Burstin bases and well-ordering the reals

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Reflections on Set Theoretic Reflection
Sant Bernat, Montseny, Nov 19, 2018
“Paradoxical” sets of reals

Definition
Let $A \subseteq \mathbb{R}$ uncountable. We say that $A$ is

- a Vitali set if $A$ is the range of a selector for the equivalence relation $\sim_V$ defined over $\mathbb{R} \times \mathbb{R}$ by $x \sim_V y \iff x - y \in \mathbb{Q}$;
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- a Luzin set if for every $M \in \mathcal{M}$ -the ideal of all meager sets- we have $|A \cap M| \leq \aleph_0$;
- a Bernstein set if for every perfect set $P \subseteq \mathbb{R}$ we have $A \cap P \neq \emptyset$ and $(\mathbb{R} \setminus A) \cap P \neq \emptyset$;
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Let $A \subseteq \mathbb{R} \times \mathbb{R}$. We say that $A$ is

- a **Mazurkiewicz set** iff $|A \cap \ell| = 2$ for every straight line $\ell \subseteq \mathbb{R} \times \mathbb{R}$. 

Suppose $V \models \text{ZF}$ and suppose that a Hamel basis $H$ exists. Then there is a Vitali set.
Basic definitions and results

Folklore and classical results

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“Paradoxical” sets and well-ordering the reals

All these classical constructions may be obtained by assuming ZF plus the existence of a well-ordering of $\mathbb{R}$ (or, ZF plus there is a well-ordering of $\mathbb{R}$ of order type $\omega_1$ in the case of Luzin and Sierpiński sets).
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**Question**

*Can we have those “paradoxical” sets of reals in the absence of a well-ordering of \( \mathbb{R} \)?
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Recall the Cohen-Halpern-Lévy model: Let $g$ be $\mathcal{C}(\omega)$-generic over $L$ ($\mathcal{C}(\omega)$ being the finite support product of $\omega$ Cohen forcings), and let $A = \{c_n : n < \omega\}$ be the set of Cohen reals added by $g$. 

$$H = \text{HOD}_{A \cup \{A\}}^L[g].$$
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**Theorem (D. Pinkus and K. Prikry, S. Feferman, 1975)**

*In the Cohen-Halpern-Lévy model \( H \), in which \( A \) is an infinite set of reals with no (infinite) countable subset (i.e., \( AC_\omega(\mathbb{R}) \) fails), there is a Luzin set as well as a Vitali set.*
"Paradoxical" sets and well-ordering the reals

Question (D. Pincus and K. Prikry, 1975)

"We would be interested in knowing whether a Hamel basis for \( \mathbb{R} \) over \( \mathbb{Q} \) (the rationals) exists in \( H \) or in any other model in which \( \mathbb{R} \) cannot be well ordered."
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Question (variant 1 of Pinkus-Prikry)

Is the existence of a Hamel basis (or, the simultaneous existence of all of those "paradoxical" sets of reals) compatible with ZF plus the negation of AC\( _\omega(\mathbb{R}) \)?
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Is the existence of a Hamel basis (or, the simultaneous existence of all of those “paradoxical” sets of reals) compatible with ZF plus DC plus the non-existence of a well-order of $\mathbb{R}$?
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Theorem (A. Blass, 1984)

In ZF, if every vector space has a basis, then the Axiom of Choice holds true.
Theorem (Beriashvili, Sch., Wu and Yu, 2018)

*In the Cohen-Halpern-Lévy model $H$ there is a Hamel basis and a Bernstein set (but there are no Sierpiński sets).*
Burstin bases and non-$\operatorname{AC}_\omega(\mathbb{R})$

Theorem (Beriashvili, Sch., Wu and Yu, 2018)

In the Cohen-Halpern-Lévy model $H$ there is a Hamel basis and a Bernstein set (but there are no Sierpiński sets).

In $H$, there is also a Hamel basis which contains a perfect set.
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I don’t know if there is a Mazurkiewicz set in $H$. 
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Let $H^*$ be the following variant of the Cohen-Halpern-Lévy model: Let $h$ be $\mathbb{S}(\omega)$-generic over $L$ ($\mathbb{S}(\omega)$ being the finite support product of $\omega$ Sacks forcings). Let $B = \{d_n : n < \omega\}$ be the set of Sacks reals added by $h$.

$$H^* = \text{HOD}_{L[h]}^{L[h]}_{B \cup \{B\}}.$$
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**Theorem**

In $H^*$ there is Sierpiński set, a Luzin set, a Hamel basis which contains a perfect set, as well as a Burstin basis.
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By replacing Sacks forcing $\mathbb{S}$ above by a refinement of Sacks forcing which is due to Jensen, one obtains a model $H^{**}$ of ZF plus non-AC$_\omega$($\mathbb{R}$) plus there is $\Delta^1_3$ Sierpiński set, a $\Delta^1_3$ Luzin set, a $\Delta^1_3$ Hamel basis which contains a perfect set, as well as a $\Delta^1_3$ Burstin basis.
Theorem (Brendle, Castiblanco, Sch., Wu, Yu)

There is a model $W$ of ZF + DC such that in $W$ the reals cannot be well-ordered and $W$ contains Luzin as well as Sierpiński sets and also a Burstin basis.
Basic definitions and results

Luzin and Sierpiński sets in the Sacks model

Lemma (Folklore)

Let \( P \) be a forcing notion satisfying the Sacks property and let \( G \) be a \( P \)-generic filter over \( V \). Then:

1. For every null set \( N \subseteq \omega^\omega \) in \( V \) there is a \( G \)-null set \( \bar{N} \subseteq \omega^\omega \) coded in \( V \) such that \( N \subseteq \bar{N} \).
2. Similarly, for every meager set \( M \subseteq \omega^\omega \) in \( V \), there is a meager set \( \bar{M} \subseteq \omega^\omega \) coded in \( V \) such that \( M \subseteq \bar{M} \).

Corollary

If \( P \) has the Sacks property, then \( P \) preserves Luzin and Sierpiński sets.
Lemma (Folklore)

Let $\mathbb{P}$ be a forcing notion satisfying the Sacks property and let $G$ be a $\mathbb{P}$-generic filter over $V$. Then:

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Corollary
If $\mathbb{P}$ has the Sacks property, then $\mathbb{P}$ preserves Luzin and Sierpiński sets.
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Let $s$ be $S(\omega_1)$-generic over $L$, and let $R^* = R \cap L[s]$. Then

(a) $L(R^*) \models$ ZF plus DC plus “there is no w.o. of the reals,”
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(a) $L(\mathbb{R}^*) \models \text{ZF plus DC plus "there is no w.o. of the reals,"}$

(b) there is a Luzin set as well as a Sierpiński set in $L(\mathbb{R}^*)$, but

(c) there is no Vitali set (and hence no Hamel basis) in $L(\mathbb{R}^*)$. 
Adding generically a Burstin set

First try. We define a partial order \( P^0_B \) adding a generic Burstin basis.
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We say $p \in \mathbb{P}_B^0$ if and only if $p$ is a countable linearly independent set of reals.
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Let $b$ be $\mathbb{P}^0_B$-generic over $L(\mathbb{R}^*)$. Then $B = \bigcup b$ is a Hamel basis in $L(\mathbb{R}^*)[b]$. 
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**Problem:** $L(\mathbb{R}^*)[b] \models \text{ZFC plus CH.}$
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Notice that $\mathbb{P}_B \neq \emptyset$. 
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Notice that $\mathbb{P}_B \neq \emptyset$. However the *extendability* of $\mathbb{P}_B$ is not obvious.
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Notice that $\mathbb{P}_B \neq \emptyset$. However the **extendability** of $\mathbb{P}_B$ is not obvious.

**Extendability:** If $p \in \mathbb{P}_B$ is such that $L[x] \models \text{"}p \text{ is a Burstin basis\"}$ and if $y \in \mathbb{R}^{L[x,y]} \setminus L[x]$, then there is some $q \leq_{\mathbb{P}_B} p$ such that $q$ is a Burstin basis in $\mathbb{R}^{L[x,y]}$. 

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**Adding generically a Burstin set**
The Marczewski ideal and new generic reals

**Definition (Marczewski)**

A set \( X \subseteq \mathbb{R} \) is in \( s^0 \) if and only if for every perfect set \( P \) there is a perfect subset \( Q \subseteq P \) with \( Q \cap X = \emptyset \).
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Let $W \subseteq V$ be an inner model such that $W \models \text{CH}$. If $\mathbb{R} \cap V \setminus W \neq \emptyset$, then

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**Corollary**
Let \( x, y \) be reals such that \( y \notin L[x] \), and let \( \{z_0, z_1, \ldots\} \in L[x, y] \cap [\mathbb{R}]^\omega \). Then
\[
\text{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \ldots\}) \in s_0^{L[x, y]}
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Corollary

Let $b \in L[x]$ be linearly independent, $x \in \mathbb{R}$. Let $y \in \mathbb{R} \setminus L[x]$. There is then some $p \supset b, p \in L[x, y]$ such that

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Extendability of $\mathbb{P}_B$

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**Lemma**

$L(\mathbb{R}^*)$ thinks that:

(a) *(Extendability)* If $p \in \mathbb{P}_B$ is such that $L[x] \models "p \text{ is a Burstin basis}"$ and if $y \in \mathbb{R}^{L[x, y]} \setminus L[x]$, then there is some $q \leq_{\mathbb{P}_B} p$ such that $q$ is a Burstin basis in $\mathbb{R}^{L[x, y]}$. 

But there is a variant of $\mathbb{P}_B$ which does add a Hamel basis over $L(\mathbb{R}^*)$ which is not a Burstin basis.
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**Lemma**

Let \( b \) be \( P_B \)-generic over \( L(R^*) \). Then

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Let $s$ be $S(\omega_1)$-generic over $L$, and let $R^* = R \cap L[s]$. Let $(b, m)$ be $P_B \times P_M$ generic over $L(R^*)$. Then $R^* = R \cap L(R)$ and

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(e) \( L(R)[b, m] \models \bigcup m \text{ is a } \text{Mazurkiewicz set}. \)
Per molts anys, Joan!