On the isometry group of the Urysohn space

joint work with M. Ziegler

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Automorphism groups of homogeneous structures

In many cases, the automorphism group of a 'very homogeneous structure' is (essentially) a simple group:

1. Riemannian symmetric spaces of non-compact type (Cartan)
2. the automorphism groups of projective spaces or, more generally, of Tits buildings (Tits)
3. the homeomorphism group of the Cantor space (Anderson)
4. a homogeneous tree (Tits)
5. the random graph, or the universal K_n-free graph (Truss)
6. $Sym(\mathbb{N})/Fin(\mathbb{N})$ where $Fin(\mathbb{N})$ is the group of permutations with finite support
7. $\text{Aut}(\mathbb{C}/\tilde{\mathbb{Q}})$ (Lascar)

Generally, there is a very close connection between the automorphism groups and the corresponding spaces: the space can be 're-discovered' from its automorphism group.

The Urysohn space

The Urysohn space $\mathbb{U}$ is the unique separable complete metric space which is homogeneous for finite subspaces and in which every finite metric space can be embedded.

homogeneous: for any two finite subsets $A, B \subset \mathbb{U}$ and isometry $f : A \to B$ there is an isometry $g \in \mathrm{Isom}(\mathbb{U})$ inducing f.

In fact, $\mathbb{U}$ embeds every separable metric space and is *strongly homogenoues*, i.e. every isometry between totally bounded subspaces extends to an isometry of $\mathbb{U}$.

A metric space is *totally bounded* if and only if for every $\epsilon > 0$ the space can be covered by a finite collection of open balls of radius ϵ.

How to construct the Urysohn space:

Let $\mathcal{F}$ be the class of finite metric spaces with rational distances. This class is countable and closed under

1. (JEP) joint embedding;
2. (AP) amalgamation;
3. (HP) hereditary: any finite subset of some A in $\mathcal{F}$ is again in $\mathcal{F}$.

By Fraïssé's Theorem, there is a countable metric space $\mathbb{U}_0$ with rational distances which is homogeneous for finite subspaces. Its completion is the Urysohn space $\mathbb{U}$.

Canonical amalgamation: For finite metric spaces $B \subseteq A, C$ we put a metric on $A \cup C$ by defining for $a \in A, c \in C$

$$d(a, c) = \min\{d(a, b) + d(b, c) \colon b \in B\}.$$

Thus, $\mathbb{U}$ is a 'very homogeneous' space. However, $\mathrm{Isom}(\mathbb{U})$ is not a simple group:

For a metric space X, we say that $g \in \mathrm{Isom}(X)$ is *bounded* if it has bounded displacement, i.e. if there is some $b \in \mathbb{N}$ such that for all $x \in X$ we have

$$d(x, g(x)) \leq b.$$

$\mathrm{Bd}(\mathbb{U}) = \{g \in \mathrm{Isom}(\mathbb{U})\colon g \text{ bounded}\}$ is a nontrivial normal subgroup of $\mathrm{Isom}(\mathbb{U})$.

Clearly, $\mathrm{Bd}(\mathbb{U})$ is normal.

$\mathrm{Bd}(\mathbb{U}) = \{g \in \mathrm{Isom}(\mathbb{U})\colon g \text{ bounded}\}$ is nontrivial:

To see that $\mathrm{Bd}(\mathbb{U}) \neq 1$, fix some $b \in \mathbb{N}$ and define a bounded isometry on $\mathbb{U}_0$ by 'back-and-forth'. Since $\mathbb{U}_0$ is dense in $\mathbb{U}$, this extends to a bounded isometry on $\mathbb{U}$.

To see that $\mathrm{Bd}(\mathbb{U}) \neq \mathrm{Isom}(\mathbb{U})$, we use a 'back-and-forth' method to move every element 'as far as possible'. (See below...)

Theorem 1

$\mathrm{Isom}(\mathbb{U})/\mathrm{Bd}(\mathbb{U})$ *is a simple group. In fact, if* $g \in \mathrm{Isom}(\mathbb{U})$ *is unbounded, then any element of* $\mathrm{Isom}(\mathbb{U})$ *is the product of* 8 *conjugates of* g.

This will follow from a more general theorem using model theoretic techniques.

Independence in metric spaces

For finite nonempty subsets A, B, C of a metric space X we say that A and C are *independent over* B

$$A \underset{B}{\downarrow} C$$

if for all $a \in A, c \in C$ there is some $b \in B$ such that

$$d(a,c) = d(a,b) + d(b,c).$$

Note that $A \downarrow_B C$ if and only if the metric space $A \cup B \cup C$ is isometric to the canonical amalgam of $A \cup B$ and $C \cup B$ over B defined above.

If A and C are independent over B then for any two isometries f of $A \cup B$ and g of $B \cup C$ with $f \upharpoonright B = g \upharpoonright B = \mathrm{id}_B$ their union $f \cup g$ is an isometry of $A \cup B \cup C$.

Notation: We write $A \cong_B C$ if and only there is an automorphism fixing B pointwise and taking A to C (as ordered tuples) and AB for $A \cup B$.

This notion of independence satisfies:

(Inv) For any automorphism g we have $A \mathop{\smile\!\!\!\!\!\!\;|}_B C \Leftrightarrow g(A) \mathop{\smile\!\!\!\!\!\!\;|}_{g(B)} g(C)$.

(Sym) $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$ if and only if $C \mathop{\smile\!\!\!\!\!\!\;|}_B A$.

(Trans) If $A \mathop{\smile\!\!\!\!\!\!\;|}_{BC} D$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$, then $A \mathop{\smile\!\!\!\!\!\!\;|}_B D$.

(Mon) If $A \mathop{\smile\!\!\!\!\!\!\;|}_B CD$, then $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_{BC} D$.

On the Urysohn space this notion of independence also satisfies

(Unique) If $A \cong_C B$ and $A \mathop{\smile\!\!\!\!\!\!\;|}_C D, B \mathop{\smile\!\!\!\!\!\!\;|}_C D$ then $A \cong_{CD} B$.

(Exist) For any finite sets A, B, C there is some C' such that

$$C' \cong_A C \quad \text{and} \quad C' \mathop{\smile\!\!\!\!\!\!\;|}_A B.$$

More examples

Other examples with such a notion of independence:

1. Let Γ be the countable universal graph, i.e. the unique countable graph such that for any finite disjoint sets of vertices A, B there is a vertex x which has an edge to every vertex of A and no edge to any vertex of B.

Then put $A \mathop{\smile\!\!\!\!\!\!|}_B C$ if there are no edges between vertices in $A \setminus B$ and $C \setminus B$.

In fact, considering Γ as a metric space, this reduces to the same notion as before, but now makes sense also for $B = \emptyset$.

2. Any *stable* theory in which all types are *stationary*.

Trivial example: a set without structure and

$$A \mathop{\smile\!\!\!\!\!\!|}_B C \quad \text{if and only if} \quad A \cap C \subseteq B.$$

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Less trivial example:
3. Let T be a tree (i.e. a connected graph without cycles) such that all vertices have infinite valency. Then for any sets of vertices $A, B, C \subset T$ we may define

$$A \underset{B}{\downarrow} C$$

if and only if for all $a \in A, c \in C$ any path from a to c passes through an element of B.

Considered as a metric space, this again agrees with the previous notion. This example generalizes to the following:

4. Let Δ be a right-angled building, i.e. a simplicial complex associated to a right angled Coxeter group (whose associated Coxeter diagram contains only 2's and ∞'s).

Then for any finite sets A, B, C in Δ we may put $A \downarrow_B C$ if for all $a \in A, c \in C$ the shortest gallery from a to c passes through an element of B.

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Automorphisms that move maximally

Let M be a countable structure with a notion of independence on finite sets which satisfies the properties given above.

We say that $g \in \mathrm{Aut}(M)$ *moves maximally* if for any finite set $X \subset M$ and $a' \in M$ there is some $a \in M$ such that $a \cong_M a'$ and

$$a \underset{X}{\mathop{\downarrow}} g(X)g(a) \quad \text{and} \quad aX \underset{g(X)}{\mathop{\downarrow}} g(a).$$

We say that g moves a maximally over X.

Examples: 1. Let $X = \{b\}$ and $a \in \mathbb{U}$. Then $g \in \mathrm{Isom}(\mathbb{U})$ moves a maximally over b if and only if

$$d(a, g(a)) = d(a, b) + d(b, g(b)) + d(g(b), g(a)).$$

2. If M is a countably infinite set without any structure, then $g \in \mathrm{Aut}(M) = \mathrm{Sym}(M)$ moves maximally if and only if g has infinite support.

Theorem 2

Let M be a countable structure with an independence relation on finite sets satisfying the conditions above and let $g \in \mathrm{Aut}(M)$ move maximally. Then any element of $\mathrm{Aut}(M)$ is the product of eight conjugates of g.

The proof relies on the fact that $G = \mathrm{Aut}(M)$ is a *polish group*, i.e. a topological group on a separable, completely metrizable space.

A basis of open sets is given by finite partial automorphisms $u : A \to B$, where

$$O_u = \{f \in G : f \text{ extends } u\}.$$

We consider the map

$$\varphi : G^4 \to G, (f_1, \ldots, f_4) \mapsto g^{f_1} \cdot \ldots \cdot g^{f_4}.$$

Clearly, $\mathrm{im}\,\varphi$ is contained in $\langle g \rangle^G$, the normal subgroup generated by $g \in G$.

The crucial proposition

Let $\varphi : G^4 \to G, (f_1, \ldots, f_4) \mapsto g^{f_1} \cdot \ldots \cdot g^{f_4}$. It suffices to prove

Proposition

The image $\operatorname{im}\varphi$ *is not meager, so not the countable union of nowhere dense sets.*

Since G contains a dense conjugacy class, this implies that $\operatorname{im}(\varphi)$ is open and $\operatorname{im}(\varphi)^2 = G$, i.e. the map

$$G^8 \to G, (f_1, \ldots, f_8) \mapsto g^{f_1} \cdot \ldots \cdot g^{f_8}$$

is surjective.

By Baire's Category Theorem, the previous proposition is equivalent to:

Proposition

For open sets $U_1, \dots U_4$ there is some open set W such that $\varphi(U_1, \dots, U_4)$ is dense in W.

We may assume that the open sets $U_i, i = 1, \dots 4$ are given by finite partial maps $u_i, i = 1, \dots 4$ and extend the partial maps in such a way that for

$$g^{u_i} : X_i \to X_{i+1}, i = 1, \dots 4$$

we have

$$X_1 \underset{X_2}{\downarrow} X_3 \quad \text{and} \quad X_3 \underset{X_4}{\downarrow} X_5.$$

Hence $\varphi(u_1, \dots, u_4)(X_1) = X_5$ is a finite partial map $w : X_1 \to X_5$ and we claim that the image is dense in O_w. For this we have to show that for any finite extension w' of w we can find finite extensions u_i' of the u_i such that

$$\varphi(u_1', \dots u_4') \supset w'.$$

A general simplicity theorem

It is in these extensions that the properties of independence play a crucial role:

(Unique) If $A \cong_C B$ and $A \mathop{\smile\!\!\!\!\!\!|}_C D, B \mathop{\smile\!\!\!\!\!\!|}_C D$ then $A \cong_{CD} B$.

(Exist) For any finite sets A, B, C there is some C' such that

$$C' \cong_A C \quad \text{and} \quad C' \mathop{\smile\!\!\!\!\!\!|}_A B.$$

Back to the Urysohn space

Theorem 1

$\mathrm{Isom}(\mathbb{U})/\mathrm{Bd}(\mathbb{U})$ is a simple group. In fact, if $g \in \mathrm{Isom}(\mathbb{U})$ is unbounded, then any element of $\mathrm{Isom}(\mathbb{U})$ is the product of 8 conjugates of g.

Needs two steps:
First show:

Proposition

If $g \in \mathrm{Isom}(\mathbb{U})$ is unbounded, then g moves maximally.

Example: Suppose g fixes b, $d(a', b) = d$. Need to find some $a \in \mathbb{U}$ with $d(a, b) = d$ and $d(a, g(a)) = 2d$.

We find such an element a by **(Exist)** as an element independent from some c with $d(c, g(c)) \geq 4d$ using *forking calculus*.

Completing the proof for the Urysohn space

In order to apply Theorem 2, we need to reduce from the uncountable Urysohn space to a countable elementary subspace $\mathbb{U}'$ which is dense in $\mathbb{U}$ by a Löwenheim-Skolem argument. If $h \restriction \mathbb{U}'$ is a product of 8 conjugates of g, then this is true for h on $\mathbb{U}$ by density.

Other applications

Theorem (Macpherson-T.)

Let M be the Fräissé limit of a free amalgamation class, not an indiscernible set. If $\mathrm{Aut}(M)$ is transitive on M, then it is simple.

The proof here yields a better bound on the number of conjugates needed.

Theorem (Haglund-Paulin)

"Many" negatively curved simply connected locally compact polyhedral complexes, admitting a discrete cocompact group of automorphisms, have automorphism groups which are locally compact, uncountable, non linear and virtually simple.

Their examples include buildings associated to right-angled and hyperbolic Coxeter groups. However, they do not obtain a bound on the number of conjugates needed.