A note on the Baldwin-Lachlan Theorem

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Abstract

We explain a variant of the proof of the Baldwin-Lachlan Theorem in [TZ] which does not use the pregeometries arising from strongly minimal sets.

In a forthcoming revised version of [TZ] we will give the following variant of the proof of the Baldwin-Lachlan Theorem. This is Theorem 5.8.1 in [TZ] and all references given below refer to that book. The new proof will be inserted right after Corollary 5.7.4. and does not use the notion of a pregeometry. We first note the following:

**Proposition 0.1.** If $T$ is strongly minimal, then $T$ is $\kappa$-categorical for all uncountable $\kappa$.

**Proof.** By Lemma 5.2.8 it suffices to show that all uncountable models are saturated. Let $\mathfrak{M}$ be a model of $T$ of cardinality $\kappa > \aleph_0$ and let $A \subset M$ with $|A| < |M|$. Let $p(x) \in S(A)$. If $p(x)$ is algebraic, then clearly $p(x)$ is realised in $M$. Otherwise $p(x)$ is realised by any $b \in M\setminus \text{acl}(A)$ since there is a unique non-algebraic type by Lemma 5.7.3. Since $|\text{acl}(A)| \leq \max\{|T|, |A|\} < |M| = \kappa$, this proves the proposition.

We also note:

**Remark 0.2.** If $T$ does not have a Vaughtian pair, then for any model $\mathfrak{M}$ of $T$ and any non-algebraic formula $\varphi(x) \in L(M)$ we have $|\varphi(\mathfrak{M})| = |M|$ by the Löwenheim-Skolem Theorem.

**Theorem 0.3.** Suppose $T$ is a countable theory and $\kappa$ is an uncountable cardinal. Then $T$ is $\kappa$-categorical if and only if $T$ is $\omega$-stable and does not have a Vaughtian pair.
Proof. The proof that a $\kappa$-categorical theory is $\omega$-stable and does not have a Vaughtian pair remains the same.

For the other direction we argue as follows: let $M_0$ be the prime model of $T$ and let $\varphi(x) \in L(M_0)$ be a minimal formula, which both exist since $T$ is totally transcendental. Since $T$ does not have a Vaughtian pair, it eliminates by Lemma 5.5.7 the quantifier $\exists^\infty x$ so that $\varphi(x)$ is in fact strongly minimal.

Now let $M, N$ be models of $T$ of cardinality $\kappa > \aleph_0$. We may assume that $M_0$ is an elementary submodel of $M, N$. Since $T$ does not have a Vaughtian pair, we have $|\varphi(M)| = |\varphi(N)| = \kappa$.

Since $M, N$ are minimal over $\varphi(M), \varphi(N)$ by Lemma 5.3.8 it suffices to define an elementary bijection from $\varphi(M)$ to $\varphi(N)$.

This can be done exactly as in Lemma 5.2.8 simplified here by the fact that over every subset of $\varphi(M), \varphi(N)$ there is a unique non-algebraic type in $\varphi(M), \varphi(N)$ as in the proof of Proposition 0.1. \qed

References