Cobordism and Chow groups

Christian Serpé

Evidently the theory $\Omega_*^{ad} := \Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z}$ is the universal additive Borel-Moore weak homology theory on Sch/k. The main goal of this talk is to show that this theory is isomorphic to the Chow theory. In the second part a filtration on cobordism is given. We closely follow section 14 of [1]. For the hole talk we assume that k is a field of characteristic zero.

By the universality of Ω_* and the fact that the group law of CH_* is additive we get a morphism

$$\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z} \to CH_*.$$

Lemma 1. *Let $X \in Sch/k$, $Y \in Sm/k$ irreducible and $f : Y \to X$ be a projective morphism. Furthermore, let $\widetilde{f(Y)} \to f(Y)$ be a resolution of singularities (i.e. the morphism $\widetilde{f(Y)} \to f(Y)$ is birational, projective and $\widetilde{f(Y)} \in Sm/k$). Then we have*

$$[Y \xrightarrow{f} X]_{\Omega_*^{ad}} = \begin{cases} deg(Y/f(Y))[\widetilde{f(Y)} \to X]_{\Omega_*^{ad}} & \text{if } dimY = dimf(Y) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply the generalised degree formula to $[Y \to f(Y)]$ and use the fact that $\Omega_*^{ad} \simeq \mathbb{Z}$. □

We denote by $Z_*(X)$ the free abelian group on the set of closed integral subschemes for a scheme $X \in Sch/k$, graded by dimension. We define a morphism

$$\phi : Z_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z}$$

by $\phi(Z) := [\tilde{Z} \to Z \to X]_{\Omega^{ad}}$, where $\tilde{Z} \to Z$ is a resolution of singularities of Z. From Lemma 1 it follows that this is independent of the chosen resolution. We have the following observation for the morphism ϕ:

- The composition $\phi : Z_*(X) \to \Omega_*(X) \otimes_{\mathbb{L}_*} \mathbb{Z} \to CH_*(X)$ is the canonical morphism.
- ϕ is surjective (by Lemma 1)
- ϕ is compatible with projective push forward (by Lemma 1)

By using resolution of singularities and the fact that Ω_*^{ad} has by definition the additive group law one can deduce

Proposition 1. *The morphism ϕ factors through rational equivalence.*

So all together we have proven

Theorem 1. *The canonical morphism*

$$\Omega_* \otimes_{\mathbb{L}_*} \mathbb{Z} \to CH_*$$

is an isomorphism of oriented Borel-Moore weak homology theories on Sch/k and ϕ induces the inverse.

From this one can deduce the following two corollaries.

Corollary 1. *For $X \in Sch/k$ the canonical morphism*

$$\Omega_0(X) \to CH_0(X)$$

is an isomorphism.

Corollary 2. *Let X be a smooth scheme over k.*

(1) *If $dim(X) = 1$ we have $\Omega_1(X) \simeq K_0(X)$.*

(2) *If $dim(X) = 2$ we have $\Omega_1(X) \simeq \widetilde{K_0(X)} := ker(K_0(X) \xrightarrow{rk} H^0_{Zar}(X, \mathbb{Z}))$.*

To get a filtration on $\Omega_*(X)$ for a $X \in Sch/k$ we define $F^{(n)}\Omega_*(X)$ as the subgroup of $\Omega_*(X)$ which is generated by classes $[Y \xrightarrow{f} X]$ with Y smooth, irreducible, and $dim(Y) - dim(f(Y)) \geq n$. We have the following observations:

- $F^{(n)}\Omega_*(X)$ is a $\Omega^*(k)$ submodule of $\Omega_*(X)$
- $F^{(n)}\Omega_*(k) \simeq \Omega_{*\geq n}(k)$
- $F^{(1)}\Omega_*(X) \simeq ker(\Omega_*(X) \to CH_*(X)$ (by theorem 1)

From the generalised degree formula one can deduce

Theorem 2. *Let $X \in Sch/k$ and $n \geq 0$. Then we have*

$$F^{(n)}\Omega_*(X) \simeq \mathbb{L}_{*\geq n}\Omega_*(X).$$

Now for the associated graded it follows

Corollary 3. *For $X \in Sch/k$ there is a surjection*

$$\mathbb{L}_* \otimes_{\mathbb{Z}} CH_* \to Gr^*\Omega_*(X)$$

of bigraded abelian groups.

References

[1] Marc Levine and Fabien Morel, *Algebraic Cobordism I*, Preprint (2002).