COMPLETELY REDUCIBLE SUBCOMPLEXES OF SPHERICAL BUILDINGS

CHRIS PARKER AND KATRIN TENT

In a spherical building Δ two chambers in Δ are *opposite* in Δ provided their convex hull is an apartment of Δ. Two simplices R and R' of Δ are *opposite* in Δ if for every chamber containing R there is an opposite chamber containing R'.

In 2005 Serre in [8] introduced the notion of complete reducibility in spherical buildings. He went on to point out the following conjecture [8, Conjecture 2.8] which he attributes to Tits from the 1950's.

Conjecture 1 (Tits' Centre Conjecture)**.** *Suppose that Δ is a spherical building and Ω is a convex subcomplex of Δ. Then (at least) one of the following holds:*

(a) for each simplex A in Ω, there is a simplex B in Ω which is opposite to A in Δ; or
(b) there exists a nontrivial simplex A' in Ω fixed by any automorphism of Δ stabilizing Ω.

If possibility (a) in the conjecture arises we say that Ω is *completely reducible* and if (b) is the case, then the simplex A' is called a *centre* of Ω. If alternative (a) holds then Ω is a possibly thin subbuilding of Δ (see [7]).

If G is an algebraic group with associated building Δ, then a subgroup H of G is called *completely reducible* provided that whenever it is a subgroup of a parabolic subgroup of G it is contained in a Levi complement of that parabolic subgroup. In this case, the convex subcomplex of Δ fixed by H is completely reducible. Conversely if the subcomplex of Δ fixed by a subgroup H of a parabolic subgroup of G is completely reducible, then so is H. This relationship between complete reducibility of subcomplexes of the building and completely reducible subgroups of parabolic subgroups has lead to a source of fruitful research of which we particularly mention [1, Theorem 3.1] in which they prove the conjecture in the case that Ω is the fixed point set of some subgroup H.

In the more general setting, for the classical buildings and buildings of rank 2 the conjecture was proved by Mühlherr and Tits [4] in 2006. For buildings of exceptional type E_6, E_7 and E_8 the conjecture has been proved by Leeb and Ramos Cuevas [3, 6] using, in part, some of the observations presented by the authors at a meeting in Oberwolfach in January 2008 [5] and which now form the content of this note. They also include the proof of the conjecture for buildings of type

F_4, which was first presented by the authors at that meeting. All of the investigations of the Centre Conjecture have used the lemma of Serre's [8] which states that Ω is completely reducible if *every* vertex of Ω has an opposite. For chamber complexes, we can prove the following stronger assertion and thereby obtain a very short proof of the Centre Conjecture for convex chamber subcomplexes of classical buildings.

Theorem 2. *Let Δ be an irreducible spherical building of type (W, I). Let Ω be a convex chamber subcomplex of Δ. If for some $k \in I$ every vertex of type k in Ω has an opposite in Ω, then Ω is completely reducible.*

Notice that the hypothesis that Δ is irreducible in Theorem 2 may not be dropped as is easily seen by taking a product of two buildings and choosing a convex subcomplex which is completely reducible in one factor and has a centre in the second factor. Our notation follows [9]. So given a simplex R of type $J \subseteq I$, the collection of all simplices containing R form a building $\mathrm{St}R$ of type $(W_{I\setminus J}, I \setminus J)$. Of particular importance to us are the *projection maps*: given simplices R and S, $\mathrm{proj}_R(S)$ is the unique simplex of $\mathrm{St}R$ which is contained in every shortest gallery from S to R (see [9, Proposition 2.29]) and is called the *projection* of S to R. Note that if Ω is a convex subcomplex of Δ then, for all simplices R and S in Ω, we have $\mathrm{proj}_R S \in \Omega$ and this is the crucial property of convexity that we use in the proof of Theorem 2. We refer the reader to [9, 2.30 and 2.31] for many properties of projection maps.

Lemma 3. *Suppose that x and y are opposite chambers in Δ. Let Σ be the convex hull of x and y in Δ and R be a simplex in Σ. Then $\mathrm{proj}_R(x)$ and $\mathrm{proj}_R(y)$ are opposite in $\mathrm{St}R$.*

Proof. Set $x_1 = \mathrm{proj}_R(x)$ and $y_1 = \mathrm{proj}_R(y)$. Then x_1 and y_1 are chambers by [9, Proposition 2.29]. Let z be opposite x_1 in $\mathrm{St}R$. Then we have $\mathrm{dist}(x, z) = \mathrm{dist}(x, x_1) + \mathrm{dist}(x_1, z)$ and $\mathrm{dist}(y, z) = \mathrm{dist}(y, y_1) + \mathrm{dist}(y_1, z)$ by [9, 2.30.6]. Therefore $\mathrm{dist}(x, y) = \mathrm{dist}(x, x_1) + \mathrm{dist}(x_1, z) + \mathrm{dist}(y_1, z) + \mathrm{dist}(y, y_1)$ as every chamber of Σ is on a shortest gallery between x and y by [9, 2.35 (iv)]. On the other hand, as x_1 and z are opposite in $\mathrm{St}R$, $\mathrm{dist}(x_1, y_1) \leq \mathrm{dist}(x_1, z)$ and so

$$\begin{aligned} \mathrm{dist}(x, y) &= \mathrm{dist}(x, x_1) + \mathrm{dist}(x_1, y_1) + \mathrm{dist}(y_1, y) \\ &\leq \mathrm{dist}(x, x_1) + \mathrm{dist}(x_1, z) + \mathrm{dist}(y, y_1). \end{aligned}$$

It follows that $\mathrm{dist}(y_1, z) = 0$ and hence $z = y_1$ as claimed. □

The following observation is especially important to us.

Corollary 4. *Suppose that R, X and Y are simplices in the apartment Σ with X opposite Y. Then either*

(a) $\mathrm{proj}_R(X)$ *is opposite* $\mathrm{proj}_R(Y)$ *in* $\mathrm{St}R$*; or*

(b) $R = \mathrm{proj}_R(X) = \mathrm{proj}_R(Y)$.

Proof. We can pair the chambers containing X and Y into opposite pairs (x, y). Then $\mathrm{proj}_R(x)$ is opposite $\mathrm{proj}_R(y)$ in $\mathrm{St}R$ by Lemma 3. This means every chamber of $\mathrm{proj}_R(X)$ has an opposite in $\mathrm{St}R$ contained in $\mathrm{proj}_R(Y)$. □

We can now prove Theorem 2. So suppose that Ω is a convex chamber subcomplex of Δ. Recall that we say Ω is a chamber complex if every simplex is contained in a chamber and if all the faces of every simplex of Ω are contained in Ω.

We repeatedly use the fact that, as Ω is convex, projections between simplices of Ω are contained in Ω.

By hypothesis, we may choose $J \subseteq I$ maximally so that every simplex of type J in Ω has an opposite in Ω. It suffices to show that $J = I$, as, if a chamber has an opposite, then so does every face of that chamber. So suppose that $J \neq I$. Since Δ is irreducible there is $i \in I \setminus J$ such that i is a neighbour of some $j \in J$ in the Dynkin diagram of Δ.

Let z be of type $J \cup \{i\}$ in Ω, x_0 be the face of z of type J, ℓ the vertex of z of type i and let C_0 be a chamber of Ω containing z. We will construct an opposite for z.

Let p be a maximal face of C_0 with missing vertex of type j and x_0^o be an opposite of x_0 in Ω. Then ℓ is a vertex of p. Put $C_0' = \mathrm{proj}_{x_0^o} C_0$ and $C_1 = \mathrm{proj}_p C_0'$. Then, by Corollary 4, $C_0 = \mathrm{proj}_p(x_0) \neq C_1$. Let x_1 be the face of C_1 of type J. So $x_1 \neq x_0$ and setting $y_0 = \mathrm{proj}_{x_1} x_0$ we see that, as the reflections corresponding to i and j do not commute, y_0 has x_1 as a face and ℓ as a vertex. We will first find an opposite of the simplex y_0.

Let $y_1 = \mathrm{proj}_{x_1} x_0^o$, so y_1 and y_0 are opposite in $\mathrm{St}x_1$ by Corollary 4. Let x_1^o be opposite x_1. By [9, Proposition 3.29], we have $y_2 = \mathrm{proj}_{x_1^o}(y_1)$ is opposite y_0. Since y_0 contains the vertex ℓ, y_2 has an opposite of ℓ as a vertex and this is contained in Ω.

In order to find an opposite for the simplex z, notice that $\mathrm{proj}_\ell x_0 = z$. Let $z_1 = \mathrm{proj}_\ell x_0^o$, so z_1 and z are opposite in $\mathrm{St}\ell$ by Corollary 4. Using [9, Proposition 3.29] again, the projection of z_1 to the opposite of ℓ in $\mathrm{St}y_2$ now yields the required opposite of z in Ω. □

Corollary 5. *The Centre Conjecture holds for convex chamber subcomplexes of irreducible spherical buildings of classical type.*

Proof. For buildings of type $\mathrm{A}_n, \mathrm{B}_n$, C_n and D_n, we identify the simplices of Δ with flags of subspaces (singular subspaces, isotropic subspaces) in the appropriate vector spaces. We then consider the vertices of Δ corresponding to

1-dimensional subspaces (for A_n) and 1-dimensional isotropic/singular subspaces in the other cases and call them type 1 vertices.

Since Ω is a chamber subcomplex, Ω contains vertices of every type. If every type 1 vertex has an opposite in Ω, then Ω is completely reducible by Theorem 2. So we suppose that this is not the case and aim to identify a centre.

Suppose that Δ has type A_n and assume that some type 1 vertex w of Ω does not have an opposite in Ω. Then w is contained in all the hyperplanes of Ω. Thus the intersection of all hyperplanes of Ω is the required centre.

Suppose that Δ has type B_n, C_n or D_n. Then a vertex of type 1 in Ω has no opposite in Ω if and only if it is collinear with every other vertex of type 1 in Ω. Hence the set of all vertices of type 1 in Ω having no opposite span a totally isotropic (singular) subspace, and this is the centre. □

References

[1] Michael Bate, Benjamin Martin and Gerhard Röhrle, On Tits' centre conjecture for fixed point subcomplexes. C. R. Math. Acad. Sci. Paris 347 (2009), no. 7-8, 353–356.

[2] A. Dress, R. Scharlau, Gated sets in metric spaces. Aequationes mathematicae, volume 34; pp. 112 - 120.

[3] B. Leeb, C. Ramos-Cuevas, The center conjecture for spherical buildings of types F_4 and E_6, arXiv:0905.0839v2.

[4] B. Mühlherr, J. Tits, The center conjecture for non-exceptional buildings, J. Algebra 300 (2), 2006, 687–706.

[5] C. Parker, K. Tent, Convexity in buildings, in: Buildings: interactions with algebra and geometry. Abstracts from the workshop held January 20–26, 2008. Organized by Linus Kramer, Bernhard Mühlherr and Peter Schneider. Oberwolfach Reports. Vol. 5, no. 1. Oberwolfach Rep. 5 (2008), no. 1, 119–172.

[6] C. Ramos-Cuevas, The center conjecture for thick spherical buildings, arXiv:0909.2761v1.

[7] L. Kramer, A completely reducible subcomplex of a spherical building is a spherical building, arXiv:1010.0083v1.

[8] J.-P. Serre, Complète réductibilité, Séminaire Bourbaki. Vol. 2003/2004, Astérisque 299, 2005.

[9] J.Tits, *Buildings of spherical type and finite BN-pairs*, Lecture Notes in Mathematics, Vol. 386, Springer-Verlag, Berlin, 1974.

Chris Parker, School of Mathematics, University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom

E-mail address: c.w.parker@bham.ac.uk

Katrin Tent, Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany

E-mail address: tent@uni-muenster.de