A note on the Baldwin-Lachlan Theorem

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Abstract

We explain a variant of the proof of the Baldwin-Lachlan Theorem in [TZ] which does not use the pregeometries arising from strongly minimal sets.

In a forthcoming revised version of [TZ] we will give the following variant of the proof of the Baldwin-Lachlan Theorem. This is Theorem 5.8.1 in [TZ] and all references given below refer to that book. The new proof will be inserted right after Corollary 5.7.4. and does not use the notion of a pregeometry. We first note the following:

Proposition 0.1. *If T is strongly minimal, then T is κ-categorical for all uncountable κ.*

Proof. By Lemma 5.2.8 it suffices to show that all uncountable models are saturated. Let $\mathfrak{M}$ be a model of T of cardinality $\kappa > \aleph_0$ and let $A \subset M$ with $|A| < |M|$. Let $p(x) \in S(A)$. If $p(x)$ is algebraic, then clearly $p(x)$ is realised in M. Otherwise $p(x)$ is realised by any $b \in M \setminus \mathrm{acl}(A)$ since there is a unique non-algebraic type by Lemma 5.7.3. Since $|\operatorname{acl}(A)| \leq \max\{|T|, |A|\} < |M| = \kappa$, this proves the proposition. □

We also note:

Remark 0.2. *If T does not have a Vaughtian pair, then for any model $\mathfrak{M}$ of T and any non-algebraic formula $\varphi(x) \in L(M)$ we have $|\varphi(\mathfrak{M})| = |M|$ by the Löwenheim-Skolem Theorem.*

Theorem 0.3. *Suppose T is a countable theory and κ is an uncountable cardinal. Then T is κ-categorical if and only if T is ω-stable and does not have a Vaughtian pair.*

Proof. The proof that a κ-categorical theory is ω-stable and does not have a Vaughtian pair remains the same.

For the other direction we argue as follows: let $\mathfrak{M}_0$ be the prime model of T and let $\varphi(x) \in L(M_0)$ be a minimal formula, which both exist since T is totally transcendental. Since T does not have a Vaughtian pair, it eliminates by Lemma 5.5.7 the quantifier $\exists^\infty x$ so that $\varphi(x)$ is in fact strongly minimal.

Now let $\mathfrak{M}, \mathfrak{N}$ be models of T of cardinality $\kappa > \aleph_0$. We may assume that $\mathfrak{M}_0$ is an elementary submodel of $\mathfrak{M}, \mathfrak{N}$. Since T does not have a Vaughtian pair, we have $|\varphi(\mathfrak{M})| = |\varphi(\mathfrak{N})| = \kappa$.

Since $\mathfrak{M}, \mathfrak{N}$ are minimal over $\varphi(\mathfrak{M}), \varphi(\mathfrak{N})$ by Lemma 5.3.8 it suffices to define an elementary bijection from $\varphi(\mathfrak{M})$ to $\varphi(\mathfrak{N})$.

This can be done exactly as in Lemma 5.2.8 simplified here by the fact that over every subset of $\varphi(\mathfrak{M}), \varphi(\mathfrak{N})$ there is a unique non-algebraic type in $\varphi(\mathfrak{M}), \varphi(\mathfrak{N})$ as in the proof of Proposition 0.1. $\square$

References

[TZ] K. Tent, M. Ziegler, *A course in model theory, ASL Lecture Notes in Logic, Cambridge University Press, 2012.*