdot \text{Ad}_C(a)) \in \{1\} \otimes C(\bar{\beta}|[2])$$.

The bidual coaction $\hat{\delta}_C$ maps the element (19) to

$$(\delta_C(c) \otimes 1 \otimes 1)(1 \otimes \hat{\Delta}(\hat{a}) \otimes 1)(1 \otimes 1 \otimes \text{Ad}_C(U\otimes 1)(\Delta(a)))$$,

and the map $\text{Ad}_{(U\otimes 1) \circ \text{Ind}_C(\delta_C)}$ sends the element (19) to

$$\text{Ad}_{(U\otimes 1) \circ \text{Ind}_C(\delta_C)}(\theta_C^{(2)}(\delta_C(c)(1 \otimes \hat{a} \cdot \text{Ad}_C(a)))) = (\delta_C(c) \otimes 1)(1 \otimes \hat{a} \otimes 1)(1 \otimes \text{Ad}_C(U\otimes 1)(\Delta(a)))$$.

Therefore, the bidual coaction $\hat{\delta}_C$ on $C \rtimes_s \hat{A} \rtimes_s A$ corresponds to the map $\text{Ad}_{(U\otimes 1) \circ \text{Ind}_C(\delta_C)}$.

ii) The proof is similar to the proof of i), simply replace the $C^*$-pseudo-Kac system $(\alpha, \hat{\alpha}, \beta, \hat{\beta}, U, V)$ by its predual $(\hat{\beta}, \beta, \alpha, \hat{\alpha}, U, V)$.

6 The $C^*$-pseudo-Kac system of a locally compact groupoid

The prototypical example of a $C^*$-pseudo-Kac system is the one associated to a locally compact groupoid. The underlying $C^*$-pseudo-multiplicative unitary was described in [10]. For background on groupoids, Haar measures, and quasi-invariant measures, see [7] or [6].

The data Let $G$ be a locally compact, Hausdorff, second countable groupoid. We denote its unit space by $G^0$, its range map by $r_G$, its source map by $s_G$, and put $G_u := r_G^{-1}(\{u\})$, $G_u := s_G^{-1}(u)$ for each $u \in G^0$.

We assume that $G$ has a left Haar system $\lambda$, and denote the associated right Haar system by $\lambda^{-1}$. Let $\mu$ be a measure on $G^0$ and denote by $\nu$ the measure on $G$ given by

$$\int_G f \, d\nu := \int_{G^0} \int_{G_u} f(x) \, d\lambda^{-1}(x) \, d\mu(u) \quad \text{for all } f \in C_c(G)$$.

The push-forward of $\nu$ via the inversion map $G \to G, x \mapsto x^{-1}$, is denoted by $\nu^{-1}$; evidently,

$$\int_G f \, d\nu^{-1} = \int_{G^0} \int_{G_u} f(x) \, d\lambda^{-1}(x) \, d\mu(u)$$.

We assume that the measure $\mu$ is quasi-invariant, i.e., that $\nu$ and $\nu^{-1}$ are equivalent. Note that there always exist sufficiently many quasi-invariant measures [7]. We denote by $D := d\nu/d\nu^{-1}$ the Radon-Nikodym derivative.
The $C^*$-base and the Hilbert space  The measure $\mu$ defines a tracial proper weight on the $C^*$-algebra $C_0(G^0)$, which we denote by $\mu$ again. Moreover, we denote by $\mathfrak{H}_\mathfrak{M}$, the $C^*$-base associated to $\mu$ (see Section 2); thus, $\mathfrak{H} = L^2(G^0, \mu)$. Note that $\mathfrak{H}_\mathfrak{M} = \mathfrak{H}_\mathfrak{M}$ because $C_0(G^0)$ is commutative. Put $H := L^2(G, \nu)$.

The $C^*$-factorizations  The space $C_c(G)$ forms a pre-$C^*$-module over $C_0(G^0)$ with respect to the structure maps

$$\langle \xi', \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)} \xi(x) d\lambda^u(x), \quad (\xi f)(x) = \xi(x) f(r_c(x)),$$

and also with respect to the structure maps

$$\langle \xi', \xi \rangle(u) = \int_{G_u} \overline{\xi'(x)} \xi(x) d\lambda^u_1(x), \quad (\xi f)(x) = \xi(x) f(s_c(x)).$$

Denote the completions of these pre-$C^*$-modules by $L^2(G, \lambda)$ and $L^2(G, \lambda^{-1})$, respectively. By [10, Proposition 5.1], there exist isometric embeddings

$$j : L^2(G, \lambda) \to L(\mathfrak{H}, H) \quad \text{and} \quad \hat{j} : L^2(G, \lambda^{-1}) \to L(\mathfrak{H}, H)$$

such that for all $\xi \in C_c(G), \zeta \in L^2(G^0, \mu), x \in G$,

$$(j(\xi) \zeta)(x) = \xi(x) \zeta(r_c(x)), \quad (\hat{j}(\xi) \zeta)(x) = \xi(x) \hat{D}^{-1/2}(x) \zeta(s_c(x)).$$

Moreover, by [10, Proposition 5.1], the images

$$\alpha := \beta := j(L^2(G, \lambda)) \quad \text{and} \quad \hat{\alpha} := \hat{\beta} := \hat{j}(L^2(G, \lambda^{-1}))$$

are compatible $C^*$-factorizations of $H$ with respect to $\mathfrak{H}_\mathfrak{M}$, the maps $j$ and $\hat{j}$ are unitary as maps of $C^*$-modules over $C_0(G^0) \cong \mathfrak{M}$, and for all $x \in G, \xi \in C_c(G)$, and $f \in C_0(G^0)$,

$$(\rho_\lambda(f) \xi)(x) := f(r_c(x)) \xi(x), \quad (\rho_\lambda(f) \xi)(x) := f(s_c(x)) \xi(x).$$

The $C^*$-pseudo-multiplicative unitary  By [12] and [10, Proposition 2.14], the Hilbert spaces $H_\beta \otimes_\beta H$ and $H_\alpha \otimes_\beta H$ can be described as follows. Define a measure $\nu_{s,r}^2$ on $G_{s,r}^2 := \{(x, y) \in G \times G \mid s(x) = r(y)\}$ by

$$\int_{G_{s,r}^2} f \, d\nu_{s,r}^2 := \int_{G^0} \int_{G^s} \int_{G^r} f(x, y) d\lambda^s(x) d\lambda^r(x) d\mu(u),$$

and a measure $\nu_{r,s}^2$ on $G_{r,s}^2 := \{(x, y) \in G^2 \mid r_c(x) = r_c(y)\}$ by

$$\int_{G_{r,s}^2} g \, d\nu_{r,s}^2 := \int_{G^0} \int_{G^s} \int_{G^r} g(x, y) d\lambda^s(x) d\lambda^r(y) d\mu(u),$$

where $f \in C_c(G_{s,r}^2)$ and $g \in C_c(G_{r,s}^2)$. By [12] and [10, Theorem 5.2], there exist isomorphisms

$$H_\beta \otimes_\beta H \cong L^2(G_{s,r}^2, \nu_{s,r}^2), \quad H_\alpha \otimes_\beta H \cong L^2(G_{r,s}^2, \nu_{r,s}^2)$$

and a $C^*$-pseudo-multiplicative unitary $V : H_\beta \otimes_\beta H \to H_\alpha \otimes_\beta H$ such that, with respect to these isomorphisms, $(V \xi)(x, y) = \zeta(x, x^{-1}y)$ for all $\xi \in L^2(G_{s,r}^2, \nu_{s,r}^2)$ and $(x, y) \in G_{s,r}^2$. 

19
The $C^*$-pseudo-Kac system

By definition of the Radon-Nikodym derivative $D = \frac{d\nu}{d\nu^{-1}}$, there exists a unitary $U \in \mathcal{L}(H)$ such that

$$(U\xi)(x) := \xi(x^{-1})D(x)^{-1/2} \quad \text{for all } x \in G, \xi \in C_c(G).$$

This unitary is a symmetry because for all $\xi \in C_c(G)$ and $\nu$-almost all $x \in G$,

$$(U^2\xi)(x) = (U\xi)(x^{-1})D(x)^{-1/2} = \xi(x)D(x)^{1/2}D(x)^{-1/2} = \xi(x);$$

here, we used the relation $D(x)^{-1} = D(x^{-1})$ [7, p. 23], [6, Eq. (3.7)].

**Theorem 6.1.** $(\alpha, \hat{\gamma}, \beta, \hat{\beta}, V, U)$ is a $C^*$-pseudo-Kac system.

**Proof.** By [10, Proposition 5.3], the $C^*$-pseudo-multiplicative unitary $V$ is regular.

We claim that $\tilde{V} = \Sigma U_1 V U_1 \Sigma$ is equal to $V^\text{op} = \Sigma V^\ast \Sigma$. Indeed, for all $\xi \in L^2(G^2_{2,r}, \nu_{\ast, r}^2)$ and $(x, y) \in G^2_{2,r}$,

$$(U_1 V U_1 \xi)(x, y) = (V U_1 \xi)(x^{-1}, y)D(x)^{-1/2} = (U_1 \xi)(x^{-1}, xy)D(x)^{-1/2} = \zeta(x, xy)D(x)^{-1/2}D(x)^{-1/2}.$$

Since $D(x)^{-1/2}D(x)^{-1/2} = 1$ for $\nu$-almost all $x \in G$ [7, p. 23], [6, Eq. (3.7)], we can conclude $U_1 V U_1 = V^\ast$, and the claim follows.

In particular, $\tilde{V} = V^\text{op}$ is a regular $C^*$-pseudo-multiplicative unitary [10, Remark 4.11].

To finish the proof, we only need to show that the map $Z := \Sigma U_2 V : \mathcal{H}_{\beta} \otimes_{\beta} H \to \mathcal{H}_{\beta} \otimes_{\beta} H$ satisfies $Z^2 = 1$. For all $\zeta \in L^2(G^2_{2,r}, \nu_{\ast, r}^2)$ and $(x, y) \in G^2_{2,r}$,

$$(\Sigma U_2 V \zeta)(x, y) = (V \zeta)(y^{-1}x^{-1})D(x)^{-1/2} = \zeta(y, y^{-1}x^{-1})D(x)^{-1/2},$$

and therefore,

$$(Z^2 \zeta)(x, y) = (Z^2 \zeta)(y, y^{-1}x^{-1})D(x)^{-1/2} = (Z \zeta)(y^{-1}x^{-1}, xy)D(x)D(y)^{-1/2} = \zeta(x, x^{-1}xy)(D(x)D(y)(D(x)^{-1}x^{-1})^{-1/2}.$$ 

Since $D(x)D(y)(D(x)^{-1}x^{-1}) = 1$ for $\nu_{\ast, r}^2$-almost all $(x, y) \in G^2_{2,r}$ [3] (but see also [6, p. 89]), we can conclude $Z = \text{id}$. 

**References**


