I'd like to report on a result of Vopěnka and Balcar, which I learned from Bukovský.

**Theorem 1 (Vopěnka–Balcar)** Let $W$ be an inner model of $V$. Let $T_P$ be a set in $W$, and let $g \in T_P$. The following are equivalent.

(1) There is $R \subseteq T_P \times T_P$, $R \in W$ such that
   (a) for each $p \in g$, $\{ q : pRq \} \subset T_P \setminus g$, and
   (b) for each $A \in W$, $A \subseteq T_P \setminus g$, there is some $p \in g$ with $\{ q : pRq \} \supset A$.

   (in other words, $\{ \{ q : pRq \} : p \in g \}$ is a $\mathsf{C}$-closed subset of $\mathcal{P}(T_P \setminus g) \cap W$.)

(2) There is some $\leq \subset T_P \times T_P$ such that $(T_P, \leq)$ is a partial order and $g$ is $(T_P, \leq)$-generic over $W$.

**Proof:** (2) $\implies$ (1). Let $R = \bot$, where $p \bot g$ if $p$ and $g$ are incompatible w.r.t. $\leq$. 
(a) is trivial.

(b) Let $A \subseteq W$, $A \subseteq \mathcal{P} \setminus \emptyset$. Let $A' = \{ p : \exists q \in A \ p \leq q \}$. Then $A' \cap \emptyset = \emptyset$.

Let $p \in \emptyset$, $p \perp \emptyset = \emptyset$. Then $p \perp q$ for all $q \in A'$ (hence for all $q \in A$).

Otherwise, if $r \leq p, q$, let $g'$ be TP-joined on $W$ with $r \in g'$; then $A' \cap g' = \emptyset$, but $r \in A' \cap g'$.

Contradiction.

Hence $A \subseteq \{ q : q \perp p \}$.

Let us now prove $(1) \Rightarrow (2)$. We may as well assume that $R$ as in $(1)$ is symmetric and non-reflexive, as we may replace $R$ by

$$(R \cup R^{-1}) \setminus \text{id} \setminus \mathcal{P}.$$  

Let us then define $p \leq q$ by $\{ r : rRp \} \supset \{ r : rRq \}$.

Claim 1. $p R q \Rightarrow p \perp q$ in the sense of $\leq$.

Proof of Claim 1: Suppose that $p R q$ and
Claim 1. \( \mathfrak{g} \) is \( \leq \)-generic on \( \langle \mathbb{P}; \leq \rangle \). Let \( r \leq p, q \).

Then \( p \mathrel{R} q, \) so \( \mathrel{R} R p \Rightarrow \mathrel{R} R r \quad \text{(Claim 1)} \)

Claim 2. \( \mathfrak{g} \subset \mathbb{P} \) is a filter.

Proof of Claim 2: First let \( \{p, q\} \leq \mathfrak{g} \). Then

\[ \{r : p \mathrel{R} r \lor q \mathrel{R} r\} \in \mathbb{P} (\mathbb{P} \setminus \mathfrak{g}) \cap W, \text{ so there is } \] by (b) some \( s \leq \mathfrak{g} \) with \( \{r : s \mathrel{R} r\} \supset \{r : p \mathrel{R} r\} \cup \{r : q \mathrel{R} r\} \), i.e., \( s \leq p, q \).

Now let \( p \leq \mathfrak{g} \), \( p \leq q \). If \( q \notin \mathfrak{g} \), then by (a) there is some \( r \leq \mathfrak{g} \) with \( r \mathrel{I} q \), hence \( r \perp \mathfrak{g} \) by Claim 1. But then \( r \perp p \) by \( p \leq q \). However, \( \{r, p\} \leq \mathfrak{g} \), so this contradicts the previous paragraph.

\[ \] (Claim 2)

Claim 3. \( \mathfrak{g} \) is \( \langle \mathbb{P}; \leq \rangle \)-generic on \( W \).

Proof of Claim 3: If \( A \subset \mathbb{P} \setminus \mathfrak{g} \) were a maximal antichain, then every \( A \in W \), then by (b) there could be some \( p \leq \mathfrak{g} \) with \( \{q : q \mathrel{R} p\} \supset A \). By Claim 1, \( q \perp p \) for all \( q \in A \).

\[ \] (Claim 3)
We have shown Theorem 1.

**Theorem 2 (Vopěnka)** Let *W* be an inner model of *V*. Suppose that for each *A* ∈ *V*, *A* ⊆ *W*, there are *X*, *Y* such that *A* = *UX*, *X* ⊆ *Y*, *Y* ∈ *W*,

\[ \overline{Y} \leq \kappa, \text{ where } \kappa \text{ is an uncountable cardinal.} \]

then there is some poset *P* ∈ *W* of size ≤ \( \kappa \) and some *g* which is *P*-generic over *W* s.t. *V* = *W*[\( g \)].

**Proof:** Let *f* : \( \theta \to \mathcal{O}(\kappa) \), \( f \in V \). Let *f* = *UX*, \( X \subseteq Y \in W \), \( Y \leq \kappa \). Let \( g : \theta \to V \) be defined by

\[ g(\bar{s}) = \{ \gamma : \exists a \in Y (\gamma \text{ is least st. } (\bar{s}, \gamma) \in a) \} \, . \]

Then \( f(\bar{s}) \in g(\bar{s}) \), \( g(\bar{s}) \leq \kappa \), for all \( \bar{s} \).

Hence by Bukovsky's theorem there is some *Q* ∈ *W* and some \( h \) Q-generic over *W* s.t. *V* = *W*[\( h \)] ad *Q* has the \( \kappa^+ \)-c.c. Fix such *Q*, \( h \).

By our hypothesis, we may write
\[ h = \bigcup X, \ x \leq y \in W, \ y \leq x. \]

Let \( t \in W^2 \), \( \tau^h = \mathcal{P}(Y \setminus X) \cap W \).

Let us define \( R \subseteq Y \times Y \) as follows. For \( a, b \in Y \) let \( a \sim b \) if

\[ \exists p \in a \cap b \exists x \in \mathcal{P}(Y) \cap W \left( (p, x) \in \tau \land b \in x \right) \]

It is easy to see that (a) + (b) of (1) in the statement of Theorem 1 are satisfied (with \( \mathcal{P} = Y, \ g = X \)).

But then there is by Theorem 1 some \( X \in \mathcal{W} \) s.t. \( X \) is \( (Y; \leq) \)-generic over \( W \) of

course, \( h = \bigcup X \in W[X] \), so that \( W[X] \supseteq W[h] = V \). As \( X \in V, \ W[X] = V. \)

\[ \square \]