A spectral sequence for equivariant K-theory

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(joint work with Marc Levine)

If X is a smooth scheme of finite type over a field, then there is a strongly convergent spectral sequence

$$E_2^{p-q} = H^{p-q}_{\mathcal{M}}(X, \mathbb{Z}(-q)) \Rightarrow K_{-p-q}(X)$$

from the motivic cohomology to the algebraic K-theory of X (compare [1], [2] [3] and [5]). Here the motivic cohomology groups can be either defined by Bloch's higher Chow groups or by the motivic cohomology groups of Voevodsky. The aim of this talk is to give an analogon of this in an equivariant situation.

For that let G be a finite group and X be a smooth G–scheme of finite type over a field k with $\frac{1}{\sharp G} \in k$. We denote by $K(G, X)$ the equivariant algebraic K–theory spectrum of X. For a closed G–invariant subset $W \subset X \times \Delta^r$ we define

$$K^W(G, X \times \Delta^r) := \mathrm{hofib}(K(G, X \times \Delta^r) \to K(G, (X \times \Delta^r) \setminus W).$$

Let $K^{(p)}(G, X, r)$ be the homotopy colimit of all $K^W(G, X \times \Delta^r)$ where $W \subset X \times \Delta^r$ has at least codimension p and has good intersection with all faces of Δ^r. With that we get an equivariant homotopy coniveau tower

$$\cdots \to K^{(p+1)}(G, X, -) \to K^{(p)}(G, X, -) \to \cdots \to K^{(0)}(G, X, -) \simeq K(G, X)$$

of simplicial spectra $K^{(p)}(G, X, \_)$. We denote by $K^{(p/p+1)}(G, X, -)$ the homotopy cofiber of the map $K^{(p+1)}(G, X, -) \to K^{(p)}(G, X, -)$. So we get:

Proposition 1. *There is a strongly convergent spectral sequence*

$$E_1^{p,q} = \pi_{-p-q}(K^{(p/p+1)}(G, X, -)) \Rightarrow K_{-p-q}(G, X).$$

To get a better description of the starting term of the spectral sequence we denote by

$$X_G^{(p)}(r) := \{[x] \in (X \times \Delta^r_k)^{(p)}/G| \begin{array}{c} \text{for all faces } F \subset \Delta^r_k \\ \mathrm{codim}(\overline{G \cdot x} \cap X \times F, X \times F) \geq q \end{array} \}$$

and define

$$z^p(G, X, r) := \bigoplus_{[x] \in X_G^{(p)}(r)} K_0(G_d(x), Spec(\kappa(x)))$$

where $G_d(x) := \{x \in G | gx = x\}$ is the decomposition group of x. This is an simplicial abelian group and an analogon of Bloch's higher cycle complex in the equivariant situation. Therefore we define the equivariant higher Chow groups (of Bredon type):

$$CH^p(G, X, r) := \pi_r(z^p(G, X, -)).$$

We get a natural cycle class map

$$cl : K^{(p/p+1)}(G, X, -) \to z^p(G, X, -),$$

where we consider both sides as simplicial spectra.

Using localization techniques of [4] and results from [5] one can show that in convenient situations the morphism cl is a weak equivalence. So all together we have the following main result.

Theorem 1. *Let G be a finite group and X be a smooth G–scheme of finite type over a field k with $\frac{1}{\sharp G} \in k$. Then there is a strongly convergent spectral sequence*

$$E_1^{p,q} = CH^p(G, X, -p-q) \Rightarrow K_{-p-q}(G, X)$$

[1]: **S. Bloch and S. Lichtenbaum**: *A spectral sequence for motivic cohomology* preprint (1995)
[2]: **E. Friedlander and A. Suslin**: *The spectral sequence relating algebraic K-theory to motivic cohomology* Ann. Sci. cole Norm. Sup. (4), 35 no.6 , 773–875 (2002).
[3]: **M. Levine**: *The homotopy coniveau filtration* preprint (2003).
[4]: **M. Levine**: *Techniques of localisation in the theory of algebraic cycles*, Journal of Algebraic Geometry 10, 299-363 (2002).
[5]: **A. Suslin**: *On the Grayson spectral sequence* preprint (2003).