Low-tech notes on group extensions

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Abstract

These are notes explaining in a low-tech way how to describe and understand group extensions.

1 Introduction

It is well known that group extensions of a group $N$ by a group $H$ can be understood via the second cohomology groups of certain associated modules. We here give an easy description of the class of possible group extensions $E$ of $N$ by $G$, i.e. groups $E$ containing $N$ (or an isomorphic copy of $N$) as a normal subgroup such that $E/N \simeq G$. We write this as

$$1 \rightarrow N \rightarrow E \rightarrow G \rightarrow 1.$$ 

Suppose that $G$ has a presentation

$$G = \langle s_1, \ldots, s_k, r_1, \ldots, r_m \rangle \cong F_k / R$$

where $F_k$ is the free group of rank $k$ on generators $s_1, \ldots, s_k$ and $R$ is the normal subgroup of $F_k$ generated (as a normal subgroup) by $r_1, \ldots, r_m$.

Now suppose we have an extension $E$ such that

$$N \rightarrow E \rightarrow G = \langle s_1, \ldots, s_k \rangle.$$

Let $\hat{s}_1, \ldots, \hat{s}_k \in E$ be lifts of $s_1, \ldots, s_k \in G$, i.e. $\pi(\hat{s}_i) = s_i, i = 1, \ldots, k$.

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Then clearly $\hat{s}_i, i = 1, \ldots, k$ act on $N$ by conjugation and hence any word $w = w(s_1, \ldots, s_k)$ in the free group $F_k$ with generators $s_1, \ldots, s_k$ acts on $N$ (as an automorphism of $N$) via the natural action of $w(\hat{s}_1, \ldots, \hat{s}_k) \in E$. Hence any group extension $E$ of $N$ by a $k$-generated group $G = \langle s_1, \ldots, s_k \rangle$ comes with an action of $F_k = F(s_1, \ldots, s_k)$ on $N$. In order to describe $E$ we will have to express this $F_k$ action on $N$.

Define $\varphi_E : R \longrightarrow N : w(s_1, \ldots, s_k) \mapsto w(\hat{s}_1, \ldots, \hat{s}_k)$.

**Lemma 1.1.** Using the previous notation we have $\varphi_E \in \text{Hom}_{F_k}(R, N)$.

**Proof.** This is clear. \(\square\)

The next lemma states that the group $E$ is determined – up to isomorphism over $N$ – by the action of $F_k$ on $N$ and the homomorphism $\varphi_E$:

**Lemma 1.2.** Using the previous notation suppose that $E^1, E^2$ are groups with a common normal subgroup $N$ such that $E^i/N \cong H, i = 1, 2$. Let $\hat{s}_i^j \in E^j, j = 1, 2, i = 1, \ldots, k$ be lifts of $s_1, \ldots, s_k$, respectively.

Suppose that the induced $F_k$-actions agree, i.e. for all $a \in N$ we have

$$a^{\hat{s}_i^1} = a^{\hat{s}_i^2}.$$

If $\varphi_{E^1} = \varphi_{E^2}$, then $E^1$ and $E^2$ are isomorphic over $N$.

**Proof.** First suppose that $\varphi_{E^1} = \varphi_{E^2}$. Define

$$f : E_1 \longrightarrow E_2, w(\hat{s}_1^1, \ldots, \hat{s}_k^1) n \mapsto w(\hat{s}_1^2, \ldots, \hat{s}_k^2) n.$$ 

Note that

$$w(\hat{s}_1^1, \ldots, \hat{s}_k^1) n = w'(\hat{s}_1^1, \ldots, \hat{s}_k^1) n'$$

if and only if

$$w(\hat{s}_1^1, \ldots, \hat{s}_k^1)(w'(\hat{s}_1^1, \ldots, \hat{s}_k^1))^{-1} \in N$$

if and only if

$$w(s_1, \ldots, s_k)(w'(s_1, \ldots, s_k))^{-1} \in R.$$ 

Since $\varphi_{E^1} = \varphi_{E^2}$, we see that indeed $f$ is well-defined and injective.

Note that $f$ is a homomorphism because the $F_2$-actions on $N$ agree. Since $E^j, j = 1, 2$, is generated by $N$ and $\hat{s}_1^j, \ldots, \hat{s}_k^j$, this now implies that $f$ is surjective and hence an isomorphism.

For the other direction, suppose that $g : E_1 \longrightarrow E_2$ is an isomorphism over $N$, so $gN = \text{id}$. For any $w(s_1, \ldots, s_k) \in R$ we thus have $g(w(\hat{s}_1^1, \ldots, \hat{s}_k^1)) =$
\[ w(s_1, \ldots, s_k) = \varphi_{E_1}(w(s, t)) = w(s_1, \ldots, s_k) = \varphi_{E_2}(w(s_1, \ldots, s_k)), \] proving the claim.

Recall that a group action is called \textit{regular} if it is transitive and point stabilizers are trivial.

\textbf{Lemma 1.3.} Let \( C = \text{Cen}(N) \). The group \( \text{Hom}_F(R, C) \) acts regularly on the set

\[ X = \{ \varphi_E : E \text{ is extension of } N \text{ by } H \text{ with prescribed } F_2\text{-action on } N \} \]

\[ \text{via } \varphi_E^\psi(w(s, t)) = \varphi_E(w(s, t))\psi(w(s, t)) \text{ for } \psi \in \text{Hom}_F(R, C) \text{ and } \varphi_E \in X \]

\textbf{Proof.} To see that the action is transitive just notice that for extensions \( E_1, E_2 \) of \( N \) by \( H \) with the given \( F_2 \)-action on \( N \), and lifts \( \hat{s}_i, \hat{t}, i = 1, 2 \) as before we have for all \( n \in N \)

\[ n^{\varphi_{E_1}(w(s, t))} = n^{w(s_1, t_1)} = n^{w(s, t)} = n^{w(s_2, t_2)} = n^{\varphi_{E_2}(w(s, t))} \]

and hence \( \varphi_{E_1}(w(s, t))(\varphi_{E_2}(w(s, t)))^{-1} \in C \) and so \( \varphi_{E_1} \) and \( \varphi_{E_2} \) differ by an element in \( \text{Hom}_F(R, C) \).

To see that \( \varphi_\psi = \varphi_E \) for some extension with prescribed \( F_2 \)-action on \( N \), define \( E' \) by choosing a transversal \( T \) for \( F_2/R \) so that any element \( w(s, t) \in F_2 \) can be written uniquely as \( w(s, t) = v(s, t)r(s, t) \) where \( v(s, t) \in T, r(s, t) \in R \).

We now define the elements of \( E' \) as \( nw(s, t) = nv(s, t)\varphi_E(r(s, t))\psi(r(s, t)) \) with the induced multiplication. Then \( E' \) is an extension with prescribed \( F(s, t) \) action \( \varphi_{E'} = \varphi_\psi \).

\[ \Box \]

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