Nonstandard methods in algebraic geometry

Christian Serpé
University of Münster, Germany

1. Juni 2008

- joint work with Lars Brünjes from Regensburg, Germany
- slides of this talk and our articles will soon be available on my homepage wwwmath.uni-muenster.de/u/serpe/

Motivation

Many problems in algebraic geometry depend on the characteristic of the base field.
The main reason for that is that

- in the characteristic zero case one can use transcendental methods,
- and in characteristic $p > 0$ case one has the Frobenius morphism.

Motivation

A link between the apparently so different worlds might be provided by the ultra product

$$\prod_{p\in M,\mathcal{U}} \mathbb{F}_p$$

of the finite fields $\mathbb{F}_p$ where M is an infinite set of primes and $\mathcal{U}$ is an ultra filter.

- $char(\prod_{p\in M,\mathcal{U}} \mathbb{F}_p) = 0$
- in some sense $\prod_{p\in M,\mathcal{U}} \mathbb{F}_p$ behaves like a finite field.

Overview

1. Enlargement of varieties
2. Enlargements of schemes
3. Étale cohomology and cycles

Enlargements

Let $\hat{M} = \cup_{i=0}^{\infty} M_i$ be a superstructure, and $* : \hat{M} \to {}^*\widehat{M}$ be an enlargement of superstructures.
We assume that $\hat{M}$ is large enough for all our purpose.
Now the ultra product $\prod_{p \in M, \mathcal{U}} \mathbb{F}_p$ is replaced by the *finite field

$$^*\mathbb{F}_P := {}^*\mathbb{Z}/P$$

where $P \in {}^*\mathbb{P} - \mathbb{P}$ is an infinite prime.
We have

- $char^{ext}({}^*\mathbb{F}_P) = 0$
- $char^{int}({}^*\mathbb{F}_P) = P$

Affine varieties

Consider:

- K algebraically closed field; $m, n \in \mathbb{N}$
- $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ polynomials
- $I := (f_1, \ldots, f_m) \subset K[x_1, \ldots, x_n]$

$$V_I(K) := \{t = (t_1, \ldots, t_n) \in K^n | f_1(t) = \cdots = f_m(t) = 0\}$$

Such subsets are called **algebraic subsets** of K^n and if the ideal I is a prime ideal the subset is called **affine variety.**
A **morphism** of two algebraic subsets/affine varieties is a map which is given by polynomials. So we have the category **AffVar/K.**

*Affine *varieties

Consider:

- K internal *algebraically closed field; $n, m \in {}^*\mathbb{N}$
- $f_1, \ldots, f_m \in K^*[x_1, \ldots, x_n]$ *finitely many *polynomials
- $I := {}^*(f_1, \ldots, f_m) \subset K^*[x_1, \ldots, x_n]$

$$V_I(K) := \{t = (t_1, \ldots, t_n) \in K^n | f_1(t) = \cdots = f_m(t) = 0\}$$

Such subsets are called ***algebraic subsets** of K^n and if the ideal I is a prime ideal the subset is called ***affine *variety.**
A **morphism** of two *algebraic subsets/*affine *varieties is a map which is given by *polynomials. So we have the category ***AffVar/K**.

The functor N

If K is an internal *algebraically closed field, it is also just an algebraically closed field. Therefore we can consider:

$$\mathit{AffVar}/K$$

An important fact is that we can construct a functor

$$N : \mathit{AffVar}/K \to {}^*\mathit{AffVar}/K$$

Doing this for $\overline{{}^*\mathbb{F}_P} := \overline{{}^*\mathbb{Z}/P}$ for an infinite prime $P \in {}^*\mathbb{P} - \mathbb{P}$ we get

$$N : \mathit{AffVar}/\overline{{}^*\mathbb{F}_P} \to {}^*\mathit{AffVar}/\overline{{}^*\mathbb{F}_P}$$

This gives a link between varieties over fields of characteristic zero and varieties over fields of charactereistic $p > 0$.

Enlargements of schemes

- affine varieties $\rightsquigarrow$ schemes

Again for an internal field K we want to have a functor

$$N : Sch^{fp}/K \to {}^{*}Sch^{fp}/K.$$

What is ${}^{*}Sch^{fp}/K$?

Enlargement of categories

- $\mathcal{C}$ category $\rightsquigarrow$ (internal) category ${}^*\mathcal{C}$
- E a property morphism of $\mathcal{C}$ can have $\rightsquigarrow$ ${}^*E$ a property morphism of ${}^*\mathcal{C}$ can have
- $\{\mathcal{C}_s\}_{s\in S}$ family of categories indexed by a set S $\rightsquigarrow$ $\{{}^*\mathcal{C}_s\}_{s\in {}^*S}$ family of categories indexed by the set ${}^*S$
- here: $\{Sch^{fp}/k\}_{k\in S} \rightsquigarrow \{{}^*Sch^{fp}/K\}_{K\in {}^*S}$

Construction of N

$$N : Sch^{fp}/{}^*\mathbb{F}_P \to {}^*Sch^{fp}/{}^*\mathbb{F}_P$$

Construction of N:
$X \in Sch^{fp}/{}^*\mathbb{F}_P$

- find a subring $A_0 \subset {}^*\mathbb{F}_P$ of finite type over $\mathbb{Z}$ and a scheme $X_0 \in Sch^{fp}/A_0$ such that $X = X_0 \otimes_{A_0} {}^*\mathbb{F}_P$
- $N(X) := {}^*X_0 {}^*\otimes_{{}^*A_0} {}^*\mathbb{F}_P$

Properties of N

Proposition (B.-S.)

- $f : X \to Y$ smooth $\Rightarrow N(f) : N(X) \to N(Y)$ *smooth
- $f : X \to Y$ étale $\Rightarrow N(f) : N(X) \to N(Y)$ *étale

For schemes X, Y over an internal field we have:

- X is a variety if and only if $N(X)$ is a *variety
  (uses a result of van den Dries/Schmidt about the map $K[x_1, \dots, x_n] \to K^*[x_1, \dots, x_n]$)
- $f : X \to Y$ is birational if and only if $N(f) : N(X) \to N(Y)$ is *birational

char 0 $\rightsquigarrow$ char p

Let Φ be a statement about schemes.
Then assume that

- Φ is true in characteristic 0.
- $\Phi(X)$ is true $\Rightarrow {}^*\Phi(N(X))$ is true

Consider an subset S of schemes over fields such that ${}^*S$ is contained in the essential image of the functor

$$N : Sch^{fp}/{}^*\mathbb{F}_P \to {}^*Sch^{fp}/{}^*\mathbb{F}_P$$

.
Then it follows:
There is a cofinite set of primes $\mathbb{P}' \subset \mathbb{P}$ such that for all schemes X over a field of characteristic $p \in \mathbb{P}'$ with $X \in S$ the statement Φ holds.

Examples

Theorem (Eklof 69)

For any pair (n, d) of natural numbers, there exists a bound $C \in \mathbb{N}$ such that for any field of characteristic $p > C$ and any closed subvariety X of $\mathbb{P}^n_k$ of degree d, there exists a resolution of singularities of X.

Theorem (B.-S.)

A similar results holds for weak factorization

Étale cohomology and algebraic cycles

Algebraic cylces and **étale cohomology** are important invariants for schemes.

$X \in Sch^{fp}/k$ a scheme over a field K and $i \in \mathbb{N}$

- $Z^i(X)$ groups of codimension i cycles
- $H^i_{et}(X, \mathbb{Z}/m)$ étale cohomology
- $H^i_{et}(X, \mathbb{Z}_l)$ l-adic cohomology

And there is a **cycle class map**

$$cl : Z^i(X) \to H^{2i}_{et}(X, \mathbb{Z}/m)$$

N for cycles and étale cohomology

Proposition (B.-S.)

It is possible to construct a canonical morphisms

$$N : H^i_{et}(X, \mathbb{Z}/m) \to {}^*H^i_{et}(N(X), {}^*\mathbb{Z}/m)$$

and

$$N : Z^i(X) \to {}^*Z^i(N(X))$$

which are compatable with *cl* and ${}^*cl$.

N for cycles and étale cohomology

Proposition (B.-S.)

Let X be a proper scheme over an internal separably closed field. Then the canonical morphism

$$N : H^i_{et}(X, \mathbb{Z}/m) \to {}^*H^i_{et}(N(X), {}^*\mathbb{Z}/m)$$

is an isomorphism.

For cycles the map N is far from being surjective.

Lifting divisors to characteristic zero

Theorem (B.-S.)

Let X be a smooth and proper variety over $\mathbb{Q}$, and let $\eta \in H^2_{et}(X_{\overline{Q}}, \mathbb{Z}_l)$ be a cohomology class. If there are infinitely many primes $p \in \mathbb{P}$ such that η lies in the image of

$$Z^1(X_{\overline{\mathbb{F}_p}}) \to H^2_{et}(X_{\overline{\mathbb{F}_p}}, \mathbb{Z}_l) \simeq H^2_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l)$$

then η lies in the image of

$$Z^1(X_{\overline{\mathbb{Q}}}) \to H^2_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_l).$$