Ampleness in the free group

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Abstract

We show that the elementary theory of the free group – and more generally the elementary theory of any torsion-free hyperbolic group – is n-ample for any $n \geq 1$. We also give an explicit description of the imaginary algebraic closure in free groups.

1 Introduction

The work of Kharlampovich-Myasnikov [4] and Sela [16], who showed that the elementary theory T_{fg} of a nonabelian free group does not depend on the rank, opened the way to a model theoretic investigation of the free group. Sela [18] also showed that this theory as well as the elementary theory of any torsion-free hyperbolic group is stable.

Stability of a theory is equivalent to the existence of a notion of independence between subsets of a model satisfying certain natural conditions, generalizing linear independence in vector spaces or algebraic independence in algebraically closed fields. Properties of the geometry induced by this independence notion reflect properties of the definable and interpretable sets in models of the theory.

Having a quantifier elimination result down to $\forall\exists$-formulas [4, 16], elimination of imaginaries down to some very restricted class of imaginaries [18], homogeneity [6, 10], a better understanding of stability theoretic independence [12, 14] and more recently a description of the algebraic closure [7] now give us the tools for studying the model theoretic geometry of independence in the free group and more generally in torsion-free hyperbolic groups.

Pillay [13] first defined the notion of ampleness in a stable theory. It is a property that reflects the existence of geometric configurations behaving very much like projective space over a field and so any theory interpreting an infinite field is in fact ample (see Pillay [13]). We use here the slightly stronger definition given by Evans in [2]. As usual we write $A \mathop{\smile\!\!\!\!\!\mid}_C B$ to express that the sets A and B are independent over the set C; we

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denote by $\mathrm{acl}^{\mathrm{eq}}(\bar{a})$ the algebraic closure of $\bar{a}$ with respect to T^{eq} (see Section 2 and [20] for more background).

Definition 1.1. [2] *Suppose T is a complete stable theory and $n \geq 1$ is a natural number. Then T is n-ample if (in some model of T, possibly after naming some parameters) there exist tuples $a_0, \ldots, a_n$ such that:*
(i) $a_n \not\perp a_0$;
(ii) $a_0 \ldots a_{i-1} \perp_{a_i} a_{i+1} \ldots a_n$ *for* $1 \leq i < n$;
(iii) $\mathrm{acl}^{\mathrm{eq}}(a_0) \cap \mathrm{acl}^{\mathrm{eq}}(a_1) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$;
(iv) $\mathrm{acl}^{\mathrm{eq}}(a_0 \ldots a_{i-1} a_i) \cap \mathrm{acl}^{\mathrm{eq}}(a_0 \ldots a_{i-1} a_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(a_0 \ldots a_{i-1})$ *for* $1 \leq i < n$.

We call T ample *if it is n-ample for all $n \geq 1$.* □

In $n+1$-dimensional projective space such a tuple $a_0, \ldots a_n$ can be chosen as a maximal *flag* of subspaces and this example can guide the intuition. It is widely conjectured that T_{fg}, the elementary theory of nonabelian free groups, does not interpret an infinite field. This conjecture would follow from knowing that this theory is non-ample. In [12] Pillay proved that the free group is 2-ample. However, his proof relied on a result by Bestvina and Feighn on on negligible sets which has not yet been completely established. Pillay also conjectured in [12] that T_{fg} is *not* 3-ample. We here refute Pillay's conjecture by showing the following more general result.

Theorem 1.2. *The elementary theory of any nonabelian torsion-free hyperbolic group is ample.*

Since nonabelian free groups of finite rank are torsion-free and hyperbolic it follows that their elementary theory is ample. In fact in our proof we go from free groups to torsion-free hyperbolic groups, we show ampleness in free groups and then we transfer this result to torsion-free hyperbolic groups. Our construction of a sequence witnessing ampleness is also very explicit. Given a basis $\{c_0, a_i, b_i, t_i \colon i < \omega\}$ of the free group of rank ω, we find witnesses for ampleness as the tuples $h_{2i} = (a_{2i}, b_{2i}, c_{2i}), i < \omega$, where the sequence $(c_i)_{i<\omega}$ is defined inductively by $c_{i+1} = t_i c_i^{-1} [a_i, b_i]^{-1} t_i^{-1}$.

We still do not know whether a torsion free hyperbolic group can interpret an infinite field although recent progress on this question has been made in [9]. While we do not believe this to be the case, ampleness is certainly consistent with the existence of a field.

We explain briefly the strategy of the proof of Theorem 1.2 in the case of free groups. By [7] the algebraic closure is closely related to graph of groups decompositions. We construct a graph of groups decomposition (over cyclic subgroups) of the free group in such a way that certain vertex groups then witnes ampleness. Since we also need to study the imaginary algebraic closure, we use Sela's elimination of imaginaries to reduce the problem to the usual algebraic closure and the algebraic closure relative to the conjugacy classes. The necessary properties of the imaginary algebraic closure are established using again its close relation to JSJ-decompositions.

The present paper is organized as follows. In Section 2 we collect the preliminaries about elimination of imaginaries, JSJ-decompositions and related notions needed in the sequel. In Section 3, we study the imaginary algebraic closure and give a geometric characterization of those conjugacy classes which are elements of the imaginary algebraic closure. Section 4 is devoted to the construction of sequences witnessing the ampleness in the free group. The last section then shows how the general theorem follows from the special case.

2 Preliminaries

In this section we put together some background material needed in the sequel. The first subsection deals with imaginaries and the two next subsections deal with splittings, JSJ-decompositions, homogeneity, algebraic closure and independence.

2.1 Imaginaries

Let T be a complete theory and M a (very saturated) model of T. Recall that T has *weak elimination of imaginaries* if for any $\emptyset$-definable equivalence relation E on M^k and any equivalence class $a_E \in M^{eq}$ there is a finite tuple $\bar{a} \subset M$ with $a_E \in \mathrm{acl}^{\mathrm{eq}}(\bar{a})$ and $\bar{a} \in \mathrm{dcl}^{\mathrm{eq}}(a_E)$. In particular, a_E and $\bar{a}$ are interalgebraic, i.e., $\mathrm{acl}^{\mathrm{eq}}(\bar{a}) = \mathrm{acl}^{\mathrm{eq}}(a_E)$ (for more details see [20]).

For a subset $\mathcal{E}$ of the set of $\emptyset$-definable equivalence relations in T, we let $M_{\mathcal{E}}$ denote the restriction of M^{eq} to the sorts in $\mathcal{E}$. That is, for every $E \in \mathcal{E}$ defined on M^k, we add a new sort S_E to the language interpreted as M^k/E and a new function $\pi_E : M^k \to S_E$ which associates to a k-tuple $x \in M^k$ its equivalence class x_E. Then the $L_{\mathcal{E}}$-structure $M_{\mathcal{E}}$ is the disjoint union of M/E for $E \in \mathcal{E}$. We say that T has weak elimination of imaginaries *relative to* $\mathcal{E}$ if $M_{\mathcal{E}}$ has weak elimination of imaginaries.

Remark 2.1. *Suppose that T has weak elimination of imaginaries relative to $\mathcal{E}$. Then for tuples $\bar{a}, \bar{b}, \bar{c} \subset M$ we have*

$$\mathrm{acl}^{\mathrm{eq}}(\bar{a}) \cap \mathrm{acl}^{\mathrm{eq}}(\bar{b}) = \mathrm{acl}^{\mathrm{eq}}(\bar{c})$$

if and only if

$$(\mathrm{acl}^{\mathrm{eq}}(\bar{a}) \cap M_{\mathcal{E}}) \cap (\mathrm{acl}^{\mathrm{eq}}(\bar{b}) \cap M_{\mathcal{E}}) = (\mathrm{acl}^{\mathrm{eq}}(\bar{c}) \cap M_{\mathcal{E}}).$$

For the theory of a nonabelian torsion-free hyperbolic group Sela established weak elimination of imaginaries relative to the following small collection of basic equivalence relations: let $\mathcal{E}$ denote the collection of the following $\emptyset$-definable equivalence relations where $C(x)$ denotes the centralizer of x and m, p, q are positive natural numbers:

$$E_0(x; y) : \exists z x^z = y$$

$$E_{1,m}(x, y; x', y') : C(x) = C(x') \wedge \exists t \in C(x) \text{ such that } y' = yt^m$$

$$E_{2,m}(x, y; x', y') : C(x) = C(x') \wedge \exists t \in C(x) \text{ such that } y' = t^m y$$

$E_{3,p,q}(x,y,z;x',y',z') : C(x) = C(x') \wedge C(y') = C(y) \wedge \exists s \in C(x) \wedge \exists t \in C(y)$ such that $z = s^p z' t^q$.

For an L-structure M we denote by $\mathcal{P}_n(M^m)$ the set of finite subsets of M^m of cardinality at most n. A function $f : M^r \to \mathcal{P}_n(M^m)$ is said to be definable if there exists a formula $\varphi(\bar{x};\bar{y})$ such that for any $\bar{a} \in M^r$, for any $\bar{b} \in M^m$, $\bar{b} \in f(\bar{a})$ if and only if $M \models \varphi(\bar{a},\bar{b})$.

Sela proved the following:

Theorem 2.2. [18, Theorem 4.6] *For any $\emptyset$-definable equivalence relation $E(\bar{x},\bar{y})$, $|\bar{x}| = n$ in the theory T of a torsion-free hyperbolic group M (in the language of groups), there exist $k,p \in \mathbb{N}$ and a function $f : M^n \to P_p({M_{\mathcal{E}}}^k)$ $\emptyset$-definable in $L_{\mathcal{E}}$ such that for all $\bar{a},\bar{b} \in M^n$*

$$M \models E(\bar{a},\bar{b}) \Leftrightarrow M_{\mathcal{E}} \models f(\bar{a}) = f(\bar{b}).$$

Clearly, this implies that for any 0-definable equivalence relation E in T with corresponding definable function f_E in $L_{\mathcal{E}}$, if $f_E(\bar{a}) = (\bar{c}_1, \ldots \bar{c}_k)$ where the $\bar{c}_i$ are tuples from $M_{\mathcal{E}}$, then $\bar{c}_1, \ldots, \bar{c}_k \in \mathrm{dcl}^{\mathrm{eq}}(\bar{a}_E)$ and $\bar{a}_E \in \mathrm{acl}^{\mathrm{eq}}(\bar{c}_1, \ldots, \bar{c}_k)$. Hence:

Corollary 2.3. [18] *The theory of a torsion-free hyperbolic group has weak elimination of imaginaries relative to $\mathcal{E}$.* □

In order to prove ampleness for torsion-free hyperbolic groups we can therefore restrict our attention to these basic equivalence relations in $\mathcal{E}$.

2.2 Splittings, JSJ-decompositions

Our notation is fairly standard and we assume that the reader familiar with basic notions of combinatorial and geometric group theory like free products, HNN-extensions, hyperbolic groups and graph of groups decompositions. We state here some properties of JSJ-decompositions needed in the sequel. Let G be a group and let $\mathcal{C}$ be a class of subgroups of G. By a $(\mathcal{C},H)$-*splitting* of G (or a splitting of G over $\mathcal{C}$ relative to H), we understand a tuple $\Lambda = (\mathcal{G}(V,E), T, \varphi)$, where $\mathcal{G}(V,E)$ is a graph of groups such that each edge group is in $\mathcal{C}$ and H is elliptic, T is a maximal subtree of $\mathcal{G}(V,E)$ and $\varphi : G \to \pi(\mathcal{G}(V,E),T)$ is an isomorphism; here $\pi(\mathcal{G}(V,E),T)$ denotes the fundamental group of $\mathcal{G}(V,E)$ relative to T. If $\mathcal{C}$ is the class of abelian groups or cyclic groups, we will just say *abelian splitting* or *cyclic splitting*, respectively. If every edge group is malnormal in the adjacent vertex groups, then we say that the splitting is *malnormal*. Recall that a subgroup $A \leq G$ is said to be *malnormal* if for any $g \in G$ if $A^g \cap A \neq 1$ then $g \in A$.

Given a group G and a subgroup H of G, G is said to be *freely H-decomposable* if G has a nontrivial free decomposition $G = G_1 * G_2$ such that $H \leq G_1$. Otherwise, G is said to be freely *H-indecomposable*.

Following [3], given a group G and two $(\mathcal{C},H)$-splittings Λ_1 and Λ_2 of G, we say that Λ_1 *dominates* Λ_2 if every subgroup of G which is elliptic in Λ_1 is also elliptic in Λ_2.

A $(\mathcal{C}, H)$-splitting of G is said to be *universally elliptic* if all edge stabilizers in Λ are elliptic in any other $(\mathcal{C}, H)$-splitting of G.

A *JSJ-decomposition of G over $\mathcal{C}$ relative to H* is an universally elliptic $(\mathcal{C}, H)$-splitting dominating all other universally elliptic $(\mathcal{C}, H)$-splittings. If $\mathcal{C}$ is the class of abelian subgroups, then we simply say *abelian JSJ-decomposition*; similarly when $\mathcal{C}$ is the class of cyclic subgroups. It follows from [15, 3] that torsion-free hyperbolic groups (so in particular nonabelian free groups of finite rank) admit (relative) cyclic JSJ-decompositions.

Given an abelian splitting Λ of G (relative to H) and a vertex group G_v of Λ, the *elliptic abelian neighborhood* of G_v is the subgroup generated by the elliptic elements that commute with nontrivial elements of G_v. It was shown in [1, Proposition 4.26] that if G is commutative transitive then any abelian splitting Λ of G (relative to H) can be transformed to an abelian splitting Λ' of G such that the underlying graph is the same as that of Λ and for any vertex v, the corresponding new vertex group $\hat{G}_v$ in Λ' is the elliptic abelian neighborhood of G_v (similarly for edges); in particular any edge group of Λ' is malnormal in the adjacent vertex groups. We call that transformation the *malnormalization* of Λ. If Λ is a (cyclic or abelian) JSJ-decomposition of G and G is commutative transitive then the malnormalization of Λ will be called a *malnormal* JSJ-decomposition. If G_v is a rigid vertex group then we call $\hat{G}_v$ also rigid; similarly for abelian and surface type vertex groups. Strictly speaking a malnormal JSJ-decomposition is not a JSJ-decomposition in the sense of [3]. However it possesses the most important properties of JSJ-decompositions that we need.

We end with the definition of *generalized malnormal JSJ-decomposition* relative to a subgroup A. First, split G as a free product $G = G_1 * G_2$, where $A \leq G_1$ and G_1 is freely A-indecomposable. Then, define a generalized malnormal (cyclic) JSJ-decomposition of G relative to A as the (cyclic) splitting obtained by adding G_2 as a new vertex group to a malnormal (cyclic) JSJ-decomposition of G_1 (relative to A). We call G_2 the *free factor*. In a similar way the notion of a *generalized cyclic JSJ-decomposition*, without the assumption of malnormality, is defined

We denote by $Aut_H(G)$ the group of automorphisms of a group G that fix a subgroup H pointwise. The *abelian modular group* of G relative to H, denoted $Mod_H(G)$, is the subgroup of $Aut_H(G)$ generated by Dehn twists, modular automorphisms of abelian type and modular automorphisms of surface type (For more details we refer the reader for instance to [7]). We will use the following property of modular automorphisms whose proof is essentially contained in [1] (see for instance [1, Proposition 4.18]).

Lemma 2.4. *Let Γ be torsion-free hyperbolic group and A a nonabelian subgroup of Γ. Let Λ be a generalized malnormal cyclic JSJ-decomposition of Γ relative to A. Then any modular automorphism $\sigma \in Mod_A(\Gamma)$ restricts to a conjugation on rigid vertex groups and on boundary subgroups of surface type vertex groups of Λ.* □

We note that when Γ is a torsion-free hyperbolic group which is freely A-indecomposable, then any abelian vertex group in any cyclic JSJ-decomposition Λ of

Γ relative to A is rigid; this a consequence of the fact that abelian subgroups of Γ are cyclic and of the fact that edge groups of Λ are universally elliptic. The same property holds also for abelian vertex groups (different from free factors) in generalized malnormal JSJ-decompositions.

Rips and Sela showed that the modular group has finite index in the group of automorphisms. We will use this result in the relative case:

Theorem 2.5. (See for instance [7]) *Let Γ be a torsion-free hyperbolic group and A a nonabelian subgroup of Γ such that Γ is freely A-indecomposable. Then $Mod_A(\Gamma)$ has finite index in $Aut_A(\Gamma)$.* □

We end this subsection with the following standard lemma needed in the sequel.

Lemma 2.6. *Let G be the fundamental group of an orientable surface S with boundaries such that $2g(S) + b(S) \geq 4$, where $g(S)$ denotes the genus of S and $b(S)$ denotes the number of boundary components. Then for any nontrivial element $g \in G$ which is not conjugate to an element of a boundary subgroup, there exists a malnormal cyclic splitting of G in which g is hyperbolic and boundary subgroups are elliptic.*

Proof. The proof is by induction on $g(S)$. If $g(S) = 0$, then $G = \langle s_1, \cdots, s_n | s_1 \cdots s_n = 1\rangle$ and $n \geq 4$. Note that $s_1, \cdots, s_{n-1}$ is a basis of G. Since g is not conjugate to an element of any boundary subgroup, the normal form of g involves at least two elements s_i, s_j with $1 \leq i < j \leq n-1$. Since $n \geq 4$, replacing the relation $s_1 \cdots s_n = 1$ by a cyclic permutation and by relabeling $s_1, \cdots, s_n$, we may assume that $1 < i < j \leq n-1$. Then g is hyperbolic in the following malnormal cyclic splitting $G = \langle s_1, \cdots, s_i | \rangle *_{s_1 \cdots s_i = s_n^{-1} \cdots s_{i+1}^{-1}} \langle s_{i+1}, \cdots, s_n | \rangle$. The geometric picture in that case is that the curve representing g intersects at least one simple closed curve which separates the surface into two subsurfaces each of which has at least two boundary components.

Now suppose that $g(S) \geq 1$. Let c be a non null-homotopic simple closed curve represented in a handle of S. Let Λ be the dual splitting induced by c which is in this case an HNN-extension. If g is hyperbolic in that splitting (in particular if g is a conjugate to a power of the element represented by c) then we are done. Otherwise g is elliptic and thus it is conjugate to an element represented by a curve in the surface obtained by cutting along c. This new surface S' has genus $g(S) - 1$ and two new boundary components and thus we have $2g(S') + b(S') = 2g(S) + b(S) \geq 4$. By induction, since g is not conjugate to any element of a boundary subgroup (also the new two boundaries) of S', there exists a malnormal cyclic splitting of S' in which g is hyperbolic and the boundary subgroups are elliptic. Since that splitting is compatible with Λ we get the required result. □

2.3 Homogeneity, algebraic closure, independence

In this subsection we collect facts about algebraic closure and independence which will be used in the sequel. We start from known facts which we cite for convenient reference and extend them for our purposes.

Whenever there is more than one group around we may write $\mathrm{acl}_G(A)$, $\mathrm{acl^{eq}}_G(A)$ and $\mathop{\smile\!\!\!\!\!\mid}^G$ to denote the algebraic closure and independence in the theory of G.

Proposition 2.7. *Let G be a torsion-free CSA-group. Let Λ be a malnormal cyclic splitting of G. Then for any nontrivial vertex group A we have $\mathrm{acl}(A) = A$.*

Proof. A consequence of [7, Proposition 4.3]. □

Theorem 2.8. [7, Theorem 4.5] *If F is a free group of finite rank with nonabelian subgroup A, then $\mathrm{acl}(A)$ coincides with the vertex group containing A in the generalized malnormal (cyclic) JSJ-decomposition of F relative to A.* □

Theorem 2.9. [7] *Let Γ be a torsion-free hyperbolic group and A a subgroup of Γ. Then $\mathrm{acl}(A)$ is finitely generated.* □

Proposition 2.10. [6, Proposition 5.9] *Let F be a nonabelian free group of finite rank and $\bar{a}$ a tuple from F such that the subgroup A generated by $\bar{a}$ is nonabelian and F is freely A-indecomposable. Then for any tuple $\bar{b}$ contained in F, the type $tp(\bar{b}/\bar{c})$ is isolated.* □

Theorem 2.11. [6, 10] *Let F be a nonabelian free group of finite rank. For any tuples $\bar{a}, \bar{b} \in F^n$ and for any subset $P \subseteq F$, if $tp^F(\bar{a}/P) = tp^F(\bar{b}/P)$ then there exists an automorphism of F fixing P pointwise and sending $\bar{a}$ to $\bar{b}$.* □

One ingredient in the proof of Theorem 1.2 is a result of Sela's which essentially allows us to work in a free group. In [Sel09], Sela shows that if Γ is a torsion-free hyperbolic group which is not elementarily equivalent to a free group, then Γ has a *minimal* elementary subgroup, denoted by $EC(\Gamma)$, called the *elementary core* of Γ (see for instance [6, Section 8] for some properties of the elementary core). It follows from the definition of the elementary core that if Γ is a nonabelian torsion-free hyperbolic group then Γ is elementarily equivalent to $\Gamma * F_n$ for any free group of rank n. Combined with [19, Theorem 7.2] this gives the following stronger result:

Theorem 2.12. *Any nonabelian torsion-free hyperbolic group Γ is elementarily equivalent to $\Gamma * F$ for any free group F.* □

We will be using the following variant of this result:

Lemma 2.13. *Let Γ be a torsion-free hyperbolic group not elementarily equivalent to a free group. Let F be a free group and C a free factor of F. Then $EC(\Gamma) * C \preceq \Gamma * F$.*

Proof. Suppose first that F has finite rank. The proof is by induction on the rank n of C. If $n = 0$ then the result is a consequence of the definition of elementary cores (see [17]). Suppose that the result holds for free factors of rank n and set $C = \langle \bar{c}_n, c_{n+1} | \rangle$. Let $\varphi(\bar{x})$ be a formula (without parameters) and $\bar{g} \in EC(\Gamma) * C$ such that $\varphi(\bar{g})$ holds in $EC(\Gamma) * C$. Then there exists a tuple of words $\bar{w}(\bar{x}_n; y)$ such that $\bar{g} = \bar{w}(\bar{c}_n; c_{n+1})$.

By induction $EC(\Gamma) * \langle \bar{c}_n | \rangle$ is an elementary subgroup of $EC(\Gamma) * C$ and thus by [7, Lemma 8.10], the formula $\varphi(\bar{w}(\bar{c}_n; y))$ is generic. Since by induction $EC(\Gamma) * \langle \bar{c}_n | \rangle$ is an elementary subgroup of $\Gamma * F$ we conclude that there exists $g_1, \cdots, g_p \in EC(\Gamma) * \langle \bar{c}_n | \rangle$ such that $\Gamma * F = \cup_i g_i X$ where $X = \varphi(\bar{w}(\bar{c}_n; \Gamma * F))$. Hence $c_{n+1} \in g_i X$ for some i. There exists an automorphism of $\Gamma * F$ fixing $EC(\Gamma) * \langle \bar{c}_n | \rangle$ pointwise and sending c_{n+1} to $g_i c_{n+1}$ and thus $\varphi(\bar{w}(\bar{c}_n; c_{n+1}))$ holds in $\Gamma * F$ as required.

Suppose now that F has infinite rank and C has finite rank. By quantifier elimination and Theorem 2.12 it suffices to show for any formula φ of the form $\forall\exists\psi$ or $\exists\forall\psi$ with parameters from $EC(\Gamma) * C$ the following: if $\Gamma * F \models \varphi$ then $EC(\Gamma) * C \models \varphi$.

Suppose that φ is of the form $\forall x \exists y \psi(x; y)$ where x and y are finite tuples and ψ is quantifier-free. If $\Gamma * F \models \varphi$ then for any $g \in EC(\Gamma) * C$ there exists a free factor C' of F of finite rank such that $C \leq C'$ and a tuple $a \in \Gamma * C'$ such that $\Gamma * C' \models \psi(g; a)$. By the previous case, $EC(\Gamma) * C$ is an elementary subgroup of $\Gamma * C'$ and thus we get $EC(\Gamma) * C \models \exists y \psi(g; y)$. Hence we conclude that $EC(\Gamma) * C \models \varphi$. The case that φ is of the form $\exists\forall$ can be treated in a similar way. Finally, the case where F and C both have infinite rank follows from the previous cases. $\square$

Corollary 2.14. *Suppose $H = \Gamma * F$ where Γ is a torsion-free hyperbolic group and F a free group of finite rank. Then $EC(\Gamma)$ and F are (in H) independent over the emptyset.*

Proof. The result is clear for $EC(\Gamma) = 1$. So suppose that $EC(\Gamma) \neq 1$, i.e., Γ is not elementarily equivalent to a free group. Combining Lemma 2.13 and [7, Lemma 8.10], if $e \in F$ is a primitive element then $tp(e/EC(\Gamma))$ is the unique generic type of H over $EC(\Gamma)$. If $\bar{h}$ is a generating tuple for $EC(\Gamma)$ we therefore have $e \mathop{\smile\!\!\!\!\!\!|}^{H} \bar{h}$ for any primitive element $e \in F$. Again combining Lemma 2.13 and [7, Lemma 8.10] we conclude that $\bar{e} \mathop{\smile\!\!\!\!\!\!|}^{H} \bar{h}$ for any basis $\bar{e}$ of F. The result now follows. $\square$

In a free group of infinite rank, we obtain the following of indepedent interest:

Lemma 2.15. *Suppose that G is a group such that the theory T_H of $H = G * F$ is simple where F is a free group of infinite rank. Then*

$$G \mathop{\smile\!\!\!\!\!\!|} F.$$

Proof. Let $\{e_i : i < \omega\}$ be part of a basis for F. By using Poizat's observation as in [12], if X is a definable generic subset of H with parameters from G, X contains all but finitely many elements of any basis of F. It follows that the e_i form a Morley sequence in the sense of T_H over G. Clearly, the e_i are indiscernible over any finite subset $A \subset G$. Therefore (see e.g. [20, Lemma 7.2.19])

$$A \mathop{\smile\!\!\!\!\!\!|} \{e_i : i < \omega\}$$

and this is enough. $\square$

The following characterization of forking independence over free factors in free groups was recently proved by Perin and Sklinos [11] using [12, Corollary 2.7] and Theorem 2.11.

Proposition 2.16. [11] Let F be a free group of finite rank, $\bar{a}, \bar{b}$ be finite tuples from F and C a free factor of F. Then

$$\bar{a} \underset{C}{\downarrow} \bar{b}$$

if and only if

$$F = A * C * B \text{ with } \ \bar{a} \in A * C \text{ and } \ \bar{b} \in C * B.$$

We will need slight extensions of the previous results to free products of a torsion-free hyperbolic group with a free group:

Proposition 2.17. (Generic homogeneity) *Let $H = \Gamma * F$ where Γ is a torsion-free hyperbolic group (possibly trivial) and F a free group of finite rank. Let $\bar{a}, \bar{b}, \bar{c}$ be finite tuples from F. If $tp_H(\bar{a}/\bar{c}) = tp_H(\bar{b}/\bar{c})$ then there exists an automorphism $f \in Aut_{\bar{c}}(F)$ such that $f(\bar{a}) = \bar{b}$.*

In particular,

$$tp_H(\bar{a}/\bar{c}) = tp_H(\bar{b}/\bar{c})$$

if and only if

$$tp_F(\bar{a}/\bar{c}) = tp_F(\bar{b}/\bar{c})$$

if and only if

$$tp_H(\bar{a}/\bar{c}\bar{h}) = tp_H(\bar{b}/\bar{c}\bar{h})$$

for any generating set $\bar{h}$ of Γ.

Proof. Let A (resp. B) be the subgroup generated by $\bar{a}$ and $\bar{c}$ (resp. $\bar{b}$ and $\bar{c}$) and let E_1 (resp. E_2) be the smallest free factor of F containing A (resp. B). By [10, Proposition 7.1], either there exists an embedding $u : E_1 \to H$ which fixes $\bar{c}$ and sends $\bar{a}$ to $\bar{b}$, or there exists a noninjective preretraction $r : E_1 \to H$ with respect to Λ, the JSJ-decomposition of E_1 relative to A. If the last case holds then by [10, Proposition 6.8], there exists a noninjective preretraction from E_1 to E_1 and by [10, Proposition 6.7] we get a subgroup $E_1' \leq E_1$ and a preretraction $r' : E_1 \to E_1$ such that (E_1, E_1', r') is a hyperbolic floor. This implies that F has a structure of hyperbolic tower over a proper subgroup contradicting [10, Proposition 6.5].

We conclude that there exists $u : E_1 \to H$ which fixes $\bar{c}$ and sends $\bar{a}$ to $\bar{b}$. Symmetrically, there exists $v : E_2 \to H$ which fixes $\bar{c}$ and sends $\bar{b}$ to $\bar{a}$. Since E_1 (resp. E_2) is freely indecomposable relative to A (resp. B) by using Grushko's Theorem $u : E_1 \to E_2$ is an isomorphism sending $\bar{a}$ to $\bar{b}$ and fixing $\bar{c}$ which can be extended to F and also to H.

This shows in particular that $tp_F(\bar{a}/\bar{c}) = tp_F(\bar{b}/\bar{c})$. Conversely, if this last property holds, then by homogeneity of F there exists an automorphism f of F fixing $\bar{c}$ and sending $\bar{a}$ to $\bar{b}$. Such an automorphism can be extended to an automorphism $\hat{f}$ of H (fixing Γ pointwise) and thus $tp_H(\bar{a}/\bar{c}\bar{h}) = tp_H(\bar{b}/\bar{c}\bar{h})$ for any generating set $\bar{h}$ of Γ. Then clearly, we also have $tp_H(\bar{a}/\bar{c}) = tp_H(\bar{b}/\bar{c})$. □

Corollary 2.18. *Let $H = \Gamma * F$ where F is a free group (of any rank) and Γ is a torsion-free hyperbolic group not elementarily equivalent to a free group. Let $\bar{a}, \bar{b}, \bar{c}$ be finite tuples from F. Then the conclusions of Proposition 2.17 hold also in this case.*

Proof. Let F_n be a free factor of finite rank of F containing the tuples $\bar{a}, \bar{b}, \bar{c}$ and set $H_n = \Gamma * F_n$. Suppose that $tp_H(\bar{a}/\bar{c}) = tp_H(\bar{b}/\bar{c})$. By Lemma 2.13, $EC(\Gamma) * F_n \preceq H_n$ and $EC(\Gamma) * F_n \preceq H$ and hence we get $tp_{H_n}(\bar{a}/\bar{c}) = tp_{H_n}(\bar{b}/\bar{c})$. By Proposition 2.17 there exists an automorphism $f \in Aut_{\bar{c}}(F_n)$ such that $f(\bar{a}) = \bar{b}$ which can be easily extended to F. □

Theorem 2.19. *Let $H = \Gamma * F$ where F is a free group and Γ is a torsion-free hyperbolic group not elementarily equivalent to a free group. For finite tuples $\bar{a}, \bar{b} \in F$ and a free factor C (possibly trivial) with finite basis $\bar{c}$ of F we have*

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{c}}^{H} \bar{b}$$

if and only if

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{c}}^{F} \bar{b}.$$

Proof. Let $\bar{h}$ be a generating tuple for $EC(\Gamma)$. By Corollary 2.14 or Lemma 2.15 we have $\bar{a}\bar{c} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{H} \bar{h}$ and $\bar{a}\bar{b}\bar{c} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{H} \bar{h}$. Using transitivity and monotonicity of forking in stable theories (see e.g. [20, Corollary 7.2.17]) we have hence $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{c}}^{H} \bar{h}$ and $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{b}\bar{c}}^{H} \bar{h}$.

Therefore, again using transitivity and monotonicity of forking we have

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{c}}^{H} \bar{b}$$

if and only if

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{c}\bar{h}}^{H} \bar{b}.$$

Thus it is sufficient to show that

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{c}\bar{h}}^{H} \bar{b}$$

if and only if

$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{c}}^{F} \bar{b}.$$

The proof is an adaptation of the argument in [11] using the characterisation of forking independence in F given in Proposition 2.16 and Proposition 2.17. Write $F = C * D$ and let $\{d_i : i < \lambda\}$ be a basis of D.

Since $EC(\Gamma) * F \preceq \Gamma * F$ we may work in $EC(\Gamma) * F$ and thus without loss of generality we assume that $\Gamma = EC(\Gamma)$. Suppose that $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{h}\bar{c}}^{H} \bar{b}$. Let D' be another copy of D with basis $\{d_i' : i < \lambda\}$ and consider $H' = \Gamma * F * D' = \Gamma * C * D * D'$. Let $\bar{w}(\bar{x}; \bar{y})$ be a tuple of words such that $\bar{a} = \bar{w}(\bar{d}, \bar{c})$ and consider $\bar{a}' = w(\bar{d}', \bar{c})$. Since $H \preceq H'$, we have $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{h}\bar{c}}^{H'} \bar{b}$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{h}\bar{c}}^{H'} \bar{b}$ and $\bar{a}' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{h}\bar{c}}^{H'} \bar{b}$ by Corollary 2.14 and Lemma 2.15.

Since by Lemma 2.13, $\Gamma * C$ is an elementary subgroup of H', it is algebraically closed in H'^{eq}. It follows that for any $\bar{a} \in F$, $tp(\bar{a}/\bar{h}\bar{c})$ is stationary; that is if $\bar{a}', \bar{a}'', \bar{b} \in F$, $tp(\bar{a}'/\bar{h}\bar{c}) = tp(\bar{a}''/\bar{h}\bar{c}) = tp(\bar{a}/\bar{h}\bar{c})$ and $a' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{h}\bar{c}}^{H} \bar{b}$, $a'' \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\bar{h}\bar{c}}^{H} \bar{b}$ then $tp(\bar{a}'/\bar{b}\bar{h}\bar{c}) = tp(\bar{a}''/\bar{b}\bar{h}\bar{c})$.

By stationarity we get $tp_{H'}(\bar{a}/\bar{h}\bar{c},\bar{b}) = tp_{H'}(\bar{a}'/\bar{h}\bar{c},\bar{b})$ and in particular $tp_{H'}(\bar{a}/\bar{c},\bar{b}) = tp_{H'}(\bar{a}'/\bar{c},\bar{b})$. By Proposition 2.17, there exists an automorphism f of $F * D'$ fixing $\bar{c},\bar{b}$ pointwise and sending $\bar{a}'$ to $\bar{a}$. We have $F * D' = f(D) * C * f(D')$, $\bar{a} \in C * f(D')$ and $\bar{b} \in f(D) * C$. The conclusion now follows from Grushko's theorem.

For the converse suppose that $F = A * C * B$, such that $\bar{a} \in A * C, \bar{b} \in C * B$. Since $\bar{a},\bar{b},\bar{c}$ are finite tuples we have a free factor F_n of finite rank with the property $F_n = (A \cap F_n) * C * (B \cap F_n)$ such that $\bar{a} \in (A \cap F_n) * C, \bar{b} \in C * (B \cap F_n)$. Set $H' = EC(\Gamma) * F_n$. Since $H' \preceq H$, it is sufficient to show that $\bar{a} \downarrow^{H'}_{\bar{h}\bar{c}} \bar{b}$. By Lemma 2.13 and [6, Lemma 8.10], if $\{e_1, \cdots, e_n\}$ is a basis of $A \cap F_n$ and $\{e'_1, \cdots, e'_m\}$ is a basis of $B \cap F_n$, then they are independent realisations of the generic type over $\bar{h}\bar{c}$. The result follows by the same argument as in the proof of Corollary 2.14. □

3 The imaginary algebraic closure

In this section we study $\mathrm{acl}^{\mathrm{eq}}(A)$ with respect to the three basic equivalence relations conjugacy, left (right) cosets of cyclic groups, and double cosets of cyclic groups. Note that the algebraic closure is independent of the model. We first prove a proposition of independent interest:

Proposition 3.1. *Suppose that G is a torsion-free CSA-group, A a nonabelian subgroup of G. Let $\Lambda = (\mathcal{G}(V,E),T,\varphi)$ be a malnormal abelian splitting of G relative to A. If $g \in G$ is hyperbolic with respect to this splitting, there exist $f_n \in Aut_A(G), n \in \mathbb{N}$ such that the $f_n(g), n \in \mathbb{N}$ are pairwise non-conjugate and $C(f_n(g)) \neq C(f_m(g))$ for $n \neq m$.*

We reduce the proof to the following basic configurations.

Lemma 3.2. *Let G be a torsion-free CSA-group and A a nontrivial subgroup of G. Suppose that one of the following cases holds.*

(i) $G = H *_C K$, C is abelian and malnormal, $A \leq H$, H is nonabelian and g is not in a conjugate of H or K.

(ii) $G = \langle H, t | C^t = \varphi(C)\rangle$, C (and $\varphi(C)$) is abelian and malnormal in H, $A \leq H$ and g is not in a conjugate of H.

Then there exists infinitely many automorphisms $f_i \in Aut_A(G)$ whose restriction to H is the identity and to K a conjugation such that $f_i(g)$ is not conjugate to $f_j(g)$ and $C(f_i(g)) \neq C(f_j(g))$ for $i \neq j$.

Proof. We treat the case (i). Write g in normal form $g = g_1 \cdots g_r$, with $r \geq 2$, where we may assume that g is cyclically reduced. If $C = 1$ choose $c \in H$ such that $[c, g_i] \neq 1$ for at least one g_i appearing in the normal form of g with $g_i \in H$. Such a choice is possible since H is nonabelian and CSA. If $C \neq 1$ we take $c \in C$ nontrivial. We define the automorphism f_n by being the identity on H and conjugation by c^n on K. We see that $f_n \in Aut_A(G)$.

Suppose towards a contradiction that the orbit $\{f_n(g); n \in \mathbb{N}\}$ is finite up to conjugacy. Hence there exists n_0 and infinitely many n such that each f_n is conjugate to f_{n_0}.

Suppose that $C \neq 1$ and thus $c \in C$. In that case, we see that $f_n(g_1) \cdots f_n(g_r)$ is a normal form of $f_n(g)$ which is moreover cyclically reduced for any n. Hence by the conjugacy theorem, there exists $d \in C$ such that $f_n(g)$ is conjugate by d to the product of a cyclic permutation of the normal form of $f_{n_0}(g)$. Since the number of that cyclic permutations is finite we conclude that there exist $n \neq m$ such that $f_n(g) = f_m(g)^d$, where $d \in C$.

Suppose that $g_r \in K$. Then $g_r c^{(n-m)} d^{-1} g_r^{-1} \in C$ and thus we must have $c^{n-m} d^{-1} = 1$ and $g_{r-1} c^{m-n} g_{r-1}^{-1} \in C$. Since G is torsion-free and C is malnormal, we conclude that $g_{r-1} \in C$; a contradiction.

Suppose that $g_r \in H$. Then $d = 1$ and $g_{r-1} c^{n-m} g_{r-1}^{-1} \in C$ and the conclusion follows as in the previous case. Hence the set of orbits of $\{f_n(g), n \in \mathbb{N}\}$ is infinite up to conjugacy. Using a similar argument, we see also that $C(f_n(g)) \neq C(f_m(g))$ for $n \neq m$.

Suppose that $C = 1$. We treat the case $g_r \in L_2$, the other case can be treated in a similar way. Then we see that $c^n f_n(g) c^{-n}$ is a cyclically reduced conjugate of $f_n(g)$. Proceeding as above, we conclude that there exists $n \neq m$ such that $c^n f_n(g) c^{-n} = c^m f_m(g) c^{-m}$. Then using normal forms, we get that $[g_i, c] = 1$ for any g_i appearing in the normal form of g with $g_i \in H$; a contradiction. Using a similar argument, we see also that $C(f_n(g)) \neq C(f_m(g))$ for $n \neq m$.

We treat now the case (ii). Let $c \in G$ be a nontrivial element of $C_H(C)$ (if $C = 1$, since A is nontrivial, then we take c to be any nontrivial element of H) and define for $n \geq 1$, f_n by being the identity on H and sending t to $c^n t$. Then $f_n \in Aut_A(G)$.

Now g can be written in a normal form $g_0 t^{\epsilon_0} g_1 \cdots g_r t^{\epsilon_r} g_{r+1}$, where $\epsilon_i = \pm 1$. Since any element is conjugate to a cyclically reduced element, we may assume that g is cyclically reduced and we may take $g_{r+1} = 1$. We have $f_n(g) = g_0 f_n(t^{\epsilon_0}) g_1 \cdots g_r f_n(t^{\epsilon_r})$, $f_n(t^{\epsilon_i}) = c^{\epsilon_i n} t^{\epsilon_i}$ if $\epsilon_i = 1$ and $f_n(t^{\epsilon_i}) = \varphi(c)^{\epsilon_i n} t^{\epsilon_i}$ if $\epsilon_i = -1$. Therefore $f_n(t^{\epsilon_i}) g_{i+1} f_n(t^{\epsilon_{i+1}}) = c^{\epsilon_i n} t^{\epsilon_i} g_{i+1} \varphi(c)^{\epsilon_{i+1} n} t^{\epsilon_{i+1}}$ if $\epsilon_i = 1$ and $\epsilon_{i+1} = -1$. Similarly we have $f_n(t^{\epsilon_i}) g_{i+1} f_n(t^{\epsilon_{i+1}}) = \varphi(c)^{\epsilon_i n} t^{\epsilon_i} g_{i+1} c^{\epsilon_{i+1} n} t^{\epsilon_{i+1}}$ if $\epsilon_i = -1$ and $\epsilon_{i+1} = 1$.

If $C \neq 1$ then $g_{i+1} c^{\epsilon_{i+1} n} \notin C$ and $g_{i+1} \varphi(c)^{\epsilon_{i+1} n} \notin \varphi(C)$ and thus replacing each $f_n(t^{\epsilon_i})$ by its value we obtain a normal form of $f_n(g)$ which is moreover cyclically reduced.

If $C = 1$ then, since G is torsion-free, for all but infinitely many n, $g_{i+1} c^{\epsilon_{i+1} n} \neq 1$ and $g_{i+1} \varphi(c)^{\epsilon_{i+1} n} \neq 1$ and thus replacing each $f_n(t^{\epsilon_i})$ by its value we obtain as above a normal form of $f_n(g)$ which is moreover cyclically reduced.

Suppose that the set $\{f_n(g) | n \in \mathbb{N}\}$ is finite up to conjugacy. Hence there exist n_0 and infinitely many n such that $f_n(g)$ is cyclically reduced and is conjugate to $f_{n_0}(g)$. By the conjugacy theorem, there exists $d \in C \cup \varphi(C)$ such that $f_n(g)$ is conjugate by d to the product of a cyclic permutation of the normal form of $f_{n_0}(g)$. Since the number of that cyclic permutations is finite we conclude that there exist $n \neq m$ such that $f_n(g) = f_m(g)^d$, where $d \in C \cup \varphi(C)$.

Since G is a CSA group, either C and $\varphi(C)$ are conjugate in H and in this case we may assume that $C = \varphi(C)$ and thus G is an extension of a centralizer; or C and $\varphi(C)$ are conjugately separated, that is $C^h \cap \varphi(C) = 1$ for any $h \in H$.

Suppose first that $C = \varphi(C) \neq 1$. Since $f_n(g)(f_m(g)^d)^{-1} = 1$, using normal forms we have

$$g_r c^{\epsilon_r n} t^{\epsilon_r} d^{-1} t^{-\epsilon_r} c^{-\epsilon_r m} g_r^{-1} \in C$$

and hence

$$g_r c^{\epsilon_r (n-m)} d^{-1} g_r^{-1} \in C.$$

By induction on $0 \leq l \leq r$, we get

$$g_l \cdots g_r c^{(\epsilon_l + \cdots + \epsilon_r)(n-m)} d^{-1} g_r^{-1} \cdots g_l^{-1} \in C$$

and thus, we conclude that

$$c^{(\epsilon_0 + \cdots + \epsilon_r)(n-m)} d^{-1} = d^{-1},$$

and we get $c^{(\epsilon_0 + \cdots + \epsilon_r)(n-m)} = 1$.

Suppose that there is no indice l such that $c^{(\epsilon_l + \cdots + \epsilon_r)(n-m)} d^{-1} = 1$. Then it follows from above that $g_i \in C$ for any i and thus $g = g_0 \cdots g_r t^{\epsilon_0 + \cdots + \epsilon_r}$. Hence $(\epsilon_0 + \cdots + \epsilon_r) \neq 0$ and since $c^{(\epsilon_0 + \cdots + \epsilon_r)(n-m)} = 1$ and G is torsion-free we get a contradiction.

Suppose that there is some indice l such that $c^{(\epsilon_l + \cdots + \epsilon_r)(n-m)} d^{-1} = 1$. Then for any $i > l$ or $i < l$ we get $g_i \in C$. Then $g = t^{\epsilon_0 + \cdots + \epsilon_{l-1}} g_0 \cdots g_r t^{\epsilon_l + \cdots + \epsilon_r}$. If we suppose that $\epsilon_0 + \cdots + \epsilon_r = 0$ then g will be a conjugate of an element of H; a contradiction. Hence $(\epsilon_0 + \cdots + \epsilon_r) \neq 0$ and since $c^{(\epsilon_0 + \cdots + \epsilon_r)(n-m)} = 1$ and G is torsion-free we get a contradiction.

Suppose that $C = \varphi(C) = 1$. Then $d = 1$ and $c^{m-n} = 1$; which is a contradiction. Suppose now that C_1 and C_2 are conjugately separated. As in the previous case, we conclude after calculation that $c^{n-m} = 1$; which is a contradiction.

We conclude that the orbit $\{f_n(g) | n \in \mathbb{N}\}$ is infinite up to conjugacy as required. Using a similar argument with normal forms we get that $[f_n(g), f_m(g)] \neq 1$ for $n \neq m$ and thus $C(f_n(g)) \neq C(f_m(g))$ for $n \neq m$. □

Proof of Proposition 3.1.

To simplify, identify G with $\pi(\mathcal{G}(V, E), T)$. For an edge e_i outside T, let $\mathcal{G}_i(V, E_i)$ be the graph of groups obtained by deleting e_i. Then G is an HNN-extension of the fundamental group $G_i = \pi(\mathcal{G}_i(V, E_i), T)$ and we can write $G = \langle G_i, t | C^t = \varphi(C) \rangle$ with $A \leq G_i$.

Suppose that for some edge e_i outside T the element g is hyperbolic in the corresponding splitting. Then we conclude by Lemma 3.2.

Now assume that there is no edge in $\mathcal{G}(V, E)$ outside T such that g is hyperbolic in the corresponding splitting as above. Let L be the fundamental group of the graph of groups $\mathcal{G}(V, E')$ obtained by deleting all the edges outside the maximal subtree T. Since g is not hyperbolic in any splitting $G = \langle G_i, t | C^t = \varphi(C) \rangle$, we may assume that g is in L

and g is hyperbolic in L. Thus we can write $L = L_1 *_C L_2$ with g is hyperbolic. We may suppose without loss of generality that $A \leq L_1$. By Lemma 3.2, there exists infinitely many automorphisms $f_i \in Aut_A(L)$ whose restriction to L_1 is the identity and to L_2 a conjugation such that $f_i(g)$ is not conjugate to $f_j(g)$ and $C(f_i(g)) \neq C(f_j(g))$ for $i \neq j$.

Since each f_n sends each boundary subgroup to a conjugate of itself, f_n has a natural extension $\hat{f}_n$ to G. If we suppose that $\{\hat{f}_n(g); n \in \mathbb{N}\}$ is finite up to conjugacy (in G), then for infinitely many n, $f_n(g)$ is conjugate to an element of a vertex group; which is a contradiction. Hence $\{\hat{f}_n(g); n \in \mathbb{N}\}$ is infinite up to conjugacy (in G) and we see also that $C(f_n(g)) \neq C(f_m(g))$ for $n \neq m$ (in G). This ends the proof of the proposition. □

Definition 3.3. *Let G be a group and A a subgroup of G, $c \in G$. We say that c is malnormaly universally elliptic relative to A if c is elliptic in any malnormal abelian splitting of G relative to A.*

Corollary 3.4. *Let G be a torsion-free CSA-group and A a nonabelian subgroup of G. If c^G or $C(c)$ is in $\mathrm{acl}^{\mathrm{eq}}(A)$, then c is malnormaly universally elliptic relative to A.*

Proof. This follows from Proposition 3.1. □

We write $\mathrm{acl}^c(\bar{a}) = \mathrm{acl}^{\mathrm{eq}}(\bar{a}) \cap S_{E_0}$, that is $\mathrm{acl}^c(\bar{a})$ is the set of conjugacy classes b^F in $\mathrm{acl}^{\mathrm{eq}}(\bar{a})$. For any subset A of a group G we also write $A^c = \{b^G \mid b \in A\}$ for the set of conjugacy classes with representatives in A.

In the special case that G is free we do have the converse of Corollary 3.4. We can formulate the following list of equivalent criteria for a conjugacy class c^F to be contained in the imaginary algebraic closure of a subset in the free group.

Proposition 3.5. *Let F be a free group of finite rank, A a nonabelian subgroup of F and $c \in F$. The following are equivalent:*

(1) $c^F \in \mathrm{acl}^c(A)$.

(2) *There exists finitely many automorphisms $f_1, \ldots, f_p \in Aut_A(F)$ such that for any $f \in Aut_A(F)$, $f(c)$ is conjugate in F to some $f_i(c)$.*

(3) *c is malnormaly universally elliptic relative to A.*

(4) *In any generalized cyclic JSJ-decomposition of F relative to A, either c is conjugate to some element of the elliptic abelian neighborhood of a rigid vertex group or it is conjugate to an element of a boundary subgroup of a surface type vertex group.*

Proof. (1) $\Rightarrow$ (2). Suppose that there exists infinitely many automorphisms $f_i \in Aut_A(F)$ such that $f_i(c)$ is not conjugate to $f_j(c)$ for $i \neq j$. Then each f_i has an unique extension $\hat{f}_i$ to F^{eq} and we get that $\hat{f}_i(c^F) \neq \hat{f}_j(c^F)$ for $i \neq j$. Hence $c^F \notin \mathrm{acl}^{\mathrm{eq}}(A)$.

(2) $\Rightarrow$ (3). This follows from Proposition 3.1 or Corollary 3.4.

(3) $\Rightarrow$ (4). We let Δ be a malnormal generalized cyclic JSJ-decomposition of F relative to A. Hence c is elliptic in Δ. Clearly c is not in a conjugate of the free factor of Δ. If c is in a conjugate of a rigid vertex group, there is nothing to prove. Otherwise c is in a conjugate of a surface type vertex group and without loss of generality we may

assume that it is included. In this case, by [3, Proposition 7.6] (or by a general version of Lemma 2.6 and Proposition 3.1) c is in a conjugate of a boundary subgroup.

(4) $\Rightarrow$ (2). Write $F = F_1 * F_2$ where F_1 contains A and freely A-indecomposable. Let Δ be a malnormal cyclic JSJ-decomposition of F relative to A. Suppose that c is in some conjugate of a rigid vertex group or it is conjugate to an element of a boundary subgroup of a surface type vertex group in a generalized malnormal cyclic JSJ-decomposition of F relative to A. W.l.o.g, we can assume that it is contained in a rigid vertex group or it is contained in a boundary subgroup of a surface type vertex group. Since $Mod_A(F_1)$ has finite index in $Aut_A(F_1)$ (Theorem 2.5), there are $f_1, \dots, f_p \in Aut_A(F)$ such that for any $f \in Aut_A(F_1)$ there exists $\sigma \in Mod_A(F_1)$ such that $f = f_i \circ \sigma$ for some i. Since F_1 is freely A-indecomposable by Grushko theorem for any $f \in Aut_A(F)$, $f_{|F_1} \in Aut_A(F_1)$. By Lemma 2.4, any $\sigma \in Mod_A(F_1)$ sends c to a conjugate of itself. Hence $f(c) = f_i(c^\alpha) = f_i(c)^{f_i(\alpha)}$ and thus for any $f \in Aut_A(F)$, $f(c)$ is conjugate to some $f_i(c)$.

(2) $\Rightarrow$ (1). Since $\mathrm{acl}(A)$ is finitely generated (Theorem 2.9), we may assume that A is finitely generated. Write $F = F_1 * F_2$ where $A \leq F_1$ and F_1 is freely A-indecomposable. Since F_1 is an elementary subgroup of F, c is a in a conjugate of F_1. We assume that $c = c'^g$ where $c' \in F_1$.

By Proposition 2.10 let $\varphi_0(x)$ be a formula isolating the type of c' over $\bar{a}$ where $\bar{a}$ is a finite generating tuple of A. Let $\varphi(z)$ be the following formula in the language L^{eq}

$$\varphi(z) := \exists x(\hat{\varphi}_0(x) \wedge \pi(x) = z),$$

where $\hat{\varphi}_0$ is the relativisation of φ_0 to the real sort of F and π is the projection from the real sort of F to the sort of the conjugacy classes. Then $F^{eq} \models \varphi(c^F)$. We claim that φ has finitely many realizations, which shows that $c^F \in \mathrm{acl}^{\mathrm{eq}}(A)$.

Let $d^F \in F^{eq}$ such that $F^{eq} \models \varphi(d^F)$. Then there exists $\alpha \in F$ such that $F \models \varphi_0(\alpha)$ and $\alpha^F = d^F$. Since φ_0 isolates the type of c' over $\bar{a}$ and F is homogeneous (Theorem 2.11), we conclude that there exists an automorphism $f \in Aut_A(F)$ such that $f(c') = \alpha$. Hence $f(c) = \alpha^{f(g)}$ and thus α is conjugate to some $f_i(c)$ and thus $d^F = f_i(c)^F$ as required. $\square$

Remark 3.6. *Let F be the free group with basis $\{a, b\}$. The following example shows that in the previous proposition we cannot remove the assumption that A be nonabelian. Let A be the subgroup generated by a. We claim that $\mathrm{acl}^c(A) = A^c$. Indeed, let $H = F * \langle c|\rangle$. Then F is an elementary subgroup of H as well as the subgroup K generated by $\langle a, c\rangle$. Hence $\mathrm{acl}^c(A) \subseteq H^c \cap K^c = A^c$. By a result of Neilsen (see for instance [5, Proposition 5.1]) any automorphism of F sends $[a, b]$ to a conjugate of $[a, b]$ or $[a, b]^{-1}$. Hence we see that the implication (2) $\Rightarrow$ (1) is not true in this case. However if we suppose that A is abelian but not contained in a cyclic free factor in Proposition 3.5 then the same proof of (2) $\Rightarrow$ (1) works in this case.*

For later reference we also note the following:

Corollary 3.7. *Let $H = \Gamma * F$ where F is a free group and Γ is a torsion-free hyperbolic group not elementarily equivalent to a free group. Let $\bar{a}$ be a finite tuple from F generating a nonabelian subgroup. Then any conjugacy class $g^H \in \mathrm{acl}^c_H(\bar{a})$ has a representative g' either in Γ or in F. If $g' \in \Gamma$, then in fact ${g'}^H \in \mathrm{acl}^c_H(1) = \mathrm{acl}^c_\Gamma(1)$ and if $g' \in F$, then ${g'}^F \in \mathrm{acl}^c_F(\bar{a})$.*

Proof. Let A be the subgroup generated by $\bar{a}$. The first part follows directly from Corollary 3.4. For the second part just note that $\mathrm{acl^{eq}}_H(EC(\Gamma)) \cap \mathrm{acl^{eq}}_H(F) = \mathrm{acl^{eq}}_H(1)$ since $EC(\Gamma)$ and F are independent by Corollary 2.14 or Lemma 2.15. Since $EC(\Gamma) \preceq \Gamma$ and $EC(\Gamma) \preceq \Gamma * F$, we get $\mathrm{acl}^c_H(1) = \mathrm{acl}^c_\Gamma(1)$. Let F_n be a free factor of F of finite rank containing A. Again since $EC(\Gamma) * F_n \preceq H$ any $g^H \in \mathrm{acl}^c_H(\bar{a})$ has a representative g' in $EC(\Gamma) * F_n$. If $g' \in F$ and if we suppose that ${g'}^{F_n} \notin \mathrm{acl}^c_{F_n}(A)$ then by Proposition 3.5, we get infinitely many $f_n \in Aut_A(F_n)$ such the $f_n(g')$ are pairwise non conjugate and each f_n has a natural extension to H; which is a contradiction. □

Remark 3.8. *Note that if Γ is elementarily equivalent to a free group then $\mathrm{acl}^c_\Gamma(1) = 1^c$. Indeed, by elementary equivalence it is sufficient to show this when Γ is free. In that case we see that any $g^\Gamma \in \mathrm{acl}^c_\Gamma(1)$ is also an element of $\mathrm{acl}^c_\Gamma(a) \cap \mathrm{acl}^c_\Gamma(b)$ where $\{a, b\}$ is a part of a basis and we see that $\mathrm{acl}^c_\Gamma(a) = \langle a \rangle^c$; which gives the required conclusion. However it may be happen that $\mathrm{acl}^c_\Gamma(1) = \Gamma^c$ when Γ is not elementarily equivalent to a free group. Indeed let Γ be a rigid torsion-free hyperbolic group. Then $Out(\Gamma)$ is finite by Paulin's theorem [8] and Γ is homogeneous and prime by [6]. The same method as in the proof of Proposition 3.5* (2) $\Rightarrow$ (1), *shows that for any $g \in \Gamma$, $g^\Gamma \in \mathrm{acl}^c_\Gamma(1)$.*

For the equivalence relations $E_{i,m}, i = 1, 2,\ m \geq 1$ and $E_{3,p,q},\ p, q \geq 1$ given in Theorem 2.2, we denote the corresponding equivalence classes by $[x]_{i,m}, i = 1, 2,\ [x]_{3,p,q}$ respectively. We start with the following lemma:

Lemma 3.9. *Let $H = \Gamma * F$ where Γ is torsion-free hyperbolic (possibly trivial) and F is a free group. For a finite tuple $\bar{a}$ from F such that $\mathrm{acl}_H(\bar{a}) = \mathrm{acl}_F(\bar{a})$ and $c \in H$ the following properties are equivalent:*
(1) *$C(c) \in \mathrm{acl^{eq}}_H(\bar{a})$.*
(2) *$c \in \mathrm{acl}_H(\bar{a})$.*

Proof. Clearly, (2) implies (1). Let A be the subgroup generated by $\bar{a}$. If A is trivial the result is clear. To prove (1) implies (2) suppose first that A is abelian; so cyclic and generated by a. Let $f_n(x) = x^{a^n}$. If $c \notin \mathrm{acl}(A) = C(a)$, then $C(f_n(c)) \neq C(f_m(c))$ for $n \neq m$. Hence $C(c) \notin \mathrm{acl^{eq}}_H(A)$.

Suppose now that A is nonabelian. Let F_p be a freely A-indecomposable free factor of finite rank of F containing A and set $F = F_p * D$. Let Δ be the malnormal cyclic JSJ-decomposition of F_p relative to A. Extend this to a decomposition Δ' of $H = (\Gamma * D) * F_p$. Then by Theorem 2.8 and Proposition 2.7 the vertex group in Δ' containing A is $\mathrm{acl}_{F_n}(A) = \mathrm{acl}_F(A) = \mathrm{acl}_H(A)$ and by Corollary 3.4 c is elliptic with respect to Δ'.

Suppose that c is in a conjugate of $\Gamma * D$ and set $c = c_0^a$ with $c_0 \in \Gamma * D$. If $a \neq 1$, let f_n denote the automorphism of H given by conjugation by a^n on $\Gamma * D$ and the identity on F_p. Then clearly c^{a^n} and c^{a^m} do not centralize each other for $n \neq m$ showing that $C(c) \notin \mathrm{acl^{eq}}_H(A)$. Hence $a = 1$ and $c \in \Gamma * D$.

Therefore we may assume that c is in a conjugate of F_p. Proceeding as above we get $c \in F_p$. Suppose now that c is not in a conjugate of $\mathrm{acl}(A)$. Write $\Delta = (\mathcal{G}(V, E), T, \varphi)$ and let L be the fundamental group of the graph of groups obtained by deleting the edges which are outside T. Hence c is in a conjugate of L and without loss of generality we may assume that $c \in L$. Let $e_1, \cdots, e_q$ be the edges adjacent to $acl(A)$ and let C_i be the edge group corresponding to e_i. Hence we can write $L = L_{i1} *_{C_i} L_{i2}$ with $\mathrm{acl}(A) \leq L_{i1}$. Since c is elliptic and c is not in a conjugate of $\mathrm{acl}(A)$, we get, without loss of generality that $c \in L_{i2}$ for some i. If $c \in C_i$ then c is in a conjugate of $\mathrm{acl}(A)$ contrary to our hypothesis. So $c \notin C_i$. Proceeding as in the proof of Proposition 3.1, we take infinitely many Dehn twists $f_n \in Aut_A(F_n)$ around C_i if $C_i \neq 1$ and a conjugation by a nontrivial element of L_{i1} if $C_i = 1$ and thus we find $C(f_n(c)) \neq C(f_m(c))$ for $n \neq m$. Now each f_n has a standard extension $\hat{f}_n \in Aut_A(H)$ and we see also that $C(\hat{f}_n(c)) \neq C(\hat{f}_m(c))$ for $n \neq m$. It follows that $C(c) \notin \mathrm{acl^{eq}}(A)$; a contradiction.

Hence c is in a conjugate of $\mathrm{acl}(A)$. Suppose that $c = d^\alpha$ with $d \in \mathrm{acl}(A)$ and $\alpha \in F_p \setminus \mathrm{acl}(A)$. Then we find infinitely many automorphisms $f_i \in Aut_A(F_p)$ such that $f_n(\alpha) \neq f_m(\alpha)$ for $n \neq m$. Clearly, these f_i extend to H. Hence $C(f_n(c)) = C(f_n(d))^{f_n(\alpha)} \neq C(f_m(c))$ for infinitely many n, m and thus $C(c) \notin \mathrm{acl^{eq}}(A)$; a contradiction. We conclude that $c \in \mathrm{acl}(A)$. □

Proposition 3.10. *Let $H = \Gamma * F$ where Γ is torsion-free hyperbolic (possibly trivial) and F is a free group. For a finite tuple $\bar{a}$ of F such that $\mathrm{acl}_H(\bar{a}) = \mathrm{acl}_F(\bar{a})$ and $c, d, e \in H$ the following properties are equivalent:*

(1) $[(c, d)]_{1,m} \in \mathrm{acl^{eq}}_H(\bar{a})$.

(2) $c, d \in \mathrm{acl}_H(\bar{a})$.

(3) $[(c, d)]_{2,m} \in \mathrm{acl^{eq}}_H(\bar{a})$.

Similarly we have $[(c, d, e)]_{3,p,q} \in \mathrm{acl^{eq}}_H(\bar{a})$ if and only if $c, d, e \in \mathrm{acl}_H(\bar{a})$.

Proof. Let A be the subgroup generated by $\bar{a}$. The implications (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) are clear. By symmetry it suffices to prove (1) $\Rightarrow$ (2). By Lemma 3.9, we have $c \in \mathrm{acl}_H(A)$.

If A is abelian and generated by a, let $f_i(x) = x^{a^i}$ for $i \in \mathbb{N}$. If $d \notin \mathrm{acl}_H(A) = C(a)$, then $f_i(d) \notin f_j(d)C(a)$ for $i \neq j$. Thus for $i \neq j$, $(f_i(c), f_i(d))$ is not equivalent to $(f_j(c), f_j(d))$ relative to the equivalence relation $E_{1,1}$ and also relative to $E_{1,m}$ for all $m \geq 1$. Hence $[c, d]_{1,m} \notin \mathrm{acl^{eq}}(A)$; a contradiction. Therefore $d \in C(a) = \mathrm{acl}(A)$.

If A is nonabelian and $d \in H \setminus \mathrm{acl}(A)$, proceeding as in the proof of Lemma 3.9, we can again find infinitely many automorphisms $f_i \in Aut_A(F)$ such that $f_i(d) \notin f_j(d)C(c)$ for $i \neq j$. Clearly, these f_i extend to H and so $[(c, d)]_{1,m} \notin \mathrm{acl^{eq}}(A)$. □

Recall that we write $\mathrm{acl}^c(\bar{a}) = \mathrm{acl^{eq}}(\bar{a}) \cap S_{E_0}$. For any subset A of a group G we also write $A^c = \{b^G \mid b \in A\}$ for the set of conjugacy classes with representatives in A.

Corollary 3.11. *Let $H = \Gamma * F$ where Γ is torsion-free hyperbolic (possibly trivial) and F is a nonabelian free group. For finite tuples $\bar{a}, \bar{b}, \bar{c} \in F$ we have*

$$\mathrm{acl^{eq}}_H(\bar{a}) \cap \mathrm{acl^{eq}}_H(\bar{b}) = \mathrm{acl^{eq}}_H(\bar{c})$$

if and only if

$$\mathrm{acl}^c_H(\bar{a}) \cap \mathrm{acl}^c_H(\bar{b}) = \mathrm{acl}^c_H(\bar{c}) \tag{1}$$

and

$$\mathrm{acl}_H(\bar{a}) \cap \mathrm{acl}_H(\bar{b}) = \mathrm{acl}_H(\bar{c}). \tag{2}$$

Proof. By Theorem 2.12 we have $\Gamma * F_2$ is elementarily equivalent to H and thus we can apply Theorem 2.2.

One direction is clear. For the other direction, by Theorem 2.2 and Remark 2.1 it suffices to show that

$$(\mathrm{acl^{eq}}(\bar{a}) \cap F_{\mathcal{E}}) \cap (\mathrm{acl^{eq}}(\bar{b}) \cap F_{\mathcal{E}}) = \mathrm{acl^{eq}}(\bar{c}) \cap F_{\mathcal{E}}$$

where $\mathcal{E}$ is the set of equivalence relations given in Theorem 2.2. For E_0 this is assumption (1), for $E_{1,m}, E_{2,m}, E_{3,p,q}$ this follows from (2) and Proposition 3.10. □

Definition 3.12. *Let G be a group and $\bar{a}$ a tuple from G. We say that $\bar{a}$* represents conjugacy (in G) if $\mathrm{acl}^c_G(\bar{a}) = \mathrm{acl}_G(\bar{a})^c$.

To verify the properties of an ample sequence in a free factor of a torsion-free hyperbolic group it now suffices to restrict to this free factor:

Lemma 3.13. *Let $H = \Gamma * F$ where F is a free group and Γ is torsion-free hyperbolic not elementarily equivalent to a free group. For finite tuples $\bar{a}, \bar{b}, \bar{c} \in F$ generating nonabelian subgroups, representing conjugacy in F and such that $\mathrm{acl}_H(\bar{a}) = \mathrm{acl}_F(\bar{a}), \mathrm{acl}_H(\bar{b}) = \mathrm{acl}_F(\bar{b}), \mathrm{acl}_H(\bar{c}) = \mathrm{acl}_F(\bar{c})$, we have*

$$\mathrm{acl^{eq}}_F(\bar{a}) \cap \mathrm{acl^{eq}}_F(\bar{b}) = \mathrm{acl^{eq}}_F(\bar{c})$$

if and only if

$$\mathrm{acl^{eq}}_H(\bar{a}) \cap \mathrm{acl^{eq}}_H(\bar{b}) = \mathrm{acl^{eq}}_H(\bar{c}).$$

Proof. By Corollary 3.11 and the assumption on $\bar{a}, \bar{b}$ and $\bar{c}$ it suffices to verify that

$$\mathrm{acl}^c_F(\bar{a}) \cap \mathrm{acl}^c_F(\bar{b}) = \mathrm{acl}^c_F(\bar{c})$$

if and only if

$$\mathrm{acl}^c_H(\bar{a}) \cap \mathrm{acl}^c_H(\bar{b}) = \mathrm{acl}^c_H(\bar{c}).$$

But this follows from Corollary 3.7 and the assumption that the considered tuples represent conjugacy: for any nonabelian subgroup $A \leq F$ the conjugacy classes in $\mathrm{acl}^c_H(A)$ have representatives either in $\mathrm{acl}^c_F(A)$ or in $\mathrm{acl}^c_H(1)$ and since $\bar{a}$ say represent conjugacy in F we have in fact that $g^H \in \mathrm{acl}^c_H(\bar{a})$ if and only if either $g^H \in \mathrm{acl}^c_H(1)$ or g has a representative $g' \in F$ such that $g'^F \in \mathrm{acl}^c_F(\bar{a})$. □

Corollary 3.14. *Let $H = \Gamma * F$ where F is a free group and Γ is torsion-free hyperbolic not elementarily equivalent to a free group. Suppose that $a_0, \ldots, a_n$ are finite tuples in F, each a_i generating a nonabelian free factor of F and witnessing that F is n-ample and such that for $0 \leq i \leq k$ we have*

$$\mathrm{acl}_F(a_0, \ldots a_i, a_k) = \mathrm{acl}_H(a_0, \ldots a_i, a_k),$$

$$\mathrm{acl}^c_F(a_0, \ldots a_i, a_k) = \mathrm{acl}_F(a_0, \ldots a_i, a_k)^c.$$

Then $a_0, \ldots, a_n$ witness the fact that $Th(\Gamma)$ is n-ample.

Proof. This follows from Lemma 3.13 and Theorem 2.19. □

4 The construction in the free group

In this section we will be working exclusively in nonabelian free groups and therefore all notions of algebraic closure and independence refer to the theory T_{fg}. Our main objective here is to construct sequences witnessing the ampleness. Corollary 3.14 then allows us to transfer the results in Section 5 to torsion-free hyperbolic groups to obtain our main theorem.

Let

$$H_i = \langle c_i, d_i, a_i, b_i \mid c_i d_i [a_i, b_i] = 1 \rangle,$$

that is H_i is the fundamental group of an orientable surface with 2 boundary components and genus 1, where c_i and d_i are the generators of boundary subgroups. Note that H_i is a free group of rank 3 with bases a_i, b_i, c_i or a_i, b_i, d_i. Let

$$P_n = H_0 * H_1 * \cdots * H_{n-1} * H_n,$$

and

$$G_0 = P_0 = H_0, \ \ G_n = \langle P_n, t_i, 0 \leq i \leq n-1 \mid d_i^{t_i} = c_{i+1} \rangle \text{ for } n \geq 1.$$

Remark 4.1. *Note that for any $k < n$ we have*

$$G_n = \langle G_k * H_{k+1} * \ldots * H_n, t_j, k \leq j \leq n-1 \mid d_j^{t_j} = c_{j+1} \rangle.$$

One of the principal properties that we will use is that gluing together surfaces on boundary subgroups gives new surfaces. For $i \geq 0$, let $\overline{h}_i = (a_i, b_i, c_i)$ be the given basis of H_i. We are going to show that the sequence $\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2n}$ is a witness for the n-ample property in G_{2n}. The proof is divided into a sequence of lemmas.

Lemma 4.2.

(1) *For* $0 \leq i \leq n$, G_i *is a free factor of* G_n *and* G_n *is a free group of rank* $3(n+1)$.

(2) *For each* $0 \leq i \leq n$, H_i *is a free factor of* G_n.

Proof. (1) The proof is by induction on n. For $n = 0$, we already noted that $G_0 = H_0$ is a free group of rank 3. For the induction step it suffices to show that G_n is a free factor of G_{n+1} with a complement which is free of rank 3. We have

$$G_{n+1} = (G_n * \langle t_n \mid \rangle) *_{d_n^{t_n} = c_{n+1}} \langle c_{n+1}, a_{n+1}, b_{n+1} \mid \rangle,$$

and since c_{n+1} is primitive in $\langle c_{n+1}, a_{n+1}, b_{n+1} \mid \rangle$, the free group generated by t_n, a_{n+1}, b_{n+1} is a free factor in G_{n+1}. This proves the claim.

(2) In view of (1) it suffices to show by induction on i that H_i is a free factor of G_i for $0 \leq i \leq n$. For $i = 0$, there is nothing to prove. For $i+1$, we have as above

$$G_{i+1} = (G_i * \langle t_i \mid \rangle) *_{d_i^{t_i} = c_{i+1}} \langle c_{i+1}, a_{i+1}, b_{i+1} \mid \rangle,$$

and since by induction H_i is a free factor of G_i, d_i is primitive in G_i. In particular, $d_i^{t_i}$ is primitive in $(G_i * \langle t_i \mid \rangle)$ and we conclude that H_{i+1} is a free factor of G_{i+1}, as required. □

Lemma 4.3. *For* $n = 2k \geq 2$, *we can write*

$$G_n = S * \langle t_0 \mid \rangle * \ldots * \langle t_{n-1} \mid \rangle. \tag{1}$$

for a surface group S *with*

$$S = \langle c_0, d'_n, a_0, b_0, a'_1, b'_1, \ldots a'_n, b'_n \mid c_0 d'_n [a'_n, n'_1] \ldots [a'_1, b'_1][a_0, b_0] = 1 \rangle$$

where d'_n *is conjugate to* d_n.

Proof. We have

$$G_n = \langle a_0, b_0 \mid \rangle * (\langle t_0 \mid \rangle * \langle a_1, b_1 \mid \rangle * \ldots * \langle t_{n-1} \mid \rangle) * \langle d_n, a_n, b_n \mid \rangle,$$

where

$$c_n = [b_n, a_n] d_n^{-1},\ d_{n-1} = t_{n-1} c_n t_{n-1}^{-1},\ c_{n-1} = [b_{n-1}, a_{n-1}] d_{n-1}^{-1}, \ldots,\ d_0 = t_0 c_1 t_0^{-1},\ c_0 = [b_0, a_0] d_0^{-1}.$$

Replacing successively and setting $s_i^{-1} = t_0 \ldots t_i$ we obtain:

$$c_0 = [b_0, a_0][b_2, a_2]^{s_1}[b_4, a_4]^{s_3} \ldots [b_n, a_n]^{s_{n-1}} (d_n^{-1})^{s_{n-1}} [a_{n-1}, b_{n-1}]^{s_{n-2}} \ldots [a_1, b_1]^{s_0}.$$

For $0 < i \leq n$ put

$$a'_i = a_1^{s_{i-1}}, b'_i = b_1^{s_{i-1}}, \text{ and } d'_n = d_2^{s_{n-1}}.$$

Then

$$c_0 = [b_0, a_0][b'_2, a'_2][b'_4, a'_4]\dots[b'_n, a'_n](d'_n)^{-1}[a'_{n-1}, b'_{n-1}]\dots[a'_1, b'_1].$$

Finally for $i = 1, \dots k$, put

$$a''_{2i-1} = a^{d'_n}_{2i-1}, \quad \text{and} \quad b''_{2i-1} = b^{d'_n}_{2i-1}.$$

Then

$$G_n = \langle a_0, b_0 \mid\rangle * \langle t_0 \mid\rangle * \langle a''_1, b''_1 \mid\rangle * \langle t_1 \mid\rangle * \langle a'_2, b'_2 \mid\rangle * \langle t_2 \mid\rangle * \dots * (\langle t_{n-1} \mid\rangle * \langle d'_n, a'_n, b'_n \mid\rangle,$$

and

$$c_0 = [b_0, a_0][b'_2, a'_2][b'_4, a'_4]\dots[b'_n, a'_n][a''_{n-1}, b''_{n-1}]\dots[a''_1, b''_1](d'_n)^{-1}.$$

With

$$S = \langle c_0, d'_n, a_0, b_0, a''_1, b''_1, a'_2, b'_2, \dots a'_n, b'_n \mid c_0 d'_n [b''_1, a''_1][b''_3, a''_3]\dots[a'_n, b'_n]\dots[a_2, b_2][a_0, b_0] = 1\rangle$$

we have

(1) $$G_n = S * \langle t_0 \mid\rangle * \dots * \langle t_{n-1} \mid\rangle.$$

□

As S is the fundamental group of an orientable surface with genus ≥ 1 and two boundary subgroups generated by c_0 and d'_n we may apply Lemma 2.6 to S and obtain the following corollary:

Corollary 4.4. *Suppose $n = 2k \geq 2$ and $g \in G_n \setminus \{1\}$ is not conjugate to a power of c_0 or of d_n. Then there exists a malnormal cyclic splitting of G_n such that c_0 and d_n are elliptic and g is hyperbolic.*

Proof. Write $g = g_1 \cdots g_k$ in normal form with respect to the splitting appearing in Lemma 4.3. W.l.o.g, we may assume that g is cyclically reduced. If $k \geq 2$, then g is hyperbolic in the given malnormal splitting. Hence we may assume that $g \in S$ or $g \in \langle t_i \mid\rangle$ for some $i \leq n-1$.

If $g \in S$, by Lemma 2.6, there exists a malnormal cyclic splitting S in which g is hyperbolic and c_0 and d_n are elliptic. This yields a refinement of the cyclic splitting of G_n in Lemma 4.3 in which g is hyperbolic.

Next suppose that $g \in \langle t_0 \mid\rangle$. For any $1 \leq i < n$ we can write $\langle t_0, t_i \mid\rangle = \langle\langle t_i, t_0 t_i t_0^{-1} \mid \rangle, t_0 \mid (t_0 t_i t_0^{-1})^{t_0} = t_i\rangle$ which is a cyclic splitting Δ in which t_0 is hyperbolic. By replacing $\langle t_0 \mid\rangle * \langle t_i \mid\rangle$ by Δ in the splitting given in Lemma 4.3 we obtain a cyclic splitting of G_n in which g is hyperbolic. If $g \in \langle t_i \mid\rangle$ for $1 \leq i < n$ the same proof works and we are done. □

Recall that for $i \geq 0$, $\overline{h}_i = (a_i, b_i, c_i)$. Having diposed by some needed properties of G_n in the previous lemmas, we are now ready to show that the sequence $\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2n}$ satisfies conditions of Definition 1.1. Since G_i is a free factor of G_k for $0 \leq i \leq k$ which implies that G_i is an elementary subgroup of G_k, in computing the algberaic closure – as well as the imganiary algebraic closure – of tuples of G_i it is enough to work in G_i.

Lemma 4.5. *For $0 \leq i < k$ we have*

$$G_{2i} \cap \operatorname{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k}) = \operatorname{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i})$$

and

$$G_{2i}^c \cap \operatorname{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k}) = \operatorname{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}).$$

Proof. Since G_{2k} is a free factor we may work in G_{2k}. Let $g \in G_{2i} \cap \operatorname{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k})$ and suppose towards a contradiction that $g \in G_{2i} \setminus \operatorname{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i})$. Then there exist infinitely many automorphisms $f_p \in Aut(G_{2i})$ which fix $\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}$ such that $f_p(g) \neq f_q(g)$ for $p \neq q$. Recall that

$$G_{2k} = \langle G_{2i} * H_{2i+1} * \dots * H_{2k-1} * H_{2k}, t_j, 2i \leq j \leq 2k-1 \mid d_j^{t_j} = c_{j+1} \rangle.$$

We extend each f_p to $G_{2i} * H_{2i+1} * \dots * H_{2k-1} * H_{2k}$ by the identity on the factor $H_{2i+1} * \dots * H_{2k-1} * H_{2k}$. Hence each f_p has a natural extension to G_{2k} that we denote by $\hat{f}_p$ and such that $\hat{f}_p(g) \neq \hat{f}_q(g)$ for $p \neq q$. We see that each f_p fixes $\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k}$ and $\hat{f}_p(g) \neq \hat{f}_q(g)$ for $p \neq q$. Therefore $g \notin \operatorname{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k})$; which is a contradiction.

Similarly if $g^{G_{2i}} \in G_{2i}^c \setminus \operatorname{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i})$ there exist infinitely many automorphisms $f_p \in Aut(G_{2i})$ which fix $\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}$ such that $f_p(g)$ and $f_q(g)$ are not conjugate for $p \neq q$. We extend these f_p to G_{2k} as in the previous paragraph. Clearly $\hat{f}_p(g)$ and $\hat{f}_q(g)$ are not conjugate in $G_{2i} * H_{2i+1} * \dots * H_{2k-1} * H_{2k}$. Suppose towards a contradiction that $\{\hat{f}_p(g) \mid p \in \mathbb{N}\}$ is finite up to conjugacy in G_{2k}. By applying [6, Lemma 3.1] it follows that for $p \neq q$ the pair $(f_p(g), f_q(g))$ is conjugate in $G_{2i} * H_{2i+1} * \dots * H_{2k-1} * H_{2k}$ to a one of the pairs (d_j^r, c_{j+1}^r) for $2i \leq j \leq 2k-1$ and $r \in \mathbb{Z}$; this is clearly a contradiction. By Proposition 3.5 $g^{G_{2k}} \notin \operatorname{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k})$; which is a contradiction. □

Lemma 4.6. *For $0 \leq i \leq k$ we have*

(i)
$$\operatorname{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k}) = H_0 * H_2 * \dots * H_{2i} * H_{2k}$$

and

(ii)
$$\operatorname{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k}) = \operatorname{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k})^c$$

Proof. We first prove *(i)*. By rewriting the splitting

$$G_{2k} = \langle H_0 * H_1 * \dots * H_{2k-1} * H_{2k}, t_i, 0 \leq i \leq 2k-1 \mid d_i^{t_i} = c_{i+1} \rangle,$$

as

$$G_{2k} = \langle (H_0 * H_2 * \cdots * H_{2i} * H_{2k}) * K, t_i, 0 \leq i \leq 2k-1 \mid d_i^{t_i} = c_{i+1} \rangle,$$

where K is the free product of the remaining of the H_i, we get a malnormal cyclic splitting in which $H_0 * H_2 * \cdots * H_{2i} * H_{2k}$ is a vertex group. Hence by Proposition 2.7

$$\mathrm{acl}(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i}, \bar{h}_{2k}) = H_0 * H_2 * \cdots * H_{2i} * H_{2k}$$

as required.

We prove (ii). We assume inductively that

$$\mathrm{acl}^c(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i}, \bar{h}_{2k}) = \mathrm{acl}(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i}, \bar{h}_{2k})^c.$$

Let $c^{G_{2k}} \in \mathrm{acl}^c(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i}, \bar{h}_{2k})$ and consider the malnormal cyclic splitting of G_{2k} with vertex groups $K = (G_{2i} * H_{2k})$ and $L = \langle H_{2i+1} * \ldots * H_{2k-1}, t_j, 2i+1 \leq j \leq 2(k-1) \mid d_j^{t_j} = c_{j+1} \rangle$ given by

$$G_{2k} = \langle K * L, t_j, j = 2i, 2k-1 \mid d_j^{t_j} = c_{j+1} \rangle.$$

By Corollary 3.4, c is in a conjugate of K or of L. Suppose first that c is in a conjugate of L and thus w.l.o.g, we may assume that $c \in L$. Note that $L \cong G_{2m}$ with $m = k-i-1$. By Corollary 4.4 when $2m \geq 2$ and Lemma 2.6 when $2m = 0$, if c is not conjugate to an element of $\langle c_{2i+1} \rangle$ or $\langle d_{2k-1} \rangle$ then L has a malnormal cyclic splitting such that c_{2i+1} and d_{2k-1} are elliptic and c is hyperbolic. Hence we get a refinement of the previous splitting of G_{2k} where K is still a vertex group and such that c is hyperbolic, contradicting Corollary 3.4.

Therefore c is either conjugate to an element of $\langle c_{2i+1} \rangle$ or of $\langle d_{2k-1} \rangle$. But since $d_{2k-1}^{t_{2k-1}} = c_{2k}, d_{2i}^{t_{2i}} = c_{2i+1}$ we conclude that either $c^{G_{2k}} \in \mathrm{acl}(\bar{h}_{2i})^c = H_{2i}^c$ or $c^{G_{2k}} \in \mathrm{acl}(\bar{h}_{2k})^c = H_{2k}^c$ and hence $c^{G_{2k}} \in \mathrm{acl}(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i}, \bar{h}_{2k})^c$ as required.

Suppose now that c is in a conjugate of K and thus w.l.o.g, we may assume that $c \in K$. Write $c = g_1 \cdots g_m$ in normal form with respect to the free product structure of $K = G_{2i} * H_{2k}$. Since any element is conjugate to a cyclically reduced one, w.l.o.g, we may assume that c is cyclically reduced. If $m = 1$, then $c \in H_{2k}$ or $c \in G_{2i}$ and hence $c^{G_{2i}} \in \mathrm{acl}(h_{2k})^c$ or $c^{G_{2i}} \in \mathrm{acl}(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i})^c$ by Lemma 4.5 and induction hypothesis. If $m > 1$, we claim that for any $1 \leq l \leq m$ if $g_l \in G_{2i}$ then $g_l \in \mathrm{acl}(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i})$ and so $c \in \mathrm{acl}(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i}, \bar{h}_{2k})$. Suppose towards a contradiction that $g_l \in G_{2i} \setminus \mathrm{acl}(\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i})$ for some $1 \leq l \leq m$. Then proceeding as in the proof of Lemma 4.5 there exist infinitely many automorphisms $f_p \in Aut(K)$ fixing $\bar{h}_0, \bar{h}_2, \ldots, \bar{h}_{2i}, \bar{h}_{2k}$ and with $f_p(c) \neq f_q(c)$ for $p \neq q$. Each f_p has a natural extension to G_{2k} that we denote by $\hat{f}_p$ and such that $\hat{f}_p(c) \neq \hat{f}_q(c)$ for $p \neq q$.

Suppose that the set $\{f_p(c) \mid p \in \mathbb{N}\}$ is finite up to conjugacy in K. Hence there exists an infinite set $I \subseteq \mathbb{N}$ and p_0 such that $f_p(c)$ is conjugate to $f_{p_0}(c)$ for every $p \in I$. We see that $f_p(c) = f_p(g_1) \cdots f_p(g_k)$ and $(f_p(g_1), \ldots, f_p(g_k))$ is a normal form and $f_p(c) = f_p(g_1) \cdots f_p(g_k)$ is cyclically reduced. Therefore $(f_p(g_1), \ldots, f_p(g_k))$ is a cyclic permutation of $(f_{p_0}(g_1), \dots, f_{p_0}(g_k))$. Since the number of such cyclic permutations is finite, we conclude that there exists $p \neq q \in I$ such that $f_p(c) = f_q(c)$ and thus $f_p(g_l) = f_q(g_l)$, a contradiction.

Hence the set $\{f_p(c) \mid p \in \mathbb{N}\}$ is infinite up to conjugacy in K. Suppose towards a contradiction that $\{\hat{f}_p(c) \mid p \in \mathbb{N}\}$ is finite up to conjugacy in G_{2k}. Since $\{\hat{f}_p(c) \mid p \in \mathbb{N}\}$ is infinite up to conjugacy in K, we conclude that there exists $p \neq q$ such that $f_p(g)$ is conjugate to $f_q(c)$ in G_{2k} and $f_p(c)$ is not conjugate to $f_q(c)$ in $K * L$. By applying [6, Lemma 3.1] it follows that the pair $(f_p(c), f_q(c))$ is conjugate in $K * L$ to a one of the pairs $(d^r_{2i}, c^r_{2i+1}), (d^r_{2k-1}, c^r_{2k})$ for some $r \in \mathbb{Z}$; this is clearly a contradiction as $f_p(c), f_q(c) \in K$. Hence the set $\{\hat{f}_p(c) \mid p \in \mathbb{N}\}$ is infinite up to conjugacy in G_{2k} and by Proposition 3.5 $c^{G_{2k}} \notin \mathrm{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2i}, \overline{h}_{2k})$; which is a contradiction. □

Lemma 4.7. *We have*

$$\mathrm{acl}(\bar{h}_0) \cap \mathrm{acl}(\bar{h}_2) = \mathrm{acl}(\emptyset)$$

and

$$\mathrm{acl}^c(\bar{h}_0) \cap \mathrm{acl}^c(\bar{h}_2) = \mathrm{acl}^c(\emptyset).$$

Proof. As G_2 is a free factor, we work in $F = G_2$. Let $a \in \mathrm{acl}(\bar{h}_0) \cap \mathrm{acl}(\bar{h}_2)$. Since $\mathrm{acl}(\bar{h}_0) = H_0$, $\mathrm{acl}(\bar{h}_2) = H_2$ and $a \in \mathrm{acl}(\bar{h}_0, \bar{h}_2) = H_0 * H_2$ we conclude that $a = 1$. We note that $\mathrm{acl}(1) = \mathrm{acl}(\emptyset)$.

Let $a^F \in \mathrm{acl}^c(\bar{h}_0) \cap \mathrm{acl}^c(\bar{h}_2)$. Therefore $a^F \in H_0^c \cap H_2^c$. Hence there exists $\alpha \in H_0, \beta \in H_2$ such that $\alpha^F = \beta^F = a^F$. Suppose that $\alpha \neq 1$. Clearly α and β are not conjugate in $P_2 = H_0 * H_1 * H_2$. By applying [6, Lemma 3.1] it follows that the pair (α, β) is conjugate in P_2 to a one of the pairs $(d^r_0, c^r_1), (c^r_1, d^r_0), (d^r_1, c^r_2), (c^r_2, d^r_1)$ for some $r \in \mathbb{Z}$; this a contradiction. Therefore $a^F = 1^F$. By Remark 3.8 we have $\mathrm{acl}^c(\emptyset) = \mathrm{acl}^c(1) = \{1^F\}$ which concludes the proof. □

Lemma 4.8. *For $i \geq 0$ we have the following:*

$$\mathrm{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2i}) \cap \mathrm{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2(i+1)}) = \mathrm{acl}\, \overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)})$$

and

$$\mathrm{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2i}) \cap \mathrm{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2(i+1)}) = \mathrm{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}).$$

Proof. We may work in $F = G_{2(i+1)}$. The first result follows from Lemma 4.6 and normal forms: if $L = A * B * C$ then $A * B \cap A * C = A$. For the second part let $c^F \in \mathrm{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2i}) \cap \mathrm{acl}^c(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2(i+1)}), c \neq 1$.

By Lemma 4.6 (ii) there exist $\alpha \in \mathrm{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2i})$ and $\beta \in \mathrm{acl}(\overline{h}_0, \overline{h}_2, \dots, \overline{h}_{2(i-1)}, \overline{h}_{2(i+1)})$ such that $\alpha^F = \beta^F = c^F$. We have

$$G_{2(i+1)} = \langle G_{2(i-1)} * H_{2i-1} * H_{2i} * H_{2i+1} * H_{2(i+1)}, t_j, 2(i-1) \leq j \leq 2i+1 \mid d_j^{t_j} = c_{j+1} \rangle.$$

First suppose that α and β are conjugate in $L = G_{2(i-1)} * H_{2i-1} * H_{2i} * H_{2i+1} * H_{2(i+1)}$. Since $\alpha \in G_{2(i-1)} * H_{2i}$ and $\beta \in G_{2(i-1)} * H_{2(i+1)}$, it follows from properties of normal forms that α is conjugate to an element of $G_{2(i-1)}$. But since $\alpha \in \operatorname{acl}(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2(i-1)}) * H_{2i}$ it follows that α is conjugate to an element of $\operatorname{acl}(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2(i-1)})$; which is the required result.

Now suppose that α and β are conjugate in $G_{2(i+1)}$ but not in $L = G_{2(i-1)} * H_{2i-1} * H_{2i} * H_{2i+1} * H_{2(i+1)}$. Then α (and similarly β) is conjugate in L to a power of one the elements d_{2i-1}, c_{2i}, $d_{2(i-1)}$, c_{2i-1}, d_{2i}, c_{2i+1}, d_{2i+1}, $c_{2(i+1)}$.

Since $\alpha \in G_{2(i-1)} * H_{2i}$, we conclude that α is conjugate to a power of c_{2i} or d_{2i} or $d_{2(i-1)}$. Similarly, since $\beta \in G_{2(i-1)} * H_{2(i+1)}$, we conclude that β is conjugate to a power of $c_{2(i+1)}$ or $d_{2(i-1)}$.

If α is conjugate to a power of c_{2i} then β is conjugate (in L) to a power of d_{2i-1}; which is a contradiction. Similarly, if α is conjugate to a power of d_{2i} then β is conjugate (in L) to a power of c_{2i+1}; which is also a contradiction. Hence α is conjugate to $d_{2(i-1)}$ and thus $c^F \in \operatorname{acl}^c(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2(i-1)})$ as required. □

By Corollary 3.11 we thus have proved the following:

Corollary 4.9. *We have*

$$\operatorname{acl}^{\mathrm{eq}}(\overline{h}_0) \cap \operatorname{acl}^{\mathrm{eq}}(\overline{h}_2) = \operatorname{acl}^{\mathrm{eq}}(\emptyset)$$

and for $i \geq 1$

$$\operatorname{acl}^{\mathrm{eq}}(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2(i-1)}, \overline{h}_{2i}) \cap \operatorname{acl}^{\mathrm{eq}}(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2(i-1)}, \overline{h}_{2(i+1)}) = \operatorname{acl}^{\mathrm{eq}}(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2(i-1)}). \quad \square$$

To finish the proof of the fact that our sequence is a witness for the n-ample property we prove the two next lemmas which yield the required properties of independence.

Lemma 4.10. *For $i = 1, \ldots, n-1$, there exists a free decomposition $G_n = K * H_{2i} * L$ such that $\overline{h}_0, \ldots, \overline{h}_{2(i-1)} \in K * H_{2i}$ and $\overline{h}_{2(i+1)} \in H_{2i} * L$.*

Proof. Let $1 \leq i \leq n-1$ and put

$$L_i = \langle H_{2i+1} * \cdots * H_n, t_j, 2i+1 \leq j \leq n-1 \mid d_j^{t_j} = c_{j+1} \rangle.$$

Then

$$G_n = \langle G_{2i-1} * H_{2i} * L_i, t_{2i-1}, t_{2i} \mid d_{2i-1}^{t_{2i-1}} = c_{2i}, d_{2i}^{t_{2i}} = c_{2i+1} \rangle$$

Since H_{2i-1} is a free factor of G_{2i-1}, d_{i-1} is primitive in G_{2i-1} and thus we can write $G_{2i-1} = K_0 * \langle d_{2i-1} \rangle$ for some free group K_0. Similarly, c_{2i+1} is primitive in L_i and thus we can write $L_i = \langle c_{2i+1} \rangle * L_0$ for some free group L_0.

Therefore

$$G_n = K_0 * \langle t_{2i-1} \mid \rangle * H_{2i} * \langle t_{2i} \mid \rangle * L_0,$$

and by setting $K = K_0 * \langle t_{2i-1} \mid \rangle$ and $L = \langle t_{2i} \mid \rangle * L_0$ we get $G_n = K * H_{2i} * L$ with $\overline{h}_0, \ldots, \overline{h}_{2(i-1)} \in K * H_{2i}$ and $\overline{h}_{2(i+1)} \in H_{2i} * L$ as required. □

Lemma 4.11. *There is no free decomposition $G_n = K * L$ such that $\overline{h}_0 \in K$ and $\overline{h}_n \in L$.*

Proof. Suppose towards a contradiction that such a free decomposition exists. Since H_i is a free factor, it follows that H_0 is a free factor of K and H_n is a free factor of L. Since c_0 is primitive in H_0 it is primitive in K and since d_n is primitive in H_n it is primitive in K. We conclude that $\{c_0, d_n\}$ is part of a basis of G_n. Therefore in the abelianisation $G_n^{ab} = G_n/[G_n, G_n]$, we get that $\{c_0, d_n\}$ is part of a basis.

However in G_n^{ab} we have $c_i d_i = 1$ and $d_i = c_{i+1}$ and thus $c_0 = d_n^{\pm 1}$ depending on whether n is odd or even, a contradiction. $\square$

By Proposition 2.16 we have thus proved the following:

Corollary 4.12. *We have*

$$\overline{h}_{2n} \not\perp \overline{h}_0$$

and for $i \geq 1$

$$\overline{h}_0 \dots \overline{h}_{2(i-1)} \perp_{\overline{h}_{2i}} \overline{h}_{2(i+1)}. \quad \square$$

Putting Corollary 4.9 and Corollary 4.12 together we therefore have proved the following:

Theorem 4.13. *The $u_i = \overline{h}_{2i} \in G_{2n}, i = 0, \dots n$ witness the fact that T_{fg} is n-ample, i.e. we have the following:*

(*i*) $u_n \not\perp u_0$;
(*ii*) $u_0 \dots u_{i-1} \perp_{u_i} u_{i+1}$ for $1 \leq i < n$;
(*iii*) $\mathrm{acl}^{\mathrm{eq}}(u_0) \cap \mathrm{acl}^{\mathrm{eq}}(u_1) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.
(*iv*) $\mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \dots, u_{i-1}, u_i) \cap \mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \dots, u_{i-1}, u_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \dots, u_{i-1})$. $\square$

Remark 4.14. *In fact, since*

$$G_{n+2} = \langle G_n * H_{n+1} * H_{n+2}, t_j, j = n, n+1 \mid d_j^{t_j} = c_{j+1} \rangle$$

any free group F of infinite rank contains a sequence $(u_n : n < \omega)$ of tuples such that

(*i*) $u_i \not\perp u_j$ *for $i \neq j$;*
(*ii*) $u_0 \dots u_{i-1} \perp_{u_i} u_{i+1}$
(*iii*) $\mathrm{acl}^{\mathrm{eq}}(u_0) \cap \mathrm{acl}^{\mathrm{eq}}(u_1) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.
(*iv*) $\mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \dots, u_{i-1}, u_i) \cap \mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \dots, u_{i-1}, u_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \dots, u_{i-1})$.

In particular, F_ω contains an explicit sequence $(u_n : n < \omega)$ such that for every n the finite sequence $u_0, u_1, \cdots, u_n$ is a witness of the n-ampleness.

5 Proof of the main theorem

We now move back to working in a torsion-free hyperbolic group. Let $H = \Gamma * G_{2n}$ where Γ is torsion-free hyperbolic and G_{2n} is as before. In order to finish the proof of the main theorem we just need the following observation:

Lemma 5.1. *With $\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2n}$ defined as in Section 4, we have for $0 \leq i \leq k$*

$$\mathrm{acl}_{G_{2n}}(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2i}, \overline{h}_{2k}) = \mathrm{acl}_H(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2i}, \overline{h}_{2k}).$$

Proof. By Lemma 4.5 we have for $0 \leq i \leq k$

$$\mathrm{acl}_{G_{2n}}(\overline{h}_0, \overline{h}_2, \overline{h}_{2i}\overline{h}_{2k}) = H_0 * H_2 * \ldots * H_{2i} * H_{2k}.$$

By Theorem 2.8 we know that $H_0 * H_2 * \ldots * H_{2i} * H_{2k}$ is the vertex group containing $\{\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2i}, \overline{h}_{2k}\}$ in the generalized malnormal cyclic JSJ-decomposition Λ of G_{2n} relative to $\{\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2i}, \overline{h}_{2k}\}$. Using Λ we obtain a malnormal cyclic splitting of $H = \Gamma * G_{2n}$. By Proposition 2.7 we now see that

$$\mathrm{acl}_H(\overline{h}_0, \overline{h}_2, \ldots, \overline{h}_{2i}, \overline{h}_{2k}) = H_0 * H_2 * \ldots * H_{2i} * H_{2k}.$$

□

Corollary 3.14 combined with Lemma 4.6 and Theorem 4.13 now yield our main theorem:

Theorem 5.2. *Let Γ be a torsion-free hyperbolic group and $T = Th(\Gamma)$. Consider the model $H = EC(\Gamma) * G_{2n}$ of T. Then $u_i = \overline{h}_{2i} \in G_{2n}, i = 0, \ldots n$ witness the fact that T is n-ample, i.e. we have the following:*

(*i*) $u_n \not\perp u_0$;
(*ii*) $u_0 \ldots u_{i-1} \perp_{u_i} u_{i+1}$ for $1 \leq i < n$;
(*iii*) $\mathrm{acl}^{\mathrm{eq}}(u_0) \cap \mathrm{acl}^{\mathrm{eq}}(u_1) = \mathrm{acl}^{\mathrm{eq}}(\emptyset)$.
(*iv*) $\mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \ldots, u_{i-1}, u_i) \cap \mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \ldots, u_{i-1}, u_{i+1}) = \mathrm{acl}^{\mathrm{eq}}(u_0, u_1, \ldots, u_{i-1})$.

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