Automorphisms with comeager conjugacy class *

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These notes are based on results by Kechris and Rosendal [1]: the automorphism group $\mathrm{Aut}(M)$ of a countable structure M contains *generic automorphisms* if and only if $\mathrm{Aut}(M)$ satisfies the *joint embedding property* JEP and the *weak amalgamation property* WAP.

1 JEP

Let M be a countable structure and $\mathcal{G} = \mathrm{Aut}(M)$. A finite partial isomorphism is a bijection between finite subsets which extends to an automorphism of M. Note that we do not assume M to be homogeneous. Then $\mathrm{Aut}(M)$ is a topological group with a basis of open sets given by $(O_u \colon u$ a finite partial automorphism of $M)$ where $O_u = \{g \in \mathcal{G} \colon u \subset g\}$. Clearly $\mathcal{G}$ acts on the finite partial automorphisms of M by conjugation. If $f \in \mathcal{G}$ fixes the domain and the range of a finite partial automorphism u pointwise, then we call a conjugate under f a u-conjugate.

Definition 1.1. $\mathrm{Aut}(M)$ *has the JEP if for any finite partial automorphisms* u, v *there is a conjugate* v' *of* v *such that* $u \cup v'$ *is a partial automorphism. We say that* u, v' *are consistent.*

Lemma 1.2. $\mathrm{Aut}(M)$ *has JEP if and only if* $\mathrm{Aut}(M)$ *contains a dense conjugacy class.*

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Proof. If $\mathrm{Aut}(M)$ has JEP we inductively construct an automorphism $f \in \mathrm{Aut}(M)$ with dense conjugacy class. If conversely $f \in \mathrm{Aut}(M)$ has a dense conjugacy class, then for any finite partial automorphisms u, v there are $g, h \in \mathcal{G}$ such that $u^g \cup v^h \subset f$. Then clearly $u \cup v^{hg^{-1}}$ is a partial automorphism. □

2 Weak generics and WAP

Definition 2.1. *An automorphism $f \in \mathcal{G}$ is weakly generic if for any $u \subset f$ there exists some $\tilde{u} \subset f$ such that any extension $v \supset \tilde{u}$ has a u-conjugate in f. In this case we call $\tilde{u}$ an f-cover of u.*

Clearly, weak genericity is invariant under conjugation.

Theorem 2.2. *If the conjugacy class of $f \in \mathcal{G}$ is non-meager, then f is weakly generic.*

Proof. Assume that f is not weakly generic. Let $u \subset f$ be a counterexample, so for any $u \subset \tilde{u} \subset f$ there is some v such that no u-conjugate of v is in f. Let $U, \tilde{U}$ and V be the basic open sets in $\mathrm{Aut}(M)$ defined by $u, \tilde{u}$ and v, respectively and let S be the stabilizer of u in $\mathrm{Aut}(M)$. By assumption on v the S-orbit f^S of f is disjoint from V. Thus any basic open set $\tilde{U}$ containing f contains an open nonempty set V disjoint from f^S. Since S acts on f^S by conjugation continuously and transitively, we conclude that the same holds for any $f' \in f^S$, i.e., any f' has a nonempty open neighbourhood which is disjoint from f^S, and hence f^S is nowhere dense. Since the index of S in $\mathrm{Aut}(M)$ is countable, the orbit $f^{\mathrm{Aut}(M)}$ of f under $\mathrm{Aut}(M)$ is the countable union of nowhere dense sets of the form $f^{Sg} = (f^S)^g$, and hence is meager. □

Definition 2.3. $\mathrm{Aut}(M)$ *has the* weak amalgamation property*, WAP, if any finite partial automorphism u has an extension $\tilde{u}$ with the following property: for any extensions v, w of $\tilde{u}$ there is a u-conjugate v' of v such that v', w are consistent. We call $\tilde{u}$ a cover of u.*

Lemma 2.4. *If $f \in \mathcal{G}$ is weakly generic, $u \subset f$ finite, and $u \subset \tilde{u} \subset f$, then $\tilde{u}$ an f-cover of u if and only if it is a cover.*

Proof. If $\tilde{u}$ is an f-cover of u, then as in the proof of Lemma 1.2 for any extensions v, w of $\tilde{u}$ there is a u-conjugate v' of v such that v', w are consistent. If $u' \subset f$ is a cover of u and $\tilde{u}$ an f-cover of u, we may assume $u' \subseteq \tilde{u}$. Let v be an extension of u'. Then there is some u-conjugate v' of v consistent with $\tilde{u}$. Since v' has a u-conjugate inside f, u' is an f-cover of u. □

Corollary 2.5. *The set of weak generics in $\mathcal{G} = \mathrm{Aut}(M)$ is a G_δ set.*

Proof. By the previous corollary, an element $f \in \mathcal{G}$ is a weak generic if and only if it satisfies the following two conditions.

1. for all $u \subset f$ there is some cover $\tilde{u} \subset f$; and
2. for all $u \subset \tilde{u} \subset f$ where $\tilde{u}$ is a cover of u and every $v \supset \tilde{u}$, there exists a u-conjugate $v' \subset f$.

Therefore, the set of weak generics can be described as

$$\bigcap_{u} \left(O_u^c \cup \bigcup_{u \subset \tilde{u}} O_{\tilde{u}} \right) \cap \bigcap_{u \subset \tilde{u} \subset v} \left(O_{\tilde{u}}^c \cup \bigcup_{v'} O_{v'} \right),$$

where $\tilde{u}$ is a cover of u and v' a u-conjugate of v. Note that for any finite partial automorphism u, the set O_u is clopen and hence so is its complement O_u^c.

□

Lemma 2.6. *If* $\mathrm{Aut}(M)$ *satisfies JEP, then all weak generics are conjugate*[1] *and if weak generics exist, their conjugacy class is dense.*

Note that $\mathrm{Aut}(M)$ has JEP if and only if the empty partial isomorphism is a cover of itself.

Proof. Let f, g be weak generics. Since the empty automorphism covers itself, any finite automorphism is conjugate to a part of f and of g, so their conjugacy classes are dense.

In order to show that f and g are conjugate we consider the family of all isomorphisms $u \to v$ between finite parts of f and g which extend to an isomorphism between $\tilde{u} \subseteq f, \tilde{v} \subseteq g$ where $\tilde{u}, \tilde{v}$ are covers of u, v, respectively. (By the previous remark we may assume that $\tilde{u}, \tilde{v}$ are covers inside f, g.) This family is nonempty since by JEP the empty isomorphism is its own cover and inside f and g.

We show the forth part: given an extension $u' \subseteq f$ of $\tilde{u}$, choose a cover $\tilde{u}'$ in f. We extend the isomorphism $\tilde{u} \to \tilde{v}$ to an isomorphism $\tilde{u}' \to \tilde{v}'$ where $\tilde{v}'$ is an extension of $\tilde{v}$ and v' denotes the image of u'. Since f satisfies 2. we

[1] Without assuming JEP one can show that there are at most countably many conjugacy classes of weak generics

may embed $\tilde{v}'$ over v into g. Let v'' be the image of u' and $\tilde{v}''$ the image of $\tilde{v}'$. This yields an extension of $u \to v$ to an isomorphism $u' \to v''$ which extends to an isomorphism $\tilde{u}' \to \tilde{v}''$. □

Theorem 2.7. $\mathrm{Aut}(M)$ *has WAP if and only if the weak generics are dense in* $\mathrm{Aut}(M)$*.*

Proof. First assume that M satisfies WAP. We have to show that any partial automorphism f_0 can be extended to a weakly generic one. We construct a sequence

$$\emptyset \subseteq f_0 \subset f_1 \ldots$$

such that

1. f_{i+1} covers f_i.
2. for any $f_{i+1} \subset h$ there is $j > i$ such that f_j extends some f_i-conjugate of h.
3. $f = \bigcup f_i$ is an automorphism.

We have to show that f is weakly generic. Let $u \subseteq f$ and i be such that $u \subseteq f_i$. We show that $\tilde{u} = f_{i+1}$ is as required. Let v be an extension of f_{i+1}. By construction there is some $j > i$ and some f_i-conjugate v' of v extending f_j and is hence extended by f.

Conversely assume that the weak generics are dense in $\mathrm{Aut}(M)$. Then any u is contained in a weak generic and hence has an f-cover. This is sufficient for WAP by Lemma 2.4. □

Definition 2.8. *An element* $f \in \mathrm{Aut}(M)$ *is called generic if its conjugacy class is dense and* G_δ*.*

Note that in Baire spaces a comeager set always *contains* a dense G_δ set, so an automorphism with comeager conjugacy class is always 'almost' generic.

Theorem 2.9. *If* $\mathrm{Aut}(M)$ *has JEP, the following are equivalent:*

1. *f is generic.*
2. *f has a comeager conjugacy class.*
3. *f has a nonmeager conjugacy class.*

4. f is weakly generic.

Clearly it follows from 1. or 2. that the conjugacy classes are dense (but possibly not from 3 without assuming JEP).

Proof. The implications 1. implies 2. implies 3. are trivial. The implication 3. implies 4. was proved above (without JEP). Let f be a weak generic. Then its conjugacy class is dense by Lemma 2.6 and G_δ by Corollary 2.5, hence 4. implies 1. □

Corollary 2.10. *The following are equivalent:*

1. *there exists $f \in \mathcal{G}$ generic;*
2. *there exists $f \in \mathcal{G}$ with comeager conjugacy class;*
3. *there exists $f \in \mathcal{G}$ with dense non-meager conjugacy class;*
4. *there exists a weak generic $f \in \mathcal{G}$ with dense conjugacy class;*
5. $\mathrm{Aut}(M)$ *has JEP and WAP.*

Proof. The implications 1. → 2. → 3. are trivial. 3. implies 4. was proved above. 4. implies WAP by Theorem 2.7 and JEP by Lemma 1.2, so we have 5.

To see that 5. implies 1., note that by WAP we have existence of a dense set of weak generics. This set is G_δ and forms a single conjugacy class by JEP.

□

References

[1] Alexander S. Kechris, Christian Rosendal. Turbulence, amalgamation, and generic automorphisms of homogeneous structures. *Proc. Lond. Math. Soc. (3)*, 94(2):3022013350, 2007.