VIRTUAL LARGE CARDINALS

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Abstract. We introduce the concept of virtual large cardinals and apply it to obtain a hierarchy of new large cardinal notions between ineffable cardinals and $0^{\#}$. Given a large cardinal notion $\mathcal{A}$ characterized by the existence of elementary embeddings $j : V_\alpha \to V_\beta$ satisfying some list of properties, we say that a cardinal is virtually $\mathcal{A}$ if the embeddings $j : V_\alpha^V \to V_\beta^V$ exist in the generic multiverse of V. Unlike their ideological cousins generic large cardinals, virtual large cardinals are actual large cardinals that are compatible with $V = L$. We study virtual versions of extendible, n-huge, and rank-into-rank cardinals and determine where they fit into the large cardinal hierarchy.

1. Introduction

The current paper introduces the theory of virtual large cardinals. Suppose $\mathcal{A}$ is a large cardinal notion that can be characterized by the existence of one or many elementary embeddings $j : V_\alpha \to V_\beta$ satisfying some list of properties. For instance, both extendible cardinals and I3 cardinals meet these requirements. Recall that κ is *extendible* if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ and $j(\kappa) > \alpha$, and recall also that κ is I3 if there is an elementary embedding $j : V_\lambda \to V_\lambda$ with critical point $\kappa < \lambda$. Let us say that a cardinal κ is *virtually* $\mathcal{A}$ if the embeddings $j : V_\alpha \to V_\beta$ needed to witness $\mathcal{A}$ can be found in set-generic extensions of the universe V; equivalently we can say that the embeddings exist in the generic multiverse of V. Indeed, as we shall see in Section 3, it suffices to only consider the collapse extensions. So we now have that κ is *virtually extendible* if for every $\alpha > \kappa$, some set-forcing extension has an elementary embedding $j : V_\alpha^V \to V_\beta^V$ with critical point κ and $j(\kappa) > \alpha$, and we have that κ is *virtually* I3 if some set-forcing extension has an elementary embedding $j : V_\lambda^V \to V_\lambda^V$ with critical point κ. As we will see in Section 2 the template of virtual large cardinals can be applied to several large cardinals notions in the neighborhood of a supercompact cardinal. We can even apply it to inconsistent large cardinal principles to obtain (consistent) virtual large cardinals that are compatible with $V = L$.

The concept of virtual large cardinals is close in spirit to generic large cardinals, but is technically very different. Suppose $\mathcal{A}$ is a large cardinal notion characterized by the existence of elementary embeddings $j : V \to M$ satisfying some list of properties. Then we say that a cardinal κ is *generically* $\mathcal{A}$ if the embeddings needed to witness $\mathcal{A}$ exist in set-forcing extensions of V. More precisely, if the existence of $j : V \to M$ satisfying some properties witnesses $\mathcal{A}$, then we want a forcing extension $V[G]$ to have a definable $j : V \to M$ with these properties, where M is an inner model of $V[G]$. So for example, κ is *generically supercompact* if for every $\lambda > \kappa$, some set-forcing extension $V[G]$ has an elementary embedding $j : V \to M$ with critical point κ and $j\,"\,\lambda \in M$. If κ is not actually λ-supercompact, the model

M will not be contained in V. Generic large cardinals are either known to have the same consistency strength as their actual counterparts or are conjectured to have the same consistency strength based on currently available evidence.[1] Most importantly, generic large cardinals need not be actually "large" since, for instance, ω_1 can be generically supercompact.

In the case of virtual large cardinals, because we consider only set-sized embeddings, the source and target of the embedding are both from V, and because the embedding exists in a forcing extension, there is no a priori reason why the target model would have any closure at all. The combination of these gives that virtual large cardinals are actual large cardinals that fit into the large cardinal hierarchy between ineffable cardinals and $0^\#$. If $0^\#$ exists, the Silver indiscernibles have (nearly) all the virtual large cardinal properties we consider in this article, and all these notions will be downward absolute to L.

The first virtual large cardinal notion, the remarkable cardinal, was introduced by the second author in [Sch00]. A cardinal κ is *remarkable* if for every $\lambda > \kappa$, there is $\bar\lambda < \kappa$ such that in a set-forcing extension there is an elementary embedding $j : V^V_{\bar\lambda} \to V^V_\lambda$ with $j(\text{crit}(j)) = \kappa$. It turns out that remarkable cardinals are virtually supercompact because, as shown by Magidor [Mag71], κ is supercompact precisely when for every $\lambda > \kappa$, there is $\bar\lambda < \kappa$ and an elementary embedding $j : V_{\bar\lambda} \to V_\lambda$ with $j(\text{crit}(j)) = \kappa$. The second author showed that the existence of a remarkable cardinal is equiconsistent with the assertion that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing [Sch00], and since then it has turned out that remarkable cardinals are equiconsistent to other natural assertions such as the third-order Harrington's principle [CS15].

The idea behind the concept of virtual large cardinals of taking a property characterized by the existence of elementary embeddings of sets and defining a virtual version of the property by positing that the embeddings exist in the generic multiverse can be extended beyond large cardinals. In [BGS], together with Bagaria, we studied a virtual version of Vopěnka's Principle (Generic Vopěnka's Principle) and a virtual version of the Proper Forcing Axiom PFA. Fuchs has generalized this approach to obtain virtual versions of other forcing axioms such as the forcing axiom for subcomplete forcing SCFA [Fuca] and resurrection axioms [Fucb]. Each of these virtual properties has turned out to be equiconsistent with some virtual large cardinal, which has so far been the main application of these ideas.

In Section 2, we will formally define several virtual large cardinal notions that we are going to study in this article. In Section 3, we will recall some standard absoluteness results about countable structures, using which we will, in particular, get useful reformulations of the definitions of virtual large cardinals. In Section 4, we will show where the virtual large cardinals we defined fit into the existing hierarchy. In Section 5, we will review some current applications of these ideas. Finally, in Section 6, we will briefly motivate the virtual large cardinal template we have chosen, by discussing some alternative definitions.

[1] Some care is necessary here, though. By a theorem of Shelah (cf. e.g. [Sch] for a writeup), a Woodin cardinal can be used to obtain that ω_1 is generically almost huge.

2. Virtual large cardinals

We will define several virtual large cardinal notions for large cardinals stronger than supercompacts, such as extendible cardinals, n-huge(-like) cardinals, and rank-into-rank cardinals. We will also introduce (consistent) virtual versions of inconsistent large cardinal notions. To simplify notation, we will drop the superscript V when talking about rank initial segments V_α^V in some forcing extension $V[G]$ with the stipulation that if we ever need to refer to the V_α of $V[G]$, we will always denote it $V_\alpha^{V[G]}$.

Definition 2.1. A cardinal κ is *virtually extendible* if for every $\alpha > \kappa$, in a set-forcing extension there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \alpha$.

In [BGS] together with Bagaria, we generalized the definition of virtually extendible cardinals to Bagaria's hierarchy of $C^{(n)}$-extendible cardinals from [Bag12]. Recall that $C^{(n)}$ is the class of all ordinals α such that $V_\alpha \prec_{\Sigma_n} V$ and that κ is $C^{(n)}$-*extendible* if the extendibility embedding $j : V_\alpha \to V_\beta$ has the additional property that $j(\kappa) \in C^{(n)}$. As we will discuss in Section 5, virtually $C^{(n)}$-extendible cardinals measure the consistency strength of fragments of Generic Vopěnka's Principle.

In the case of extendible cardinals we can drop the requirement on the embeddings that $j(\kappa) > \alpha$ because if for every $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$, then for every $\alpha > \kappa$, there is some such embedding j^* with $\text{crit}(j^*) = \kappa$ *and* $j^*(\kappa) > \alpha$ (see, for instance, Proposition 23.15 (b) in [Kan09]). All standard proofs of this fact make use of Kunen's Inconsistency [Kun71]. We will see in Section 4 that Kunen's Inconsistency does not hold for virtual embeddings, so that in a forcing extension we can have elementary embeddings $j : V_\lambda \to V_\lambda$ with λ much larger than the supremum of the critical sequence. In [GH], it is shown that if we remove the requirement that $j(\kappa) > \alpha$ for virtually extendible cardinals, we do not get an equivalent notion.

Recall that a cardinal κ is *n-huge* for a natural number $n \geq 1$ if there is an elementary embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $M^{j^n(\kappa)} \subseteq M$, and that 1-huge cardinals are simply called *huge*. Equivalently, κ is n-huge if there is a normal κ-complete ultrafilter U on some $\mathcal{P}(\lambda)$ and cardinals $\kappa = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda$ such that for each $i < n$, $\{x \in \mathcal{P}(\lambda) \mid \text{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in U$ (see, for instance, Theorem 24.8 in [Kan09]). It follows that we can assume without loss of generality that the ultrafilter U is on $[\lambda]^{\lambda_{n-1}}$ instead of on the entire $\mathcal{P}(\lambda)$.

We were not able to find a characterization of n-huge cardinals that fits the virtual large cardinals template (namely, that there are elementary embeddings $j : V_\alpha \to V_\beta$). (See Section 6 for possible alternative definitions of virtually n-huge cardinals.) So instead we will introduce a hierarchy of n-huge* cardinals which intertwines with the n-huge cardinals, such that the n-huge* cardinals have a suitable elementary embedding characterization to produce a virtual version. We define that a cardinal κ is *n-huge** if for some $\alpha > \kappa$, there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.

Proposition 2.2.

(1) *An n-huge* cardinal is an n-huge limit of n-huge cardinals.*
(2) *An $n+1$-huge cardinal is n-huge*.*

Proof. Suppose that κ is n-huge*, and so for some $\alpha > \kappa$, there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$. Let $\lambda_0 = \kappa$ and for $1 \leq i \leq n$, let $\lambda_i = j^i(\kappa)$ with $\lambda_n = \lambda$. Let's first suppose the worst possible case that $\alpha = \lambda + 1$. Then $\beta = j(\lambda) + 1$ by elementarity, and so $j\,"\,\lambda \subseteq j(\lambda) \in V_\beta$. Fix $A \subseteq [\lambda]^{\lambda_{n-1}}$. Since λ is clearly inaccessible, it follows that every $x \in A$ is in V_λ. So $A \subseteq V_\lambda$, and hence $A \in V_\alpha$. Now, it is easy to see that the ultrafilter U on $[\lambda]^{\lambda_{n-1}}$ defined by $A \in U$ if and only if $j\,"\,\lambda \in j(A)$ witnesses that κ is n-huge. If $\lambda + 1 < \alpha$, the argument to produce U is even easier. Now observe that U is an element of V_β, which means V_β sees that κ is n-huge, and so κ is a limit of n-huge cardinals.

Now suppose that κ is $n+1$-huge and fix an embedding $j : V \to M$ with $\text{crit}(j) = \kappa$ and $M^\lambda \subseteq M$, where $\lambda = j^{n+1}(\kappa)$. Observe that $V^M_{\lambda+1} = V^V_{\lambda+1}$, and so j restricts to $j : V_{j^n(\kappa)+1} \to V_{\lambda+1}$, which witnesses that κ is n-huge*. □

Now we can define the virtual version of n-huge* cardinals.

Definition 2.3. A cardinal κ is *virtually n-huge** if for some $\alpha > \kappa$, in a set-forcing extension there is an elementary embedding $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$.

We move on to rank-into-rank cardinals. Since Kunen's Inconsistency does not hold for virtual embeddings, we will not stratify the virtual rank-into-rank large cardinals.

Definition 2.4. A cardinal κ is *virtually rank-into-rank* if for some $\lambda > \kappa$, in a set-forcing extension there is an elementary embedding $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$.

Going beyond the consistent large cardinal notions, we can define a virtual large cardinal that captures the properties of some Silver indiscernibles in L under $0^\#$.

Definition 2.5. A cardinal κ is *Silver* if in a set-forcing extension there is a club in κ of generating indiscernibles for V_κ of order-type κ.

3. Absoluteness results

When investigating properties of embeddings of structures from V that exist in a forcing extension, we often rely on the following folklore result about the absoluteness of embeddings on countable structures.

Lemma 3.1. *Suppose M is a countable first-order structure and $j : M \to N$ is an elementary embedding. If W is a transitive (set or class) model of (some sufficiently large fragment of)* ZFC *such that M is countable in W and $N \in W$, then for any finite subset of M, W has some elementary embedding $j^* : M \to N$, which agrees with j on that subset. Moreover, if both M and N are transitive $\in$-structures and j has a critical point, we can additionally assume that crit$(j^*) =$ crit(j).*

Proof. Let $\{a_i \mid i < \omega\}$ be an enumeration of M in W. In W, we can build the tree T of all partial finite isomorphisms between M and N with domain $\{a_i \mid i < n\}$ for some $n < \omega$. The tree T is ill-founded in V, as witnessed by the branch constructed from the embedding j, and hence it is ill-founded in W as well by the absoluteness of well-foundedness. The branch of T in W gives the desired embedding j^*. We can ensure that j and j^* agree on some finitely many values or on the critical point by imposing the corresponding requirements on the finite partial isomorphisms in T. □

Corollary 3.2. *Suppose M are N are first-order structures in the same first-order language. If some set-forcing extension has an elementary $j : M \to N$, then in $V^{\mathrm{Coll}(\omega,M)}$ there is an elementary $j^* : B \to A$, and we can additionally assume that j and j^* agree on a fixed finite set of values and where appropriate have the same critical point.*

Proof. Suppose there is a forcing extension $V[G]$ by $\mathbb{P}$ with an elementary embedding $j : M \to N$. Let $H \subseteq \mathrm{Coll}(\omega, M)$ be $V[G]$-generic. Clearly $j \in V[G][H] \supseteq V[G]$, and so it follows by Lemma 3.1, that there is some elementary $j^* : M \to N$ with the desired properties in $V[H] \subseteq V[G][H]$ because M is countable in $V[H]$. Since $\mathrm{Coll}(\omega, M)$ is a weakly homogeneous[2] forcing notion, it follows that every $\mathrm{Coll}(\omega, M)$-extension has some such embedding. □

Thus, we can restate the definitions of all virtual large cardinals to say that the required embedding exists in $V^{\mathrm{Coll}(\omega,V_\alpha)}$, where V_α is the source of embedding. Note that the partial orders $\mathrm{Coll}(\omega, M)$ and $\mathrm{Coll}(\omega, |M|)$ are isomorphic using a bijection between M and $|M|$. In what follows we will often abuse notation by conflating the two posets.

We show in [BGS], together with Bagaria, that the existence of an elementary embedding between V-structures M and N in a forcing extension also has a game-theoretic formulation. Let $G(M, N)$ be an Ehrenfeucht-Fraïssé-like game of length ω in which player I plays elements of M and player II plays elements of N, and player II wins if she is able to maintain a finite partial isomorphism at each step. Then some set-forcing extension has an elementary embedding $j : M \to N$ if and only if player II has a winning strategy in $G(M, N)$.

We will now use Lemma 3.1 to show that if $0^\#$ exists, then there are γ such that $L_\gamma = V_\gamma^L$ and in $L^{\mathrm{Coll}(\omega,L_\gamma)}$ there is an elementary embedding $j : L_\gamma \to L_\gamma$ with γ much larger than the supremum of the critical sequence of j. So suppose that $0^\#$ exists and let $\{i_\xi \mid \xi \in \mathrm{ORD}\}$ be the Silver indiscernibles. Let $j : L \to L$ be the elementary embedding generated by a shift of indiscernibles such that $j(i_n) = i_{n+1}$ for $n < \omega$ and $j(i_\xi) = i_\xi$ for $\xi \geq \omega$. Let $\gamma = i_\xi$ for some $\xi > \omega$. Then j restricts to $j : L_\gamma \to L_\gamma$. Let $H \subseteq \mathrm{Coll}(\omega, L_\gamma)$ be V-generic. Then j is an element of $V[H] \supseteq V$, and so by Lemma 3.1, in $L[H]$ there is some $j^* : L_\gamma \to L_\gamma$ with $\mathrm{crit}(j^*) = i_0$ and $j^*(i_\omega) = i_\omega$. The requirement that $j^*(i_\omega) = i_\omega$ ensures that the supremum of the critical sequence of j^* is at most i_ω.

Some other absoluteness results about the existence of countable objects are also useful in the study of virtual large cardinals. For instance, it is a standard argument that existence of a homogeneous set of a countable order-type for a coloring is absolute between transitive models of set theory.

Lemma 3.3. *Suppose $f : [\kappa]^{<\omega} \to 2$ and there is a homogeneous set $H \subseteq \kappa$ of order-type some countable λ. If W is a transitive (set or class) model of (a sufficiently large fragment of)* ZFC *such that $f \in W$ and λ is countable in W, then W has some homogeneous set of order-type λ for f.*

Proof. We work in W. Let $\{a_n \mid n < \omega\}$ be some enumeration of λ. Let T be the tree of all finite tuples $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle$ of elements of κ such that we have $\alpha_i < \alpha_j$

[2]A forcing notion $\mathbb{P}$ is *weakly homogeneous* if for every pair $p, q \in \mathbb{P}$, there is an automorphism π of $\mathbb{P}$ such that $\pi(p)$ is compatible to q. It is easy to see that if $\mathbb{P}$ is weakly homogeneous and there is $p \in \mathbb{P}$ such that $p \Vdash \varphi(\check{a})$, then $\mathbb{1} \Vdash \varphi(\check{a})$.

if and only if $a_i < a_j$ and $\{\alpha_i \mid i < n\}$ is homogeneous for f, with the tuples being ordered by extension. The homogeneous set H witnesses that T is ill-founded in V, and hence it is also ill-founded in W by absoluteness of well-foundedness. Clearly a branch of T in W is a homogeneous set for f of order-type λ. □

For results about Silver cardinals we will need that the existence of a club of generating indiscernibles of a certain order-type for a countable structure is absolute between transitive models of set theory.

Lemma 3.4. *Suppose M is a countable first-order structure with an ordinal $\kappa \subseteq M$ and suppose there is a club in κ of generating indiscernibles for M of order-type κ. If W is a transitive (set or class) model of (a sufficiently large fragment of)* ZFC *such that M is countable in W, then W has some club in κ of generating indiscernibles for M of order-type κ.*

This follows by Shoenfield's Absoluteness applied to the statement above with reals coding M and κ as parameters.

4. The hierarchy

The virtual large cardinals form a hierarchy that exactly mirrors the hierarchy of their actual counterparts. Virtually rank-into-rank cardinals are virtually n-huge* limits of virtually n-huge* cardinals for every $n < \omega$; virtually n-huge* cardinals form a hierarchy, and a virtually huge* cardinal implies the consistency of a proper class of virtually extendible cardinals; finally the virtually extendible cardinals are remarkable limits of remarkable cardinals. Silver cardinals imply the existence of virtually rank-into-rank cardinals because every element of a club C witnessing that κ is Silver is virtually rank-into-rank. Assuming $0^\#$ exists, the Silver indiscernibles have (nearly) every one of the virtual large cardinal properties we consider, but all these properties are downward absolute to L (only some Silver indiscernibles are Silver cardinals). Roughly, the virtual large cardinals sit between ineffable cardinals and $0^\#$ in the large cardinal hierarchy. Their more precise placement can be determined by considering the α-iterable cardinals hierarchy introduced in [Git11] with which the virtual large cardinals intertwine. Below we briefly review the relevant properties of α-iterable cardinals.

The motivation for introducing α-iterable cardinals comes from the iterability properties of mini-measures whose existence characterizes several large cardinals below a measurable. Although we often associate smaller large cardinals with their combinatorial properties, most of them have characterizations in terms of the existence of elementary embeddings on *weak κ-models*, which are transitive models of ZFC$^-$ of size κ and height above κ. Weakly compact cardinals have the simplest such embedding characterization: a cardinal κ is weakly compact if and only if ($2^{<\kappa} = \kappa$ and) every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is an elementary $j : M \to N$ with N transitive and $\text{crit}(j) = \kappa$. The embedding characterization can be restated in terms of the existence of mini-measures, called M-ultrafilters, on κ. If $M \models \text{ZFC}^-$ is transitive and κ is a cardinal in M, then an *M-ultrafilter U* on κ is a filter measuring all sets in $\mathcal{P}(\kappa)^M$, which is κ-complete and normal for sequences from M; more precisely, the structure $\langle M, \in, U\rangle$, with a predicate for U, satisfies that U is a normal κ-complete ultrafilter on κ. Supposing M is a transitive model of ZFC$^-$ with a cardinal κ, it is easy to see that the existence of an M-ultrafilter on κ with a well-founded ultrapower is equivalent to the existence of an elementary $j : M \to N$ with $\text{crit}(j) = \kappa$ and N transitive. In one direction, the ultrapower by U gives the desired embedding and in the other direction we define $A \in U$ whenever $\kappa \in j(A)$ (we say that U is *generated* by κ via j). So an equivalent characterization of weakly compact cardinals is that ($2^{<\kappa} = \kappa$ and) every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is an M-ultrafilter on κ with a well-founded ultrapower. If an M-ultrafilter U is countably complete[3], then the ultrapower by U is obviously well-founded, but, unlike the case of κ-complete ultrafilters on κ, the converse fails to hold, so that an M-ultrafilter need not be countably complete to have a well-founded ultrapower.

To iterate the ultrapower construction with an M-ultrafilter, the filter must satisfy an additional property, which makes it partially internal to M. We say that an M-ultrafilter on κ is *weakly amenable* to M (for a transitive model M with a cardinal κ) if for every $X \in M$ of size κ in M, $X \cap U \in M$. A weakly amenable M-ultrafilter on κ with a well-founded ultrapower produces an embedding $j : M \to N$ where M and N have the same subsets of κ ($\mathcal{P}^M(\kappa) = \mathcal{P}^N(\kappa)$), and conversely such an embedding gives rise to a weakly amenable M-ultrafilter. With weakly amenable M-ultrafilters we can construct the iterated ultrapowers in the standard fashion. But while a κ-complete ultrafilter on κ has all well-founded iterated ultrapowers, a weakly amenable M-ultrafilter can have any ordinal $\alpha < \omega_1$ many well-founded iterated ultrapowers or ω_1-many, and hence all, well-founded iterated ultrapowers.

Definition 4.1 ([Git11])**.** A cardinal κ is *α-iterable* for $1 \leq \alpha \leq \omega_1$ if every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is a weakly amenable M-ultrafilter on κ with α-many well-founded iterated ultrapowers. A cardinal κ is *$<\alpha$-iterable* if it is β-iterable for all $\beta < \alpha$.

Note that a 1-iterable cardinal differs in its embeddings characterization from a weakly compact cardinal only in that the M-ultrafilter is required to be weakly amenable (equivalently the embedding $j : M \to N$ must have $\mathcal{P}^M(\kappa) = \mathcal{P}^N(\kappa)$). This additional assumption greatly pushes up the consistency strength so that a 1-iterable cardinal is a weakly ineffable limit of completely ineffable cardinals [Git11]. The α-iterable cardinals form a hierarchy: an α-iterable cardinal is a limit of β-iterable cardinals for every $\beta < \alpha$ [GW11]. It is easy to see that an ω_1-iterable cardinal implies $0^\#$, but for $\alpha < \omega_1$, α-iterable cardinals are downward absolute to L [GW11].

We can always assume without loss of generality in the definition of α-iterable cardinals that the weak κ-model M believes all its sets have transitive size κ because if this is not the case, we can take $H^M_{\kappa^+}$ instead. In this case, if $j : M \to N$ is the ultrapower by a weakly amenable M-ultrafilter U on κ, we get that $M = H^N_{\kappa^+}$ (because M and N have the same subsets of κ) and so in particular (if $V_\kappa \in M$), $M \in N$.

Let us call a commuting system $\{j_{\xi\nu} : M_\xi \to M_\nu \mid \xi < \nu < \alpha\}$ of embeddings *suitable* if, defining κ_ξ to be the critical point of $j_{\xi\xi+1}$ and U_ξ to be the M_ξ-ultrafilter generated by κ_ξ via $j_{\xi\xi+1}$, we have that for all $\xi < \nu < \alpha$, if $A \in M_\xi$ and $A \subseteq U_\xi$, then $j_{\xi\nu}(A) \subseteq U_\nu$. The following lemma will allow us to use such systems to obtain weak κ_0-models M with a weakly amenable α-iterable M-ultrafilter on κ_0.

[3]An M-ultrafilter U is said to be *countably complete* if for every sequence $\langle A_n \mid n < \omega\rangle$ (possibly not in M) with each $A_n \in U$, $\bigcap_{n<\omega} A_n \neq \emptyset$. Note that we do not require that $\bigcap_{n<\omega} A_n$ is an element of U.

Lemma 4.2 (Lemma 3.8 in [GW11]). *Suppose* $\{j_{\xi\nu} : M_\xi \to M_\nu \mid \xi < \nu < \alpha\}$ *is a suitable commuting system of embeddings. If* $M^0 \prec M^1 \prec \cdots \prec M^i \prec \cdots$ *for* $i < \omega$ *is an elementary chain with union* M *of elementary submodels of* M_0 *with the property that* $U_0 \cap M^i, M_i \in M^{i+1}$*, then* $U_0 \cap M$ *is a weakly amenable* M*-ultrafilter on* κ_0 *with* α*-many well-founded iterated ultrapowers.*

The following elementary fact will be useful in several arguments below.

Proposition 4.3. *Suppose a forcing extension* $V[G]$ *has an elementary embedding* $j : V_\alpha \to V_\beta$ *with* $crit(j) = \kappa$*. If* $X \in V_\alpha$ *has size* κ*, then* $j \restriction X \in V_\beta$*.*

Proof. Fix any bijection $f : \kappa \xrightarrow[1-1]{\text{onto}} X$. Using a flat pairing function[4] if neccessary, we can assume that $f \in V_\alpha$. Observe that for all $x \in X$, we have $j(x) = j(f)(f^{-1}(x))$. Since both f and $j(f)$ are elements of V_β it must have $j \restriction X$ as well. □

Note the the above proof goes through for an embedding $j : V_\alpha \to N$ with $N \in V$ provided that N is closed under κ-sequences in V.

Finally, we would like to discuss the connection between the α-iterable cardinals and the α-Erdős cardinals hierarchy. Welch and Sharpe showed that an ω_1-Erdős cardinal is a limit of ω_1-iterable cardinals [SW11]. In [GW11] it is shown that for additively indecomposable $\lambda < \omega_1$, a λ-Erdős cardinal implies the consistency of a δ-iterable cardinal for every $\delta < \lambda$. We can optimally improve this result and show the following.

Theorem 4.4. *Let* $\lambda \leq \omega_1$ *be additively indecomposable. Then* $\kappa(\lambda)$*, the least* λ*-Erdős cardinal, is a limit of* λ*-iterable cardinals.*

Proof. Write $\kappa = \kappa(\lambda)$, and fix $\beta < \kappa$. We aim to verify that there is a λ-iterable cardinal between β and κ. Fix $D \subseteq \kappa$ coding V_κ. By a theorem of Silver, cf. Proposition 7.15 (a) of [Kan09],

$$\kappa \to (\lambda)^{<\omega}_{2^{\max(\lambda,\beta)}},$$

so that we may let $I \subseteq (\beta, \kappa)$ be a set of indiscernibles for the structure

$$(L_{\kappa^+}[D]; \in, (\xi : \xi \leq \max(\lambda, \beta)))$$

of order-type λ. Let $L_\alpha[B]$ be isomorphic to $\text{Hull}^{L_{\kappa^+}[D]}((\max(\lambda,\beta)+1) \cup I)$. Writing $\bar{I}$ for the pointwise preimage of I under the Mostowski isomorphism, $\bar{I}$ is a set of generating indiscernibles for $L_\alpha[B]$ relative to $\max(\lambda,\beta)+1$. As $\max(\lambda,\beta)+1 \subseteq \text{Hull}^{L_{\kappa^+}[D]}((\max(\lambda,\beta)+1) \cup I)$, $\lambda < \omega_1^{L_\alpha[B]}$ and by elementarity it suffices to prove that $L_\alpha[B]$ has a λ-iterable cardinal between β and $\bar{\kappa}$, the preimage of κ.

Let $\{\iota_i : i < \lambda\}$ denote the monotone enumeration of $\bar{I}$. It is not difficult to see that $\bar{I}$ can be used to produce a suitable commuting system of embeddings $\{\sigma_{ij} : L_\alpha[B] \to L_\alpha[B] \mid i < j < \lambda\}$. Let us give some details.

For $i \leq j < \lambda$, let π_{ij} denote the "minimal" order-preserving map from λ to λ which sends i to j, i.e., $\pi_{ij}(k) = k$ for $k < i$ and $\pi_{ij}(i+k) = j+k$ for $k < \lambda$. (In order for π_{ij} to be well-defined, we need that λ is additively indecomposable.) We let $\sigma_{ij} : L_\alpha[B] \to L_\alpha[B]$ denote the map which sends

$$\tau^{L_\alpha[B]}(\vec{\xi}, \iota_{i_0}, \ldots, \iota_{i_{k-1}}) \text{ to } \tau^{L_\alpha[B]}(\vec{\xi}, \iota_{\pi_{ij}(i_0)}, \ldots, \iota_{\pi_{ij}(i_{k-1})}),$$

[4]A *flat pairing function* $\langle a, b\rangle$ has the property that if $a, b \in V_\alpha$, then also $\langle a, b\rangle \in V_\alpha$.

where τ stands for a Skolem term, $\vec{\xi} \leq \max(\lambda, \beta)$, and $\iota_{i_0}, \ldots, \iota_{i_{k-1}} \in \bar{I}$. It is easy to see that $\sigma_{jk} \circ \sigma_{ij} = \sigma_{ik}$ for $i \leq j \leq k < \lambda$.

For $i < \lambda$, let $\kappa_i \leq \iota_i$ denote the critical point of σ_{ii+1}. Notice that $\kappa_i > \max(\lambda, \beta)$. By indiscernibility arguments, $\sigma_{ij}(\kappa_i) = \kappa_j$ for $\omega \leq i \leq j < \lambda$. Let

$$U_i = \{X \in \mathcal{P}^{L_\alpha[B]}(\kappa_i) : \kappa_i \in \sigma_{ii+1}(X)\}$$

be the $L_\alpha[B]$-ultrafilter on κ_i derived from σ_{ii+1}. We have $L_\alpha[B] = \mathrm{Hull}^{L_\alpha[B]}(\{\kappa_i\} \cup \mathrm{ran}(\sigma_{ii+1}))$, so that σ_{ii+1} is actually equal to the ultrapower map given by forming $\mathrm{Ult}(L_\alpha[B]; U_i)$. By Kunen's argument (see Proposition 4.3), every U_i is weakly amenable to $L_\alpha[B]$, so that U_i is well-defined for all $i < \lambda$.

We claim that in fact $\sigma_{ij}(U_i) = U_j$ for $\omega \leq i \leq j < \lambda$. To see this, fix $k_0, \ldots, k_{m-1} < \lambda$. Then $U_i \cap \mathrm{Hull}^{L_\alpha[B]}(\kappa_i + 1 \cup \{\iota_{k_0}, \ldots, \iota_{k_{m-1}}\})$ is the measure derived from the map which sends

$$\tau^{L_\alpha[B]}(\vec{\xi}, \kappa_i, \iota_{k_0}, \ldots, \iota_{k_{m-1}}) \text{ to } \tau^{L_\alpha[B]}(\vec{\xi}, \kappa_{i+1}, \iota_{\pi_{ii+1}(k_0)}, \ldots, \iota_{\pi_{ii+1}(k_{m-1})}),$$

where τ stands for a Skolem term and $\vec{\xi} < \kappa_i$, so that by elementarity of σ_{ij} and the definition of σ_{ij} as well as by $\sigma_{ij}(\kappa_i) = \kappa_j$, $\sigma_{ij}(U_i \cap \mathrm{Hull}^{L_\alpha[B]}(\kappa_i + 1 \cup \{\iota_{k_0}, \ldots, \iota_{k_{m-1}}\}))$ is the measure derived from the map which sends

$$\tau^{L_\alpha[B]}(\vec{\xi}, \kappa_j, \iota_{\sigma_{ij}(k_0)}, \ldots, \iota_{\sigma_{ij}(k_{m-1})}) \text{ to } \tau^{L_\alpha[B]}(\vec{\xi}, \kappa_{j+1}, \iota_{\pi_{ij+1}(k_0)}, \ldots, \iota_{\pi_{ij+1}(k_{m-1})}),$$

where τ stands for a Skolem term and $\vec{\xi} < \kappa_j$, and hence

$$\begin{aligned} \sigma_{ij}(U_i \cap \mathrm{Hull}^{L_\alpha[B]}(\kappa_i + 1 \cup \{\iota_{k_0}, \ldots, \iota_{k_{m-1}}\})) &= \\ U_j \cap \mathrm{Hull}^{L_\alpha[B]}(\kappa_j + 1 \cup \{\iota_{\pi_{ij+1}(k_0)}, \ldots, \iota_{k_{m-1}}\}) &, \end{aligned}$$

so that indeed $\sigma_{ij}(U_i) = U_j$.

It is then easy to verify that $((L_\alpha[B]; U_i) : \omega \leq i < \lambda)$ is in fact an iteration of $(L_\alpha[B]; U_\omega)$ by U_ω and its images. Writing $\tau = \kappa_\omega^{+L_\alpha[B]}$, the structure $(L_\tau[B]; U_\omega)$ is now λ-iterable.

Now let $A \in \mathcal{P}^{L_\alpha[B]}(\kappa_\omega)$. Recall that $\lambda < \omega_1^{L_\alpha[B]}$. An ill-founded tree argument as in the proof of Theorem 3.11 of [GW11] will then give that $L_\alpha[B]$ contains a weak κ_ω-model $M \prec L_\tau[B]$ for which there is a weakly amenable M-ultrafilter on κ_ω with λ many well-founded ultrapowers. But certainly $\kappa_\omega > \beta$. □

Theorem 4.5. *For an additively indecomposable λ, if there is a $\lambda + 1$-iterable cardinal, then there is a λ-Erdős cardinal below it.*

Proof. Suppose κ is $\lambda + 1$-iterable. We will show that every $f : [\kappa]^{<\omega} \to 2$ has a homogeneous set of order-type λ. Since there are weak κ-models M containing V_κ with an embedding $j : M \to N$ with $\mathrm{crit}(j) = \kappa$, κ cannot be the least cardinal with this property. So some smaller cardinal must be λ-Erdős. Fix some $f : [\kappa]^{<\omega} \to 2$ and let M be a weak κ-model containing f for which there is a weakly amenable M-ultrafilter U on κ with $\lambda + 1$-many well-founded iterated ultrapowers. Let $M = M_0$, let $\{j_{\xi\nu} : M_\xi \to M_\nu \mid \xi < \nu \leq \lambda\}$ be the directed system of embeddings obtained from iterating the ultrapower construction with U and let $\{\kappa_\xi \mid \xi \leq \lambda\}$ be the critical sequence. In M_λ, fix some $g : [\kappa_\lambda]^{<\omega} \to 2$. Since λ is a limit, there is some $\xi < \lambda$ and $x \in M_\xi$ such that $g = j_{\xi\lambda}(x)$. By standard results about iterated ultrapowers (see, for instance, [Kan09] Lemma 19.9), $\{\kappa_\nu \mid \xi < \nu < \lambda\}$ are indiscernibles for the structure M_λ with a constant for g. Thus, by the indecomposability of λ, g has a homogeneous set of order-type λ. But, by Lemma 3.3, the existence of a homogeneous set of a countable order-type is absolute between transitive models of set theory, and so M_λ has a homogeneous set of order-type λ for g as well. It follows, by elementarity, that M satisfies that every $g : [\kappa]^{<\omega} \to 2$ has a homogeneous set of order-type λ, and so in particular f has one. □

In the following subsections, we will prove results about where each of the virtual large cardinals we introduced fits into the large cardinal hierarchy.

4.1. **Virtually extendible cardinals.**

Theorem 4.6. *Every virtually extendible cardinal is a remarkable limit of remarkable cardinals. If κ is virtually extendible, then V_κ is a model of proper class many remarkable cardinals.*

Proof. Suppose κ is virtually extendible. Fix some $\lambda > \kappa$ and let $\alpha \gg \lambda$. Let $H \subseteq \mathrm{Coll}(\omega, V_\alpha)$ be V-generic. In $V[H]$, fix an elementary $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. Consider the restriction $j : V_\lambda \to V_{j(\lambda)}$. It follows, by Lemma 3.1, that in $V^{\mathrm{Coll}(\omega, V_\lambda)}$ there is an elementary $j^* : V_\lambda \to V_{j(\lambda)}$ with $j^*(\mathrm{crit}(j^*)) = j(\kappa)$. By choosing V_α to satisfy a large enough fragment of ZFC to make forcing work, we can assume that V_β satisfies the above conclusion as well. Since $\lambda < j(\kappa)$, by elementarity, V_α satisfies that there is $\bar{\lambda} < \kappa$ such that in $V^{\mathrm{Coll}(\omega, \bar{\lambda})}$ there is an elementary $j^* : V_{\bar{\lambda}} \to V_\lambda$ with $j^*(\mathrm{crit}(j^*)) = \kappa$. This witnesses the remarkability of κ for λ, and since λ was arbitrary, we have verified that κ is remarkable.

The same argument which shows that for extendible γ, $V_\gamma \prec_{\Sigma_3} V$, extends to show the same conclusion for virtually extendible cardinals. So once we show that V_κ is a model of proper class many remarkable cardinals, it will follow that κ is a limit of remarkable cardinals since being remarkable is a Π_2 property. Since κ is actually remarkable, it is in particular remarkable in $V_{j(\kappa)}$. Thus, fixing $\xi < \kappa$, we have that V_β satisfies that there is a $\gamma > \xi$ which is remarkable in $V_{j(\kappa)}$. So, by elementarity, V_α satisfies that there is $\gamma > \xi$ which is remarkable in V_κ. Since $\xi < \kappa$ was arbitrary, we have shown that V_κ is a model of proper class many remarkable cardinals. □

Theorem 4.7. *Assuming $0^\#$ exists, every Silver indiscernible is virtually extendible in L. Virtually extendible cardinals are downward absolute to L.*

Proof. Let $\{i_\xi \mid \xi \in \mathrm{ORD}\}$ be the Silver indiscernibles. Fix some i_ξ and $\alpha > i_\xi$. Let $j : L \to L$ be a shift of indiscernibles embedding with $\mathrm{crit}(j) = i_\xi$ and $j(i_\xi) = i_\gamma > \alpha$. Consider the restriction $j : V^L_\alpha \to V^L_{j(\alpha)}$. Let $H \subseteq \mathrm{Coll}(\omega, V^L_\alpha)$ be V-generic. Clearly $j \in V \subseteq V[H]$. So by Lemma 3.1, in $L[H]$ there is an elementary embedding $j^* : V^L_\alpha \to V^L_{j(\alpha)}$ with $\mathrm{crit}(j^*) = i_\xi$ and $j^*(\kappa) = i_\gamma$. Since α and i_ξ were arbitrary, we have verified that every Silver indiscernible is virtually extendible.

Suppose κ is virtually extendible and fix an ordinal α. Let $\bar{\alpha} \gg \alpha$. So in $V^{\mathrm{Coll}(\omega, V_{\bar{\alpha}})}$, there is an elementary $j : V_{\bar{\alpha}} \to V_{\bar{\beta}}$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \bar{\alpha}$. Consider the restriction $j : V^L_\alpha \to V^L_{j(\alpha)}$ and proceed as in the argument with Silver indiscernibles. □

Virtually extendible cardinals fit between 1-iterable and 2-iterable cardinals in the consistency strength hierarchy.

Theorem 4.8. *Virtually extendible cardinals are* 1*-iterable limits of* 1*-iterable cardinals.*

Proof. Suppose κ is virtually extendible. So for some $\alpha > \kappa$, in $V[H]$, where $H \subseteq \mathrm{Coll}(\omega, V_\alpha)$ is V-generic, there is an elementary $j : V_\alpha \to V_\beta$ with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \alpha$. In $V[H]$, define a H_{κ^+}-ultrafilter on κ by $A \in U$ if and only if $\kappa \in j(A)$. Since $\mathcal{P}(\kappa) \subseteq V_\alpha$, U is clearly weakly amenable to H_{κ^+}. Let $\{a_n \mid n < \omega\}$ be an enumeration of H_{κ^+} in $V[H]$. We can find some weak κ-model $M_0 \prec H_{\kappa^+}$ in V such that $A, a_0 \in M_0$. Since U is weakly amenable, $U_0 = M_0 \cap U$ is in V. So we can find some weak κ-model $M_1 \prec H_{\kappa^+}$ in V such that $M_0, U_0, a_1 \in M_1$. Continuing in this manner, we obtain a sequence $\langle M_n \mid n < \omega\rangle \in V[H]$ of weak κ-models $M_n \prec H_{\kappa^+}$ with $M_n \in V$ such that $H_{\kappa^+} = \bigcup_{n<\omega} M_n$. Let $j_n : M_n \to j(M_n)$ be the restriction of j to M_n, and observe that $j(M_n) \prec H_{j(\kappa)^+}$ by elementarity. Also, by Proposition 4.3, every j_n is in V_β. Now let's make the following definition. Call a sequence $\langle h_0, h_1, \ldots, h_n\rangle$ of embeddings $h_i : N_i \to K_i$ *good* if

1. $\mathrm{crit}(h_0) = \kappa$ and $A \in N_0$,
2. $N_i \prec H_{\kappa^+}$ is a weak κ-model and $K_i \prec H_{j(\kappa)^+}$ for all $i \leq n$,
3. $h_i \subseteq h_{i+1}$ for all $i < n$,
4. $N_i, W_i \in N_{i+1}$, where W_i is the N_i-ultrafilter generated by κ via h_i for all $i < n$.

Notice that all sequences $\langle j_0, j_1, \ldots, j_n\rangle$ are good and, as we observed above, they are all in V_β. In V, let T be the tree of all good sequences of embeddings ordered by extension. The tree T is ill-founded in $V[H]$ as witnessed by $\langle j_n \mid n < \omega\rangle$. So by absoluteness of well-foundedness, T is ill-founded in V. Let $\langle h_n \mid n < \omega\rangle$ be a branch of T in V. Let $N = \bigcup_{n<\omega} N_i$, $K = \bigcup_{n<\omega} K_i$ and $h : K \to N$ be the union of the h_n. Clearly $W = \bigcup_{n<\omega} W_n$ is the N-ultrafilter generated by κ via h and W is weakly amenable by construction. Thus, we have found a weak κ-model N containing A and a weakly amenable N-ultrafilter W on κ with a well-founded ultrapower. Since A was arbitrary, we have shown that κ is 1-iterable. Since V_β sees that κ is 1-iterable, it follows that κ is a limit of 1-iterable cardinals. □

Theorem 4.9. *Suppose κ is* 2*-iterable. Then V_κ is a model of proper class many virtually extendible cardinals.*

Proof. Let M_0 be a weak κ-model containing V_κ for which there is a weakly amenable M_0-ultrafilter U_0 on κ with 2 well-founded iterated ultrapowers. We will assume without loss that $M_0 = H^{M_0}_{\kappa^+}$. It is a standard fact that iterated ultrapowers give rise to the following commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{j_0} & M_1 \\ \downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle h_0} \\ M_1 & \xrightarrow{j_1} & M_2 \end{array}$$

where j_0 and h_0 are ultrapowers by U_0 and j_1 is the ultrapower by U_1, the first iterate of U_0. Note that U_0 is a weakly amenable M_1-ultrafilter because M_0 and M_1 have the same subsets of κ by weak amenability. The model M_2 is well-founded because U_0 has 2 well-founded iterated ultrapowers by assumption. Since M_0 contains all functions $f : \kappa \to M_0$ from M_1 (because $M_0 = H^{M_1}_{\kappa^+}$), it follows that

$h_0 \restriction M_0 = j_0$. So in particular $V^{M_1}_{j_0(\kappa)} = V^{M_1}_{h_0(\kappa)} = V^{M_2}_{h_0(\kappa)}$. Consider the restriction

$$h_0 : V^{M_2}_{h_0(\kappa)} \to V^{M_2}_{h_0^2(\kappa)}.$$

Let $M = V^{M_2}_{h_0(\kappa)}$ and $N = V^{M_2}_{h_0^2(\kappa)}$, and let's rename h_0

$$j : M \to N.$$

Observe that $M \prec N \models \text{ZFC}$ (since $\text{crit}(j_1) = j_0(\kappa)$ and $j_1(j_0(\kappa)) = j_0^2(\kappa)$) and $M = V^N_{j(\kappa)}$.

First, we argue that κ is virtually extendible in M. Fix $\alpha > \kappa$ in M, and consider the restriction $j : V^M_\alpha \to V^N_{j(\alpha)}$. Let $H \subseteq \text{Coll}(\omega, V^M_\alpha)$ be V-generic. Since $V^M_\alpha = V^N_\alpha$ is countable in $N[H]$, by Lemma 3.1, $N[H]$ has some elementary $j^* : V^N_\alpha \to V^N_{j(\alpha)}$ with $\text{crit}(j^*) = \kappa$ and $j^*(\kappa) > \alpha$. So N satisfies that in $V^{\text{Coll}(\omega, V_\alpha)}$ there is an elementary $j^* : V_\alpha \to V_\beta$ (namely, $\beta = j(\alpha)$) with $\text{crit}(j^*) = \kappa$ and $j^*(\kappa) > \alpha$. Since M is elementary in N by our assumptions, it satisfies the same statement. Since $\alpha \in M$ was arbitrary, we have shown that κ is virtually extendible in M. Again using $M \prec N$, we get that κ is virtually extendible in N. It follows, using the j-elementarity, that M thinks that κ is a limit of virtually extendible cardinals. Now fix some $\delta < \kappa$ which M thinks is virtually extendible, and observe that since $V_\kappa \prec V^N_{j(\kappa)} = M$, it agrees that δ is virtually extendible. So V_κ is a model of proper class many virtually extendible cardinals. □

It follows from the above proof that if in a set-forcing extension, we have an embedding $j : M \to N$, with $\text{crit}(j) = \kappa$, of transitive ZFC-models from V such that $V_\kappa \in M$, $M \prec N$, and $M = V^N_{j(\kappa)}$, then V_κ is a model of proper class many virtually extendible cardinals.

In [BGS] together with Bagaria, we generalized Theorem 4.9 to show that if κ is 2-iterable, then V_κ is a model of a proper class of $C^{(n)}$-extendible cardinals for every $n < \omega$.

4.2. **Virtually n-huge* cardinals.**

Theorem 4.10. *Suppose κ is virtually huge*. Then V_κ is a model of proper class many virtually extendible cardinals.*

Proof. Fix some $\xi < \kappa$. We will show that there is $\bar\kappa > \xi$ which V_κ thinks is virtually extendible. Fix $\alpha > \kappa$ and $H \subseteq \text{Coll}(\omega, V_\alpha)$ such that in $V[H]$ there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) < \alpha$. Let $\kappa < \gamma < j(\kappa)$ and consider the restriction $j : V_\gamma \to V_{j(\gamma)}$. It follows, by Lemma 3.1, that in $V^{\text{Coll}(\omega, V_\gamma)}$ there is an elementary $j^* : V_\gamma \to V_\delta$ ($\delta = j(\gamma)$) with $\text{crit}(j^*) = \kappa$ and $j^*(\kappa) = j(\kappa) > \gamma$. Note that since $j(\kappa) < \alpha$, $j^2(\kappa)$ exists and $V_{j^2(\kappa)} \models \text{ZFC}$. Since γ was arbitrary, $V_{j^2(\kappa)}$ satisfies that for every $\kappa < \gamma < j(\kappa)$, in $V^{\text{Coll}(\omega, V_\gamma)}$ there is an elementary $j^* : V_\gamma \to V_\delta$ as above. So $V_{j^2(\kappa)}$ satisfies that there is $\xi < \bar\kappa < j(\kappa)$ (namely κ) such that for every $\bar\kappa < \gamma < j(\kappa)$ in $V^{\text{Coll}(\omega, V_\gamma)}$ there is an elementary j^* as above. By elementarity, $V_{j(\kappa)}$ satisfies that there is $\xi < \bar\kappa < \kappa$ such that for every $\bar\kappa < \gamma < \kappa$, there is an elementary $j^* : V_\gamma \to V_\delta$ with $\text{crit}(j^*) = \bar\kappa$ and $j^*(\bar\kappa) > \gamma$. So fix some such $\bar\kappa < \kappa$ above ξ. Since $V_\kappa \prec V_{j(\kappa)}$, it follows that V_κ satisfies that for all $\gamma > \bar\kappa$, in $V^{\text{Coll}(\omega, V_\gamma)}$ there is an elementary j^* as above. But this means that $\bar\kappa > \xi$ is virtually extendible in V_κ. □

Theorem 4.11. *A virtually $n+1$-huge* cardinal is a limit of virtually n-huge* cardinals.*

Proof. Suppose κ is $n+1$-huge* and fix some $\alpha > \kappa$ and $H \subseteq \text{Coll}(\omega, V_\alpha)$ such that in $V[H]$ there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^{n+1}(\kappa) < \alpha$. Fix some $j^n(\kappa) < \delta < j^{n+1}(\kappa)$ and consider the restriction $j : V_\delta \to V_{j(\delta)}$. By Lemma 3.1, in $V^{\text{Coll}(\omega, V_\delta)}$ there is an elementary $j^* : V_\delta \to V_\gamma$ with $\text{crit}(j^*) = \kappa$ and $j^{*n}(\kappa) < \delta$. So this statement must hold in $V_{j^{n+2}(\kappa)} \models \text{ZFC}$ as well. So κ must be a limit of virtually n-huge* cardinals. □

Theorem 4.12. *Virtually n-huge* cardinals are downward absolute to L for every $n < \omega$.*

Proof. Suppose κ is virtually n-huge*. Fix $\alpha > \kappa$ and $H \subseteq \text{Coll}(\omega, V_\alpha)$ such that in $V[H]$ there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j^n(\kappa) < \alpha$. Observe that $V^L_{j^n(\kappa)+1}$ is definable in V_α. Since $j^n(\kappa)$ is inaccessible $V^L_{j^n(\kappa)} = L_{j^n(\kappa)}$. So V_α just needs to determine which subsets of $L_{j^n(\kappa)}$ are in L. But since L_ξ for $\xi < (j^n(\kappa)^+)^L$ can be coded by subsets of $j^n(\kappa)$, it follows that V_α can construct codes of the L_ξ, and so it can determine when a subset of $L_{j^n(\kappa)}$ is in L. Let $\bar{\alpha} = j^n(\kappa) + 1$ and $\bar{\beta} = j^{n+1}(\kappa) + 1$. So j restricts to $j : V^L_{\bar{\alpha}} \to V^L_{\bar{\beta}}$. It follows that in $L^{\text{Coll}(\omega, V^L_{\bar{\alpha}})}$ there is an elementary $j^* : V^L_{\bar{\alpha}} \to V^L_{\bar{\beta}}$ with $\text{crit}(j^*) = \kappa$ and $j^{*n}(\kappa) < \bar{\alpha}$, and so κ is virtually n-huge* in L. □

Virtually n-huge* cardinals fit between $n+1$-iterable and $n+2$-iterable cardinals in the consistency strength hierarchy.

Theorem 4.13. *A virtually n-huge* cardinal is an $n+1$-iterable limit of $n+1$-iterable cardinals.*

The proof is an easier version of the argument given in the proof of Theorem 4.20.

Theorem 4.14. *Suppose κ is $n+2$-iterable. Then V_κ is a model of proper class many virtually n-huge* cardinals.*

Proof. We will argue that if κ is 3-iterable, then V_κ is a model of proper class many virtually huge* cardinals. Then we will explain how to extend the argument to the general case. So suppose κ is 3-iterable. Let M_0 be a weak κ-model containing V_κ for which there is a weakly amenable M_0-ultrafilter on κ with 3 well-founded iterated ultrapowers. We can assume without loss of generality that $M_0 = H^{M_0}_{\kappa^+}$. Thus, we get the following commutative diagram

$$\begin{array}{ccccc}
M_0 & \xrightarrow{j_0} & M_1 & \xrightarrow{h_0} & M_2 \\
\downarrow{\scriptstyle j_0} & & \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle l_0} \\
M_1 & \xrightarrow{j_1} & M_2 & \xrightarrow{h_1} & M_3 \\
\downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle h_1} & & \\
M_2 & \xrightarrow{j_2} & M_3 & &
\end{array}$$

where j_0, h_0, l_0 are ultrapowers by U_0, j_1, h_1 are ultrapowers by U_1, the first iterate of U_0, and j_2 is the ultrapower by U_2, the second iterate of U_0. This is because the

h_i ultrapowers are also iterates of U_0, and so in particular, the ultrapower of M_2 by U_0 is M_3. Since $j_0(\kappa) = h_0(\kappa) = l_0(\kappa)$ and $j_0^2(\kappa) = h_0^2(\kappa) = l_0^2(\kappa)$, it follows that $l_0^2(\kappa) \in M_2$ and $V^{M_2}_{l_0^2(\kappa)} = V^{M_3}_{l_0^2(\kappa)}$. So we can consider the restriction

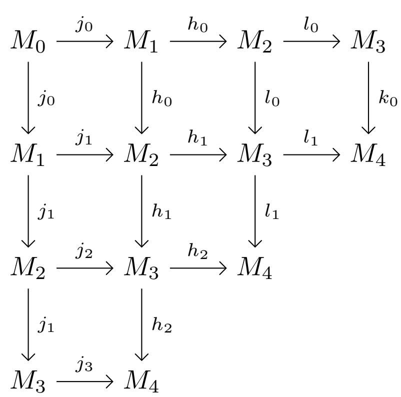
$$l_0 : V^{M_3}_{l_0^2(\kappa)} \to V^{M_3}_{l_0^3(\kappa)}.$$

Now fix some α such that $l_0(\kappa) < \alpha < l_0^2(\kappa)$ and consider the restriction

$$l_0 : V^{M_3}_{\alpha} \to V^{M_3}_{l(\alpha)}.$$

Observe that $l_0(\alpha) < l_0^3(\kappa)$, and so $V^{M_3}_{l(\alpha)} \in V^{M_3}_{l_0^3(\kappa)}$. Thus, $V^{M_3}_{l_0^3(\kappa)}$ satisfies that in $V^{\text{Coll}(\omega,V_\alpha)}$ there is an elementary $j : V_\alpha \to V_\beta$ with $\text{crit}(j) = \kappa$ and $j(\kappa) < \alpha$. Now fix some $\xi < \kappa$. The model $V^{M_3}_{l_0^3(\kappa)}$ satisfies that there are $\xi < \bar\kappa < \bar\alpha < l_0^2(\kappa)$ such that in $V^{\text{Coll}(\omega,V_{\bar\alpha})}$ there is an elementary $j : V_{\bar\alpha} \to V_{\bar\beta}$ with $\text{crit}(j) = \bar\kappa$ and $j(\bar\kappa) < \bar\alpha$. Thus, $V^{M_3}_{l_0(\kappa)}$ satisfies that there are $\xi < \bar\kappa < \bar\alpha < \kappa$ such that in $V^{\text{Coll}(\omega,V_{\bar\alpha})}$ there is an elementary $j : V_{\bar\alpha} \to V_{\bar\beta}$ with $\text{crit}(j) = \bar\kappa$ and $j(\bar\kappa) < \bar\alpha$. Fixing some such $\bar\kappa$ and $\bar\alpha$, and using that V_κ is elementary in $V^{M_3}_{l_0(\kappa)}$, it follows that $\bar\kappa$ is virtually huge* in V_κ.

To argue that if κ is 4-iterable, then V_κ is a model of proper class many virtually 2-huge* cardinals, we would use the embedding k_0 from the following commutative diagram.



Now it should be clear how to argue for the general case. $\square$

4.3. **Virtually rank-into-rank cardinals.**

Theorem 4.15. *Every virtually rank-into-rank cardinal is a virtually n-huge* limit of virtually n-huge* cardinals for every $n < \omega$.*

Proof. Clearly every virtually rank-into-rank cardinal is virtually n-huge* for every $n < \omega$. Suppose κ is virtually rank-into-rank and fix $\lambda > \kappa$ and $H \subseteq \text{Coll}(\omega, V_\lambda)$ such that in $V[H]$ there is an elementary $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$. Fix $n < \omega$ and let $\alpha > j^n(\kappa)$. Consider the restriction $j : V_\alpha \to V_{j(\alpha)}$. By Lemma 3.1, V_λ satisfies that in $V^{\text{Coll}(\omega,V_\lambda)}$ there is an elementary $j^* : V_\alpha \to V_\beta$ with $\text{crit}(j^*) = \kappa$ and $j^{*n}(\kappa) < \alpha$. So κ must be a limit of virtually n-huge* cardinals. $\square$

Theorem 4.16. *Assuming $0^\#$ exists, every Silver indiscernible is virtually rank-into-rank, and hence also virtually n-huge* for every $n < \omega$.*

This follows from the argument in Section 3 that Kunen's Inconsistency does not hold for virtual embeddings.

Theorem 4.17. *The least ω-Erdős cardinal $\kappa(\omega)$ is a limit of virtually rank-into-rank cardinals.*

Proof. Suppose $\kappa = \kappa(\omega)$ and fix $\xi < \kappa$. Fix $D \subseteq \kappa$ coding V_κ. Let $I = \{\alpha_i \mid i < \omega\} \subseteq (\xi, \kappa)$ be indiscernibles for the structure $(L_{\kappa^+}[D]; \in, (\eta : \eta \leq \xi))$. Let $L_\alpha[B]$ be isomorphic to $\mathrm{Hull}^{L_{\kappa^+}[D]}((\xi+1) \cup I)$. Observe that κ is in X because it is definable without parameters. Let $\bar\kappa$ be the image of κ under the Mostowski collapse and let β_i be the image of α_i for $i < \omega$. Then $\bar I = \{\beta_i \mid i < \omega\}$ are generating indiscernibles for $L_\alpha[B]$ relative to $\xi + 1$. Let $j : L_\alpha[B] \to L_\alpha[B]$ be any shift of indiscernibles embedding. Observe that $\mathrm{crit}(j) > \xi$ and $j(\bar\kappa) = \bar\kappa$. Thus, we can consider the restriction $j : V_{\bar\kappa}^{L_\alpha[B]} \to V_{\bar\kappa}^{L_\alpha[B]}$. By Lemma 3.1, in $L_\alpha[B]^{\mathrm{Coll}(\omega, V_{\bar\kappa})}$ there is an elementary $j^* : V_{\bar\kappa}^{L_\alpha[B]} \to V_{\bar\kappa}^{L_\alpha[B]}$ with $\mathrm{crit}(j^*) > \xi$. So by elementarity via the collapse map, $L_{\kappa^+}[A]$ satisfies that in $L_{\kappa^+}[A]^{\mathrm{Coll}(\omega, V_\kappa)}$ there is an elementary $j^* : V_\kappa \to V_\kappa$ with $\mathrm{crit}(j^*) > \xi$. Thus, we have shown that there is a virtually rank-into-rank cardinal above ξ. □

Since in the argument above $\bar\kappa$ could end up being the supremum of $\bar I$, we don't necessarily get the strong version of virtually rank-into-rank cardinals from this argument. The consistency of the strong version of virtually rank-into-rank cardinals can be obtained from a slightly stronger assumption.

Theorem 4.18. *Suppose $L_\delta \models \mathrm{ZFC}$ with $\kappa = \kappa(\omega) < \delta$. Then in L, κ is a limit of virtually rank-into rank cardinals with embeddings $j : V_\lambda \to V_\lambda$ where λ much larger than the supremum of the critical sequence of j.*

Proof. Fix $\xi < \kappa$ and let $I = \{\alpha_i \mid i < \omega\} \subseteq (\xi, \kappa)$ be indiscernibles for $(L_\delta; \in, (\eta \colon \eta \leq \xi))$. Let $L_{\bar\delta}$ be isomorphic to $\mathrm{Hull}^{L_\delta}((\xi+1) \cup I)$. Let $\bar\kappa$ and β_i be images under the Mostowski collapse of κ and α_i respectively, and let $\bar I = \{\beta_i \mid i < \omega\}$. Let λ be the supremum of $\bar I$. Since every element of $L_{\bar\delta}$ is definable from elements in $(\xi+1) \cup \bar I$, $\lambda = \varphi(\vec\eta, \beta_0, \ldots, \beta_n)$ for some $\vec\eta \leq \xi$, $n < \omega$, and a formula $\varphi(x_0, \ldots, x_n)$. Let $j : L_{\bar\delta} \to L_{\bar\delta}$ be a shift of indiscernibles embedding such that $j(\beta_i) = \beta_i$ for $i \leq n$. By construction, $j(\lambda) = \lambda$, and so $j(\beta) = \beta$ for any β that is definable without parameters from λ in $L_{\bar\delta}$, e.g. $\beta = \lambda + i$, the i-th ordinal successor of λ, or $\beta = \lambda^{(+i)}$, the i-th cardinal successor of λ ($i \in \omega$). Thus, for any such β, j restricts to $j : V_\beta^{L_{\bar\delta}} \to V_\beta^{L_{\bar\delta}}$. Now we proceed as in the proof of Theorem 4.17 above. □

Virtually rank-into-rank cardinals fit between ω-iterable and $\omega+1$-iterable cardinals in the consistency strength hierarchy.

Theorem 4.19. *An $\omega + 1$-iterable cardinal implies the consistency of a virtually rank-into-rank cardinal.*

Proof. Suppose κ is $\omega + 1$-iterable. By Theorem 4.17, there is an ω-Erdős cardinal below κ. Thus, by Theorem 4.17, there are many virtually rank-into-rank cardinals below κ. □

Indeed an $\omega + 1$-iterable cardinal implies the consistency of the strong version of virtually rank-into-rank cardinals because if κ is $\omega + 1$-iterable, then L_κ is a model of ZFC with an ω-Erdős cardinal.

Theorem 4.20. *Every virtually rank-into-rank cardinal is an ω-iterable limit of ω-iterable cardinals.*

Proof. Suppose κ is virtually rank-into-rank and fix $\lambda > \kappa$ and $H \subseteq \text{Coll}(\omega, V_\lambda)$ such that in $V[H]$ there is an elementary $j : V_\lambda \to V_\lambda$ with $\text{crit}(j) = \kappa$. To argue that κ is ω-iterable, we must show that every $A \subseteq \kappa$ is contained in a weak κ-model M for which there is a weakly amenable M-ultrafilter U on κ with ω-many well-founded iterated ultrapowers (but not necessarily a well-founded direct limit).

Working in $V[H]$, we are going to construct a suitable commuting system

$$\{j_{mn} : M_m \to M_n \mid m < n < \omega\}$$

of embeddings with $M_0 = H_{\kappa^+}$ (of V) such that U_0, the M_0-ultrafilter generated by κ_0 via j_0, is weakly amenable. So let U be the weakly amenable H_{κ^+}-ultrafilter on κ generated by j. Let $\{a_n \mid n \in \omega\}$ be an enumeration of H_{κ^+} in $V[H]$. As in the proof of Theorem 4.8, we construct in $V[H]$ an increasing sequence $\langle M_0^n \mid n < \omega\rangle$ of weak κ-models $M_0^n \prec H_{\kappa^+}$, each of which is in V_λ, whose union is H_{κ^+} such that for all $n < \omega$, we have $a_{n+2}, M_0^n, M_0^n \cap U \in M_0^{n+1}$. Recall that every restriction $j : M_0^n \to j(M_0^n)$ is in V_λ (but the sequence is not), and hence in $H_{j(\kappa)^+}$. Let $j(M_0^n) = M_1^n$ and let $M_1 = \bigcup_{n<\omega} M_1^n$ be the union of the elementary chain of the M_1^n. Finally, let $j_0 : M_0 \to M_1$ be defined by $j_0(x) = j(x)$ for all $x \in M_0$. Let $\kappa_0 = \text{crit}(j_0) = \kappa$. Now we explain how to define the next embedding j_1. Let $M_2^n = j(M_1^n)$ and let $M_2 = \bigcup_{n<\omega} M_2^n$. We define $j_1 = \bigcup_{n<\omega} j(j_0 \restriction M_0^n)$, so that $j_1 : M_1^n \to M_2^n$. Observe first of all that the definition makes sense because each restriction $j_0 : M_0^n \to M_1^n$ is in $H_{j(\kappa)^+} \subseteq V_\lambda$, and all the restrictions cohere. So we get that $j_1 : M_1 \to M_2$. The new embedding j_1 again has the property that all restrictions $j_1 : M_1^n \to M_2^n$ are in V_λ, more precisely, in $H_{j^2(\kappa)^+}$. Let $\kappa_1 = \text{crit}(j_1) = j(\kappa)$. Continuing in this manner we obtain (in the obvious fashion) a commuting system of embeddings $\{j_{mn} : M_m \to M_n \mid m < n < \omega\}$.

We claim that this commuting system is suitable. Let U_n be the M_n-ultrafilter generated by $\kappa_n = j^n(\kappa)$ via j_n and let $U_n^m = U_n \cap M_n^m$. We will show that $U_{n+1}^m = j_n(U_n^m)$, which suffices. Since U_n^m is, by definition, the ultrafilter generated by κ_n via $j_n \restriction M_n^m$, we get that, by elementarity, $j(U_n^m)$ is the ultrafilter generated by κ_{n+1} via $j(j_n \restriction M_n^m)$, which is by definition U_{n+1}^m. In particular, $j_0(U_0^m) = U_1^m$ for all $m < \omega$. So we can assume inductively that $j_n(U_n^m) = U_{n+1}^m$. Now we compute

$$j_{n+1}(U_{n+1}^m) = j(j_n)(U_{n+1}^m) = j(j_n)(j(U_n^m)) = j(j_n(U_n^m)) = j(U_{n+1}^m) = U_{n+2}^m.$$

Now we use an ill-founded tree argument exactly as in the proof of Theorem 3.11 of [GW11] to show that a suitable system with the same properties exists in V. So by Lemma 4.2, for every $A \subseteq \kappa$, we can construct a weak κ-model $M \prec H_{\kappa^+}$ for which there is a weakly amenable M-ultrafilter on κ with ω-many well-founded iterated ultrapowers. So finally, it follows by elementarity that κ is a limit of ω-iterable cardinals. □

It follows from the above proof that if in a set-forcing extension we have an elementary $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $j^n(\kappa) < \alpha$ such that $N \in V$ and N is closed under κ-sequences in V, then κ is an $n+1$-iterable limit of $n+1$-iterable cardinals.

4.4. **Silver cardinals.**

Theorem 4.21. *Assuming $0^\#$ exists, there are Silver cardinals in L.*

Proof. Let κ be any uncountable cardinal of V. Then V has a club C in κ of generating indiscernibles for $V_\kappa^L = L_\kappa$ of order-type κ. Suppose $H \subseteq \text{Coll}(\omega, L_\kappa)$ is V-generic. Clearly $C \in V[G]$, and so since κ is countable in $L[H]$, by Lemma 3.4, $L[H]$ has some club C^* in κ of generating indiscernibles for L_κ of order-type κ. □

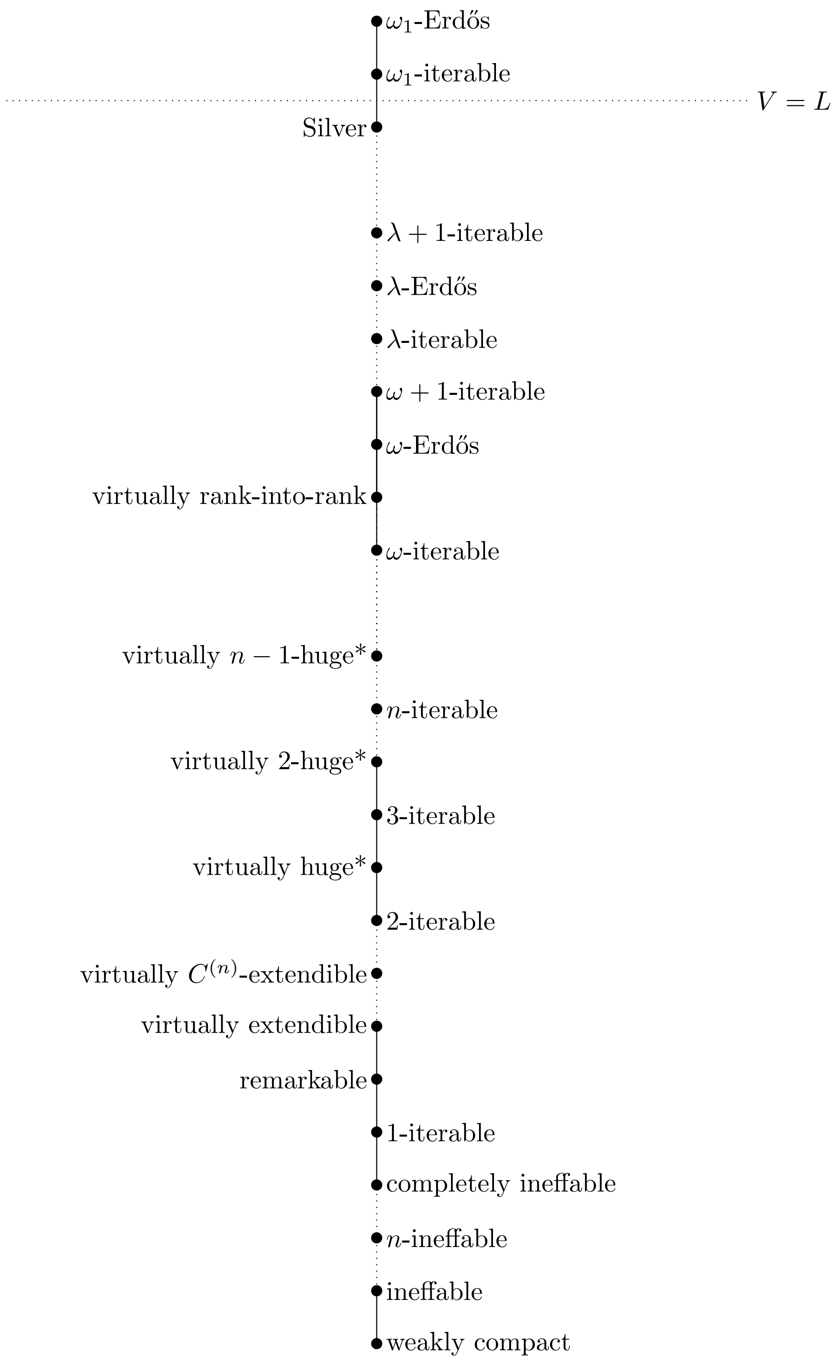
Theorem 4.22. *Silver cardinals are downward absolute to L.*

Proof. Suppose that κ is a Silver cardinal. Let $H \subseteq \text{Coll}(\omega, V_\kappa)$ be V-generic and fix, in $V[H]$, a club C in κ of generating indiscernibles for V_κ of order-type κ. Elements of C are clearly inaccessible, and so we have $V_\kappa^L = L_\kappa$. Since L_κ is definable in V_κ it follows that elements of C are indiscernibles for L_κ as well. Also, C generates L_κ because the Skolem closure of C collapses to some L_β, and if $\beta < \kappa$, then κ embeds into β, which is impossible. So, by Lemma 3.4, $L[H]$ has some club C^* in κ of generating indiscernibles of order-type κ for L_κ. □

Theorem 4.23. *If $C \in V[H]$, a forcing extension by* $\text{Coll}(\omega, V_\kappa)$, *is a club in κ of generating indiscernibles for V_κ of order-type κ, then each $\xi \in C$ is $<\omega_1$-iterable.*

Proof. By indiscernibility, it suffices to show that the first element, call it δ, of C is $<\omega_1$-iterable. In $V[H]$, let's define the following commuting system of embeddings $\{j_{\alpha\beta} \mid \alpha < \beta < \omega_1^V\}$. Let $C = \{\xi_\eta \mid \eta < \kappa\}$ be the enumeration of C in order-type κ, so that $\delta = \xi_0$. Let $j_{\alpha\beta}$ be a shift of indiscernibles embedding such that $j(\xi_0) = \xi_\beta$ and, more generally, $j_{\alpha\beta}(\xi_\eta) = \xi_\nu$, where if $\eta = \alpha + \gamma$, then $\nu = \beta + \gamma$. It is not difficult to see that this commuting system is suitable (see Theorem 4.4). So by the argument in the end of the proof of Theorem 4.20, δ is α-iterable for every countable α of V, and hence $<\omega_1$-iterable (by definition). □

Here is a diagram summarizing the consistency strength hierarchy around the virtual large cardinals. The dotted line is meant to indicate that there is some hierarchy in between.



5. Overview of applications

The notion of taking a property characterized by the existence of elementary embeddings of certain set-sized structures and considering a virtual version of the property where the embeddings exist in the generic multiverse extends beyond large cardinal axioms.

In [BGS], together with Bagaria, we considered a virtual version of Vopěnka's Principle, the *Generic Vopěnka's Principle*. Recall that Vopěnka's Principle is a scheme stating that for every proper class $\mathcal{C}$ of first-order structures in some fixed language, there are $B \neq A \in \mathcal{C}$ such that B elementarily embeds into A. Let's call $\mathrm{VP}(\boldsymbol{\Sigma}_\mathrm{n})$, the first-order expressible fragment of Vopěnka's Principle for Σ_n-definable (with parameters) classes. The Generic Vopěnka's Principle is then the scheme stating that for every proper class $\mathcal{C}$ of first-order structures in some fixed language, there are $B \neq A \in \mathcal{C}$ such that B elementarily embeds into A in some set-forcing extension. Let's call $\mathrm{gVP}(\boldsymbol{\Sigma}_\mathrm{n})$ the fragment of Generic Vopenka's Principle for Σ_n-definable (with parameters) classes. Bagaria showed in [Bag12] that $\mathrm{VP}(\boldsymbol{\Sigma}_2)$ holds if and only if there is a proper class of supercompact cardinals and for $n \geq 1$, $\mathrm{VP}(\boldsymbol{\Sigma}_{\mathrm{n+2}})$ holds if and only if there is a proper class of $C^{(n)}$-extendible cardinals. It turns out that $\mathrm{gVP}(\boldsymbol{\Sigma}_2)$ is equiconsistent with a proper class of remarkable (virtually supercompact) cardinals and for $n \geq 1$, $\mathrm{gVP}(\boldsymbol{\Sigma}_{\mathrm{n+2}})$ is equiconsistent with a proper class of virtually $C^{(n)}$-extendible cardinals [BGS]. But interestingly in the virtual case the schemes are not equivalent: it is consistent that there is a model of Generic Vopenka's Principle in which there are no remarkable cardinals [GH].

In [CS12], Claverie and the second author provided the following characterization of the Proper Forcing Axiom PFA: whenever $\mathcal{M} = \langle M, \in, \{R_i \mid i < \omega_1\}\rangle$ is a transitive model with some ω_1-many relations, $\varphi(x)$ is a Σ_1-formula, and $\mathbb{Q}$ is a proper forcing such that $\Vdash_{\mathbb{Q}} \varphi(\mathcal{M})$, then there is in V some transitive $\bar{\mathcal{M}} = \langle \bar{M}, \in, \{\bar{R}_i \mid i < \omega_1\}\rangle$ together with some elementary embedding $j : \bar{\mathcal{M}} \to \mathcal{M}$ such that $\varphi(\bar{\mathcal{M}})$ holds. The second author introduced the weak Proper Forcing Axiom wPFA, a virtual version of PFA, defined by stating that the embedding j exist in some set-forcing extension. The axiom wPFA is equiconsistent with a remarkable cardinal [BGS]. Fuchs used this idea to define a virtual version wSCFA of the forcing axiom for subcomplete forcing SCFA, which has a characterization identical to the one above with proper $\mathbb{Q}$ replaced by subcomplete $\mathbb{Q}$. He showed that wSCFA is also equiconsistent with a remarkable cardinal. He also defined a virtual version of the resurrection axiom for a class Γ of forcing notions (introduced in [HJ14]). The resurrection axiom for a class Γ states that for every $\kappa \geq \omega_2$ and forcing notion $\mathbb{P} \in \Gamma$, in every forcing extension $V[G]$ by $\mathbb{P}$ there is a forcing notion $\mathbb{Q} \in \Gamma^{V[G]}$ and a λ such that in every further forcing extension $V[G][H]$ by $\mathbb{Q}$ there is an elementary $j : H_\kappa^V \to H_\lambda^{V[G][H]}$. In the virtual version, the embedding j is required to exist in a forcing extension of $V[G][H]$. Fuchs showed in [Fucb] that for Γ being the class of proper, semi-proper, countably closed, or subcomplete forcing, the resurrection axiom for Γ is equiconsistent with a virtually extendible cardinal.

In a current work [SW], Wilson and the second author show that another virtual large cardinal, the virtually Shelah for supercompactness cardinal, is equiconsistent with the statement that every universally Baire set has a perfect subset. A cardinal κ is *virtually Shelah for supercompactness* if for every function $f : \kappa \to \kappa$ and $B \subseteq \kappa$, there is $\bar{\kappa} < \lambda$ such that in a set-forcing extension there is an elementary

$j : V_{\max(f(\bar{\kappa}),\bar{\kappa}+1)} \to V_\lambda$ with $\text{crit}(j) = \bar{\kappa}$ and $j(\bar{\kappa}) = \kappa$. It is not difficult to see that if κ is virtually Shelah for supercompactness, then V_κ is a model of proper class many $C^{(n)}$-extendible cardinals for every $n < \omega$, and that if κ is 2-iterable, then V_κ is a model of proper class many virtually Shelah for supercompactness cardinals.

6. Alternative definitions of virtual large cardinals

Our template for the definition of virtual large cardinals requires the large cardinal notion to be characterized by the existence of elementary embeddings $j : V_\alpha \to V_\beta$. This template is quite restrictive as we saw for example in the case of n-huge cardinals. Its main advantage is that it gives a hierarchy of large cardinal notions that mirrors the hierarchy of its actual counterparts, and the large cardinals have other desirable properties such as being downward absolute to L.

It is natural to explore what happens if we drop the requirement that the target of the virtual large cardinal embedding has to be a rank initial segment V_β. For instance, a natural alternative definition of a virtually supercompact cardinal κ is that for every $\lambda > \kappa$, there is $\alpha > \lambda$ and a transitive model N closed under λ-sequences such that in a set-forcing extension there is an elementary $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. It turns out that this is an equivalent characterization of remarkable cardinals. Yet another equivalent characterization of remarkables is that for every $\lambda > \kappa$, there is $\alpha > \lambda$ and a transitive model N with $V_\lambda \subseteq N$ such that in a set-forcing extension there is an elementary $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$ and $j(\kappa) > \lambda$. The last characterization suggests that closure requirements on the target model in V may not play a significant role in the theory of virtual large cardinals. This is illustrated also by the following potential definition of virtually n-huge cardinals. Dropping the requirement that the target of the virtual embedding should be a rank initial segment V_β, we can define that a cardinal κ is virtually n-huge whenever there is $\alpha > \kappa$ and a transitive model N such that in a set-forcing extension there is an elementary $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $j^n(\kappa) < \alpha$ and $N^{j^n(\kappa)} \subseteq N$ in V. While it seems natural to require that $N^{j^n(\kappa)} \subseteq N$ in V, this closure does not give additional strength to virtual embeddings. Indeed, by the remark following Theorem 4.9, the existence of a virtual embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $j(\kappa) < \alpha$ such that N is only closed under κ-sequences in V suffices to prove the consistency of a proper class of virtually extendible cardinals (the restriction of j to $V^N_{j(\kappa)}$ has the desired properties). Even worse, the existence of a virtual embedding $j : V_\alpha \to N$ with $\text{crit}(j) = \kappa$, $j^{n+1}(\kappa) < \alpha$ such that N is closed under κ-sequences in V gives the consistency of virtually $n+2$-iterable cardinals (by remark following Theorem 4.20), and hence the consistency of virtually n-huge* cardinals. Yet another issue with this proposed definition of virtually n-huge cardinals is that it is not clear that they are downward absolute to L.

The difficulties described above with the alternative definitions together with the desirable properties exhibited by the virtual large cardinals under the $j : V_\alpha \to V_\beta$ template suggest that this template is the natural one to adapt.

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