Low-tech notes on group extensions

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Abstract

These are notes explaining in a low-tech way how to describe and understand group extensions.

1 Introduction

It is well known that group extensions of a group N by a group H can be understood via the second cohomology groups of certain associated modules. We here give an easy description of the class of possible group extensions E of N by G, i.e. groups E containing N (or an isomorphic copy of N) as a normal subgroup such that $E/N \cong G$. We write this as

$$1 \longrightarrow N \longrightarrow E \longrightarrow G \longrightarrow 1.$$

Suppose that G has a presentation

$$G = \langle s_1, \ldots, s_k, r_1, \ldots, r_m \rangle \cong F_k/R$$

where F_k is the free group of rank k on generators $s_1, \ldots s_k$ and R is the normal subgroup of F_k generated (as a normal subgroup) by $r_1, \ldots, r_m$.

Now suppose we have an extension E such that

$$N \longrightarrow E \longrightarrow G = \langle s_1, \ldots s_k \rangle.$$

Let $\hat{s}_1, \ldots \hat{s}_k \in E$ be *lifts* of $s_1, \ldots, s_k \in G$, i.e. $\pi(\hat{s}_i) = s_i, i = 1, \ldots, k$.

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Then clearly $\hat{s}_i, i = 1, \dots k$ act on N by conjugation and hence any *word* $w = w(s_1, \dots, s_k)$ in the free group F_k with generators $s_1, \dots, s_k$ acts on N (as an automorphism of N) via the natural action of $w(\hat{s}_1, \dots, \hat{s}_k) \in E$. Hence any group extension E of N by a k-generated group $G = \langle s_1, \dots, s_k \rangle$ comes with an action of $F_k = F(s_1, \dots, s_k)$ on N. In order to describe E we will have to express this F_k action on N.

Define $\varphi_E : R \longrightarrow N$: $w(s_1, \dots, s_k) \mapsto w(\hat{s}_1, \dots, \hat{s}_k)$.

Lemma 1.1. *Using the previous notation we have* $\varphi_E \in \mathrm{Hom}_{F_k}(R, N)$.

Proof. This is clear. □

The next lemma states that the group E is determined – up to isomorphism over N – by the action of F_k on N and the homomorphism φ_E:

Lemma 1.2. *Using the previous notation suppose that* E^1, E^2 *are groups with a common normal subgroup* N *such that* $E^i/N \cong H, i = 1, 2$. *Let* $\hat{s}_i^j \in E^j, j = 1, 2, i = 1, \dots, k$ *be lifts of* $s_1, \dots, s_k$, *respectively.*

Suppose that the induced F_k*-actions agree, i.e. for all* $a \in N$ *we have*

$$a^{\hat{s}_i^1} = a^{\hat{s}_i^2}.$$

If $\varphi_{E^1} = \varphi_{E^2}$, *then* E^1 *and* E^2 *are isomorphic over* N.

Proof. First suppose that $\varphi_{E_1} = \varphi_{E_2}$. Define

$$f : E_1 \longrightarrow E_2, w(\hat{s}_1^1, \dots, \hat{s}_k^1)n \mapsto w(\hat{s}_1^2, \dots, \hat{s}_k^2)n.$$

Note that

$$w(\hat{s}_1^1, \dots, \hat{s}_k^1)n = w'(\hat{s}_1^1, \dots, \hat{s}_k^1)n'$$

if and only if

$$w(\hat{s}_1^1, \dots, \hat{s}_k^1)(w'(\hat{s}_1^1, \dots, \hat{s}_k^1))^{-1} \in N$$

if and only if

$$w(s_1, \dots, s_k)(w'(s_1, \dots, s_k))^{-1} \in R.$$

Since $\varphi_{E^1} = \varphi_{E^2}$, we see that indeed f is well-defined and injective.

Note that f is a homomorphism because the F_2-actions on N agree. Since $E^j, j = 1, 2$, is generated by N and $\hat{s}_1^j, \dots, \hat{s}_k^j$, this now implies that f is surjective and hence an isomorphism.

For the other direction, suppose that $g : E_1 \longrightarrow E_2$ is an isomorphism over N, so $gN = \mathrm{id}$. For any $w(s_1, \dots, s_k) \in R$ we thus have $g(w(\hat{s}_1^1, \dots, \hat{s}_k^1)) =$

$w(\hat{s}_1^1, \ldots, \hat{s}_k^1) = \varphi_{E_1}(w(s,t)) = w(\hat{s}_1^2, \ldots, \hat{s}_k^2) = \varphi_{E_2}(w(s_1, \ldots, s_k))$, proving the claim.

□

Recall that a group action is called *regular* if it is transitive and point stabilizers are trivial.

Lemma 1.3. *Let $C = Cen(N)$. The group $\mathrm{Hom}_F(R, C)$ acts regularly on the set*

$$X = \{\varphi_E \colon E \text{ is extension of } N \text{ by } H \text{ with prescribed } F_2\text{-action on } N\}$$

via $\varphi_E^\psi(w(s,t)) = \varphi_E(w(s,t))\psi(w(s,t))$ for $\psi \in Hom_F(R,C)$ and $\varphi_E \in X$

Proof. To see that the action is transitive just notice that for extensions E_1, E_2 of N by H with the given F_2-action on N, and lifts $\hat{s}_i, \hat{t}, i = 1, 2$ as before we have for all $n \in N$

$$n^{\varphi_{E_1}(w(s,t))} = n^{w(\hat{s}_1,\hat{t}_1)} = n^{w(s,t)} = n^{w(\hat{s}_2,\hat{t}_2)} = n^{\varphi_{E_2}(w(s,t))}$$

and hence $\varphi_{E_1}(w(s,t))(\varphi_{E_2}(w(s,t)))^{-1} \in C$ and so φ_{E_1} and φ_{E_2} differ by an element in $\mathrm{Hom}_F(R, C)$.

To see that $\varphi_E^\psi = \varphi_{E'}$ for some extension with prescribed F_2-action on N, define E' by choosing a transversal T for F_2/R so that any element $w(s,t) \in F_2$ can be written uniquely as $w(s,t) = v(s,t)r(s,t)$ where $v(s,t) \in T, r(s,t) \in R$.

We now define the elements of E' as $nw(s,t) = nv(s,t)\varphi_E(r(s,t))\psi(r(s,t))$ with the induced multiplication. Then E' is an extension with prescribed $F(s,t)$ action $\varphi_{E'} = \varphi_E^\psi$.

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References

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