From Wadge comparability to determinacy

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Abstract

We describe a scenario for proving that if any two projective sets are Wadge comparable, then Projective Determinacy holds.

0 Introduction

For A, $B \in {}^\omega\omega$ we write $A \leq_{\mathsf{W}} B$ iff A is Wadge reducible to B, i.e., if there is some continuous $f: {}^\omega\omega \to {}^\omega\omega$ such that for all $x \in {}^\omega\omega$, $x \in A$ iff $f(x) \in B$. We say that A, $B \in {}^\omega\omega$ are *Wadge comparable* iff $A \leq_{\mathsf{W}} B$ or else $B \leq_{\mathsf{W}} {}^\omega\omega \setminus A$.

In this paper, we shall be interested in *projective* sets of reals. Wadge's classical theorem (cf. [9]) yields that if Projective Determinacy holds, then any two projective sets are Wadge comparable. On the other hand, Harrington (cf. [1, Theorem 4.3]) showed that if any two analytic sets are Wadge comparable, then every analytic set is determined. Later, Hjorth (cf. [2, Theorem 3.15]) proved that if any two $\mathbf{\Pi}^1_2$ sets are Wadge comparable, then every $\mathbf{\Pi}^1_2$ set is determined.

It is a well-known open problem to show that following.

Conjecture. (Folklore) If any two projective sets are Wadge comparable, then Projective Determinacy holds.

In what follows we shall describe a scenario for verifying this conjecture. A key building block of this scenario is a formulation of two "correctness hypotheses for the core model K" one of which generalizes Steel's Σ^1_3 correctness result for K (cf. [8]), which in turn was used by Hjorth in his proof of [2, Theorem 3.15]. We'll present a proof of Hjorth's theorem which, in our opinion and in contrast to his own proof from [2], brings in the right perspective to hopefully enable us finding a generalization of his argument.

1 Two correctness conjectures

Definition 1.1 *Let $n < \omega$. We let*

$$\mathcal{I}_{n+1} = \{x \in {}^\omega\omega : x \textit{ codes a } \Pi^1_{n+1}\textit{-iterable premouse } \mathcal{M} \textit{ with } \mathcal{M} \models \mathsf{ZFC}\}.$$

Obviously, $\mathcal{I}_{n+1}$ is Π^1_{n+1}. (Cf. [7] on the concept of Π^1_{n+1} iterability.) We shall also be interested in the following theories in the language of premice over a real, which express "I'm $M_n^\#(x)$" and "I'm $M_n^\dagger(x)$," respectively.

Definition 1.2 *Let $n < \omega$.*

$$\Psi_n \equiv \text{ I am an } \dot{x}\text{-premouse which is not } n\text{-small, but whose proper initial segments are all } n\text{-small.}$$

$\Phi_n \equiv$ *I am an active $\dot{x}$-premouse with n Woodin cardinals and a measurable cardinal above, but there is no active proper initial segment with a Woodin cardinal and a measurable cardinal above.*

The following statement, $\clubsuit_n$, is the first one of our correctness hypotheses.

Definition 1.3 *Let $n < \omega$. By $\clubsuit_n$ we denote the following statement.*

Suppose that for every $x \in {}^\omega\omega$, $M_n^\#(x)$ exists, but also suppose that $M_n^\dagger$ does not exist (or just assume that $M_n^\P$ does not exist). Let $z \in {}^\omega\omega$, and let $(\mathcal{M}_k, x_k : k < \omega)$ be a sequence with the following properties.

1. *$x_0 = z$, and for all $k < \omega$, x_{k+1} is a real which codes the transitive structure $\mathcal{M}_k$,*
2. *$\mathcal{M}_k$ is an x_k-premouse which is a model of Ψ_n,*
3. *$\mathcal{M}_{k+1} \models$ "$\mathcal{M}_k$ is iterable, i.e., $\mathcal{M}_k = (x_k)^\#$," and*
4. *$K^{\mathcal{M}_k} \in \mathcal{I}_{n+2}$.*

Then for each $k < \omega$, $\mathcal{M}_k$ is n-Neeman iterable.

In the situation of Definition 1.3, $K^{\mathcal{M}_k}$ is the core model of $\mathcal{M}_k$ of height κ, where κ is the critical point of the top measure of $\mathcal{M}_k$, in the sense of [6]. Notice that if $n > 0$, then $K^{\mathcal{M}_k}$ will have Woodin cardinals. The concept of being "n-Neeman iterable" is basically due to I. Neeman, cf. [3].

Definition 1.4 *Let $\mathcal{M}$ be a premouse, and let $\delta \in \mathcal{M}$. Then $\mathcal{M}$ is* 0-Neeman iterable above *δ iff there is some measure $E_\alpha^{\mathcal{M}}$ with critical point above δ such that $\mathcal{M}$ is linearily iterable with respect to $E_\alpha^{\mathcal{M}}$ and its images. For $n < \omega$, $\mathcal{M}$ is* $(n+1)$-Neeman iterable above *δ iff for every iteration tree $\mathcal{T}$ on $\mathcal{M}$ of length ω (sic!) which is above δ there is some cofinal branch b through $\mathcal{T}$ such that $\mathcal{M}_b^{\mathcal{T}}$ is transitive and n-Neeman iterable above $\delta(\mathcal{T})$.*

The following is basically due to Neeman, cf. [3].

Theorem 1.5 *Let $n < \omega$, and let $\mathcal{M}$ be an n-Neeman iterable premouse. Suppose that $\mathcal{M}$ is a model of Ψ_n if n is even, and suppose that $\mathcal{M}$ is a model of Φ_n if n is odd. Then $\mathcal{M} \prec_{\Sigma^1_{n+2}} V$.*

PROOF. See [6, Lemma 2.4]. □

We would like to have that the iterability of $K^{\mathcal{M}_k}$ witnesses the iterability of $\mathcal{M}_k$, but this would probably be too much to ask for. We showed in [4] that $\clubsuit_n$ is true for $n = 0$.

Theorem 1.6 ([4]) $\clubsuit_0$ *holds true.*

PROOF. Let us fix $(\mathcal{M}_k, x_k : k < \omega)$, a sequence as in $\clubsuit_0$. For an ordinal α, we write $(M_k^i, \pi_k^{ij} : i \leq j \leq \alpha)$ for the putative iteration of $\mathcal{M}_k$ of length α (if it exists); and if so, then we write κ_k^i for the critical point of the top measure of M_k^i (or for $\sup\{\kappa_k^j : j < i\}$ if $i = \alpha$ is a limit ordinal and M_k^α is not well-founded). We say that α is a *uniform indiscernible* iff for every $k < \omega$, $\alpha = \kappa_k^i$ for some i (in fact $i = \alpha$). By $\binom{\alpha}{k}$, we denote the statement that M_k^α, the α^{th} iterate of $\mathcal{M}_k$ (via its unique measure and its images) is well-founded.

We are now going to prove, by induction on α, simultaneously for all $k < \omega$, that $\binom{\alpha}{k}$ holds true for every $\alpha < \omega_1$.

Let us first suppose that α is not a uniform indiscernible. Let $k < \omega$. Let us write $\kappa = \kappa_{k+1}^0$ for the moment. Let $(\mathcal{T}, \mathcal{U})$ denote the coiteration of $K^{J_\kappa[x_k]}$ with $K^{\mathcal{M}_{k+1}}$. As $\mathcal{M}_{k+1} \models$ "$\mathcal{M}_k = (x_k)^\#$," there must be some *proper* initial segment of $K^{\mathcal{M}_{k+1}}$ which iterates past $K^{J_\kappa[x_k]}$. (It is not hard to verify that some such $\mathcal{P}$ must exist, as otherwise there is an inner model with a strong cardinal. One can even show that otherwise $0^\P$ would exist, cf. [4].) We may therefore construe $\mathcal{U}$ as an iteration tree (of length κ) on $\mathcal{P}$.

Now let, for $\beta \leq \alpha$, $\mathcal{T}_\beta = \pi_{k+1}^{0\beta}(\mathcal{T})$ and $\mathcal{U}_\beta = \pi_{k+1}^{0\beta}(\mathcal{U})$. We then have that for each $\beta < \alpha$, $(\mathcal{T}_\beta, \mathcal{U}_\beta)$ is the coiteration of $K^{J_{\kappa_{k+1}^\beta}[x_k]}$ with $\mathcal{P}$. (Notice that $\alpha = \kappa_{k+1}^\alpha$.) Let us write $(\mathcal{T}^*, \mathcal{U}^*) = (\mathcal{T}_\alpha, \mathcal{U}_\alpha)$. The key point is that, because $K^{\mathcal{M}_{k+1}}$ (and hence $\mathcal{P}$) is in $\mathcal{I}_2$, $\mathcal{P}$ is iterable, so that $\mathcal{M}_\alpha^{\mathcal{U}^*}$, the last model of $\mathcal{U}^*$, is well-founded.

Let us now fix $\gamma < \beta < \alpha$, and let us write $\kappa = \kappa_{k+1}^\gamma$ and $\lambda = \kappa_{k+1}^\beta$ for the moment. Let us also assume that γ and β are "typical," i.e., that $\kappa = \kappa_k^\gamma$ and $\lambda = \kappa_k^\beta$ as well.

Claim 1. $\pi_k^{\gamma\beta} \restriction (\mathcal{P}(\kappa) \cap J_\alpha[x_k]) = \pi_{k+1}^{\gamma\beta} \restriction (\mathcal{P}(\kappa) \cap J_\alpha[x_k])$.

PROOF. Let $X \in \mathcal{P}(\kappa) \cap J_\alpha[x_k]$, and let $X = \pi_k^{i\gamma}(\bar{X})$, where $i < \gamma$. Then $\pi_k^{\gamma\beta}(X) = \pi_k^{i\beta}(\bar{X}) = \pi_{k+1}^{\gamma\beta}(\pi_k^{i\gamma})(\bar{X}) = \pi_{k+1}^{\gamma\beta}(\pi_k^{i\gamma}(\bar{X})) = \pi_{k+1}^{\gamma\beta}(X)$.

Claim 2. $\pi_{\kappa\lambda}^{\mathcal{U}^*} \restriction (\mathcal{P}(\kappa) \cap \mathcal{M}_\kappa^{\mathcal{U}^*}) = \pi_{k+1}^{\gamma\beta} \restriction (\mathcal{P}(\kappa) \cap \mathcal{M}_\kappa^{\mathcal{U}^*})$.

PROOF. Let $X \in \mathcal{P}(\kappa) \cap \mathcal{M}_\kappa^{\mathcal{U}^*}$, and let $X = \pi_{i\kappa}^{\mathcal{U}^*}(\bar{X})$, where $i < \kappa$. Then $\pi_{\kappa\lambda}^{\mathcal{U}_\beta}(X) = \pi_{i\lambda}^{\mathcal{U}_\beta}(\bar{X}) = \pi_{k+1}^{\gamma\beta}(\pi_{i\kappa}^{\mathcal{U}_\gamma})(\bar{X}) = \pi_{k+1}^{\gamma\beta}(\pi_{i\kappa}^{\mathcal{U}_\gamma}(\bar{X})) = \pi_{k+1}^{\gamma\beta}(X)$.

Now notice that

$$\kappa^{+M_k^\kappa} = \kappa^{+K^{J_\alpha[x_k]}} \leq \kappa^{+\mathcal{M}_\kappa^{\mathcal{T}^*}} = \kappa^{+\mathcal{M}_\kappa^{\mathcal{U}^*}}.$$

(Here, the first equality is true by covering.)

Therefore, Claims 1 and 2 imply that for "typical" $\gamma < \beta < \alpha$,

$$\pi_k^{\gamma\beta} \restriction (\kappa_{k+1}^{\gamma})^{+M_k^{\kappa}} = \pi_{k+1}^{\gamma\beta} \restriction (\kappa_{k+1}^{\gamma})^{+M_k^{\kappa}} = \pi_{\kappa\lambda}^{\mathcal{U}^*} \restriction (\kappa_{k+1}^{\gamma})^{+M_k^{\kappa}}.$$

This buys us that M_k^{α} must be well-founded, i.e., $\binom{\alpha}{k}$ is true. □

We now turn to our second correctness conjecture, which comes from [6].

Definition 1.7 *Let $n < \omega$. By IIH_n we denote the following statement.*

Suppose that for every $x \in {}^{\omega}\omega$, $M_n^{\dagger}(x)$ exists, but also suppose that $M_{n+1}^{\#}$ does not exist.

Let $\mathcal{P}$ be a z-premouse such that $\mathcal{P} \models \Phi_n$ and such that $K^{\mathcal{P}} \in \mathcal{I}_{n+2}$. Then $\mathcal{P}|\Omega$ is n-Neeman iterable, where Ω is the critical point of the top measure of $\mathcal{P}$.

Here, "IIH" stands for "Iterability Inheritance Hypothesis." In the situation of Definition 1.7, by K^P we mean the core model of $\mathcal{P}$ of height Ω in the sense of [6].

Steel showed in [8] that IIH_n holds true for $n = 0$. Later, we gave a different proof of IIH_0 in [5] (cf. also [6]).

Theorem 1.8 ([8]) IIH_0 *holds true.*

2 The even case

The proof of the above Conjecture will be a core model induction on the levels of the projective hierarchy. The first two steps of this induction are provided by [1] and [2]. An appropriate generalization of [1] should give a general version of the odd steps of this induction. We here sketch how to run the even steps, modulo our two correctness hypotheses.

Theorem 2.1 *Let $n < \omega$. Assume both $\clubsuit_{2n}$ and IIH_{2n} to hold. Suppose that all $\mathbf{\Pi}^1_{2n+1}$ sets are determined. Suppose further that for all A, $B \subset {}^{\omega}\omega$ such that A, B are both $\mathbf{\Pi}^1_{2n+2}$, either $A \leq_{\mathsf{W}} B$ or $B \leq_{\mathsf{W}} {}^{\omega}\omega \setminus A$. Then all $\mathbf{\Pi}^1_{2n+2}$ sets are determined.*

Proof. The hypothesis of Theorem 2.1 yields that for all $x \in {}^{\omega}\omega$, $M_{2n}^{\#}(x)$ exists. Let us assume that (the conclusion of) Theorem 2.1 does not hold. There is then some $x_0 \in {}^{\omega}\omega$ such that $M_{2n+1}^{\#}(x_0)$ does not exist. We aim to derive a contradiction. We'll present the argument for the case $x_0 = 0$ and leave it to the reader to verify that the argument to follow easily relativizes.

Claim 1. There is some $z \in {}^{\omega}\omega$ such that $\mathcal{I}_{n+2}$ is $\Sigma^1_{2n+2}(z)$.

Proof. Set $\mathcal{I} = \mathcal{I}_{n+2}$. We may easily define a Π^1_{2n+2}-norm φ on $\mathcal{I}$. Let U be a complete Π^1_{2n+2}-set. If $U \leq_{\mathsf{W}} \mathcal{I}$, then φ induces a Π^1_{2n+2}-norm on U. But $\mathbf{\Pi}^1_{2n+1}$-determinacy implies that there is no such norm on U. Therefore $\mathcal{I} \leq_{\mathsf{W}} {}^{\omega}\omega \setminus U$, which yields that $\mathcal{I}$ is $\Sigma^1_{2n+2}(z)$ for some $z \in {}^{\omega}\omega$. □ (Claim 1)

Claim 2. For every $x \in {}^\omega\omega$, $M_{2n}^\dagger(x)$ exists.

PROOF. Suppose not, and let $x_1 \in {}^\omega\omega$ be such that $M_{2n}^\dagger(x_1)$ does not exist. We aim to derive a contradiction. We'll present the argument for the case $x_1 = 0$ and leave it to the reader to verify that the argument to follow easily relativizes.

We let

$$\begin{aligned} B = \; & \{x \in {}^\omega\omega : (x)_0 = z \text{ , and for all } k < \omega \text{ ,} \\ & (x)_{k+1} \text{ codes a } (x)_k\text{-premouse } \mathcal{M}_k \text{ which is a model of } \Psi_{2n} \text{ ,} \\ & \mathcal{M}_{k+1} \models \mathcal{M}_k \text{ is iterable, and} \\ & K^{\mathcal{M}_k} \in \mathcal{I}_{2n+2}\}. \end{aligned}$$

Of course, $B \neq \emptyset$. Let $x \in B$, and let $\mathcal{M}$ be the premouse coded by $(x)_1$. Using $\clubsuit_{2n}$, $\mathcal{M}$ is $2n$-Neeman iterable.

By Claim 1, B is $\Sigma^1_{2n+2}(z)$. But $\mathcal{M} \prec_{\Sigma^1_{2n+2}} V$, and hence there is some $y \in {}^\omega\omega \cap \mathcal{M}$ such that $y \in B$. But then $\mathcal{N} \in \mathcal{M}$ by the above Subclaim, where $\mathcal{N}$ is the premouse coded by $(y)_1$. We have shown that

$$\{\mathcal{M} : \exists x \in B \; \mathcal{M} \text{ is coded by } x\}$$

is non-empty and does not have an $\in$-least element. Contradiction! $\square$ (Claim 2)

Let

$$\begin{aligned} B^* = \; & \{x \in {}^\omega\omega : x \text{ codes a } z\text{-premouse } \mathcal{M} \text{ which is a model of } \Phi_{2n} \text{ and} \\ & K^{\mathcal{M}} \in \mathcal{I}_{n+2}\}. \end{aligned}$$

We have that $B^* \neq \emptyset$, and by IIH_{2n}, if $x \in B^*$, then x codes a $2n$-Neeman-iterable z-premouse which is a model of Φ_{2n}.

Now let $x \in B^*$, and let $\mathcal{M}$ be the premouse coded by x. By Claim 1, B^* is $\Sigma^1_{2n+2}(z)$. But $\mathcal{M} \prec_{\Sigma^1_{2n+2}} V$, and hence there is some $y \in {}^\omega\omega \cap \mathcal{M}$ such that $y \in B^*$. In particular, if $\mathcal{N}$ is the premouse coded by y, then $\mathcal{N} \in \mathcal{M}$.

We have shown that

$$\{\mathcal{M} : \exists x \in B^* \; \mathcal{M} \text{ is coded by } x\}$$

is non-empty and does not have an $\in$-least element. $\square$ (Theorem 2.1)

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