# SET THEORY COURSE WINTER TERM 2020-21, EXERCISE SHEET NO. 1, NOV 03, 2020

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**Hand in by Nov 08, 2020.**

For sets $X$, $Y$, $X \sim Y$ was defined in the lecture.

**Problem 1.** For reals $a \leq b$ let $(a, b) = \{x \in \mathbb{R} \colon a < x < b\}$. Show that if $a < b$, then $(a, b) \sim \mathbb{R}$.

**Problem 2.** Show naively that the countable union of countable sets is countable, i.e., if $x = \bigcup_{n \in \mathbb{N}} x_n$ and every $x_n$ is countable, then $x$ is countable.

**Problem 3.** Show that there are countably many rational numbers (i.e., $\mathbb{Q} \sim \mathbb{N}$). Also show that there are only countably many algebraic real numbers. (A real number $x$ is algebraic iff it is a solution to a non-zero polynomial with rational coefficients, otherwise $x$ is called transcendental.) Conclude that there are uncountably many transcendental numbers.

**Problem 4.** Show that $\mathcal{P}(\mathbb{N}) \sim \mathbb{R}$. Here $\mathcal{P}(X)$ is the power set of $X$.

For a set $A$ of reals, we let $A' = \{x \in \mathbb{R} \colon \forall a, b(a < x < b \to (a, b) \cap (A \setminus \{x\}) \neq \emptyset\}$ be the set of accumulation points of $A$. Define $A^0 = A$ and $A^{n+1} = (A^n)'$ for natural numbers $n$. Define $A^\infty = \bigcap_{n \in \mathbb{N}} A^n$. Also define $A^{\infty+(n+1)} = (A^{\infty+n})'$ for natural numbers $n$ (where $\infty + 0 = \infty$).

**Problem 5.** Show that $A'$ is always closed. Show that if $A$ is closed, then $A' \subset A$. Construct closed sets $A$ and $B$ with $A' = A$ and $B' \subsetneq B$. In fact, given $n \in \mathbb{N}$, construct a closed set $A_n$ with $(A_n)^n \neq \emptyset$ and $(A_n)^{n+1} = \emptyset$. Construct a close set $A$ with $A^n \neq \emptyset$ for all $n \in \mathbb{N}$, but $A^\infty = \emptyset$. Given $n \in \mathbb{N}$, construct a closed set $B_n$ with $(B_n)^{\infty+n} \neq \emptyset$ and $(B_n)^{\infty+(n+1)} = \emptyset$.

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