SET THEORY COURSE WINTER TERM 2020-21, EXERCISE SHEET NO. 4, NOV 16, 2020

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Let A be any set. We say that $S \subset \mathcal{P}(A)$ is a *chain* iff for all B, $C \in S$, $B \subset C$ or $C \subset B$. We say that $P \subset \mathcal{P}(A)$ is *inductive* iff for all chains $S \subset P$, $\bigcup P \in P$. We say that $P \subset \mathcal{P}(A)$ is of *finite character* iff for all $B \subset A$, $B \in P$ iff every finite subset of B is in P.

Problem 1. Show in ZF that the following statements are equivalent.

(a) AC, the axiom of choice.
(b) If A is a set, and $P \subset \mathcal{P}(A)$ is a nonempty inductive set, then there is some $B \in P$ such that no $B' \supsetneq B$ is also in P.
(c) If A is a set, and $P \subset \mathcal{P}(A)$ is a nonempty set of finite character, then there is some $B \in P$ such that no $B' \supsetneq B$ is also in P.
(d) If A is a set, then there is some well-ordering of A.

If you're more ambitious, show the equivalence in Z.

Let $(a, \leq)$ be an ordered set, where $\leq$ is linear. We call a *dense with no endpoints* iff for all x, $y \in a$ with $x < y$ there are z, u, $v \in a$ such that $z < x < u < y < v$. $\mathbb{Q}$, equipped with the natural order, is dense without endpoints.

Problem 2. Show that if $(a, \leq)$, $(a', \leq')$ are dense linear orders with no endpoints, where both a and a' are countable, then

$$(a, \leq) \cong (a', \leq').$$

[Hint. Construct an isomorphismus in a back and forth fashion.] Show that this becomes false if e.g. $a \sim \mathbb{R} \sim a'$.

Problem 3. Show in ZF that for every set A there is some well-ordered set $(a, \leq)$ such that there is no injection $f \colon a \to A$. [Hint. Consider the set of all well-orderings of subsets of A.]

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