Stacking mice

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Abstract

We show that either of the following hypotheses imply that there is an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals. 1) There is a countably closed cardinal $\kappa \geq \aleph_3$ such that $\square_\kappa$ and $\square(\kappa)$ fail. 2) There is a cardinal κ such that κ is weakly compact in the generic extension by $\mathrm{Col}(\kappa, \kappa^+)$. Of special interest is 1) with $\kappa = \aleph_3$ since it follows from PFA by theorems of Todorcevic and Velickovic. Our main new technical result, which is due to the first author, is a weak covering theorem for the model obtained by stacking mice over $K^c \| \kappa$.

0 Introduction

It is a well-known conjecture that the consistency strength of the Proper Forcing Axiom is a supercompact cardinal. In this paper, we show that PFA implies the existence of an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals. In fact, we get indiscernibles for a proper class model of this large cardinal property. For the reader interested in determinacy, this is significantly beyond the consistency strength of $\mathrm{AD}_\mathbb{R}$ by theorems of Woodin.

As one might expect from [11], the only two consequences of PFA that are used to prove our lower bound are $2^{\aleph_0} = \aleph_2$ (Todorcevic [1] and Velickovic [22]) and the failure of $\square(\kappa)$ at all (regular) $\kappa \geq \aleph_2$ (Todorcevic [21]). Recall that $\square_\kappa$ implies $\square(\kappa^+)$. The

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papers Schimmerling [10], Schimmerling-Steel [13] and Steel [18] include steps towards measuring the large cardinal consistency strength of the existence of a singular cardinal κ such that $\square_\kappa$ fails. In Schimmerling [11], it is shown that if $\kappa \geq 2^{\aleph_0} \cdot \aleph_2$ is a regular cardinal and both $\square_\kappa$ and $\square(\kappa)$ fail, then for every $n < \omega$, there is an inner model with n Woodin cardinals; Steel (unpublished) extended the conclusion to infinitely many Woodin cardinals. Hypotheses about regular cardinals are more to our taste than singular cardinals because we need only apply PFA to posets of cardinality $(2^{\aleph_0})^+$ to see $2^{\aleph_0} = \aleph_2$ and the failure of $\square(\aleph_2)$ and $\square_{\aleph_2}$. For technical reasons, the least κ to which the results of this paper apply is not $\aleph_2$ but $\aleph_3$.

The papers mentioned above use the true core model, K. In the theory of K, one first builds the background certified core model, K^c, then defines K to be the Mostowski collapse of a certain elementary substructure of K^c. Many of the basic core model tools involving K are unknown or false for K^c. In this sense, K^c is less useful than K. On the other hand, in the current stage of knowledge, the anti-large-cardinal hypothesis under which one can establish the basic properties of K^c is much less severe than for K. So, in those instances in which we can make do with K^c, the conclusions are stronger. This was among our main inspirations.

Our work also builds on Andretta, Neeman and Steel [2] where the theory of K^c was developed under the assumptions 1) there is a measurable cardinal and 2) all premice are domestic. A non-domestic premouse N is one that has an initial segment $M \trianglelefteq N$ with a top extender F^M such that the strong cardinals of $M \| \mathrm{crit}(F^M)$ are unbounded in $\mathrm{crit}(F^M)$ and so are the Woodin cardinals of M. The relevant corollary in [2] is that if κ is a measurable cardinal and $\square_\kappa$ fails, then there is a non-domestic premouse. The corresponding M from their proof is linearly iterable by its top extender and, in this way, generates indiscernibles for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals. We will refer to such an M as a sharp.

The main new element in this paper is a technique, due to the first author, for producing a K^c-like fine structural model with the weak covering property at a given regular cardinal. We call it "stacking mice." We shall combine this technique with the argument of [11] to show the following.

Theorem 0.1 *Let $\kappa \geq \aleph_3$ be a regular cardinal. Assume that κ is countably closed in the sense that $\eta^{\aleph_0} < \kappa$ for every $\eta < \kappa$. Suppose that $\square(\kappa)$ and $\square_\kappa$ both fail. Then there is a sharp for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

Corollary 0.2 PFA *implies that there is a sharp for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

In subsequent work, cf. [7], the first and fourth authors used the mouse-stacking technique to develop the theory of K below a Woodin cardinal without assuming that there is a measurable cardinal or anything other than ZFC. (This was one of the main problems left open in Steel [17].)

The effect of the proof of Theorem 0.1 can also be expressed as follows.

Theorem 0.3 *Let $\kappa \geq \aleph_3$ be a regular cardinal. Assume that κ is countably closed in the sense that $\eta^{\aleph_0} < \kappa$ for every $\eta < \kappa$. Suppose that $\square(\kappa)$ and $\square_\kappa$ both fail. If the certified K^c exists in $V^{\mathrm{Col}(\kappa,\kappa)}$, then there is a subcompact cardinal in the certified K^c of $V^{\mathrm{Col}(\kappa,\kappa)}$.*

Concerning the phrase "the certified K^c exists" we refer the reader to Definition 2.7.

Another application of the methods developed here is given by the following set of theorems.

Theorem 0.4 *If κ is a weakly compact cardinal in $V^{\mathrm{Col}(\kappa,\kappa^+)}$, then there is a sharp for a proper class model with a proper class of strong cardinals and a proper class of Woodin cardinals.*

Theorem 0.5 *Suppose that κ is a weakly compact cardinal in $V^{\mathrm{Col}(\kappa,\kappa^+)}$. If the certified K^c exists in $V^{\mathrm{Col}(\kappa,\kappa^+)}$, then there is a superstrong cardinal in the certified K^c of $V^{\mathrm{Col}(\kappa,\kappa^+)}$.*

The paper is organized as follows. In the first section, we recall some necessary fine structural tools (which are taken from [10] and [5]). In the second section, we develop our K^c construction, the *certified* K^c; it is constructed by joining the approach of [2] with the one of [8]. Nothing is really new in the second section. The third section contains the new technique of producing a fine structural model which satisfies weak covering at a given regular cardinal κ. The key result will be Theorem 3.4 which says that if $\kappa \geq \aleph_3$ is an ω-closed regular cardinal with $2^{<\kappa} = \kappa$, and if the certified K^c exists, but K^c does not have a superstrong cardinal, then there is a mouse $\mathcal{S}$ end-extending $K^c || \kappa$ such that $\mathrm{cf}^V(\kappa^{+\mathcal{S}}) \geq \kappa$. The results in the third section are due to the first author. Similar in spirit to [11], the fourth section will then show how the proof of Theorem 3.4 gives a proof of Theorem 4.1 and thus proofs of Theorems 0.1 and 0.3; this application was discovered by the second, third, and fourth authors. The last section will produce proofs of Theorems 0.4 and 0.5 by exploiting an argument of the third author.

1 Some fine structure

In this section we summarize key fine structural facts which shall be exploited in the proofs of Theorems 0.1, 0.3, 0.4, and 0.5.

In much the same way and for the same reason as in [2], we shall work here with the *Jensen premice* of [5] (rather than with the *Mitchell-Steel premice* from [9]).[4] In what follows, the term "extender" will refer to an extender in the sense of [5, §1] (cf. also [24, 2.1, p. 48]), and term "premouse" will refer to a premouse in the sense of [5, §4] (cf. also [24, 9.1, p. 284]).

[4] We could have worked with Mitchell-Steel premice as well, but we would then have produced results which are weaker than Theorems 0.1 and 0.4. Of course, Theorems 0.3 and 0.5 would not have been affected, though.

An *extender* F will thus be a partial map from $\mathcal{P}(\kappa)$ to $\mathcal{P}(\lambda)$, where $\kappa = \mathrm{crit}(F)$ is the *critical point* of F and $\lambda = F(\kappa)$ is the *length* of F. If F is an extender on $\mathcal{M}$ with length λ, and if $\xi \leq \lambda$, then we write $F|\xi$ for $\{(X, Y \cap \xi) : (X, Y) \in F\}$; $\xi < \lambda$ is called a *cutpoint* of F (cf. [24, Definition p. 282]) iff for all $f \in {}^{\kappa}\kappa \cap \mathcal{M}$ and for all $\bar{\xi} < \xi$, $i_F(f)(\bar{\xi}) < \xi$, where i_F is the ultrapower map induced by F. The concept of a "premouse" is defined with the help of the *Initial Segment Condition* (*ISC*) which says that if F is the top extender of $\mathcal{M}$, and if ξ is a cutpoint of F, then $F|\xi \in \mathcal{M}$ (cf. [24, p. 283]). If there are no premice with superstrong extenders, then a potential premouse $\mathcal{M}$ (cf. [24, Definition p. 281]) is a premouse if and only if no extender on the sequence of $\mathcal{M}$ has any cutpoints (cf. [24, Corollary 9.13]). If $\mathcal{M}$ is a premouse, say $\mathcal{M} = (J_\alpha[E]; \in, E, E_\alpha)$, and if $\beta \leq \alpha$, then we write $\mathcal{M}||\beta$ for $\mathcal{M}$ cut off at β, i.e., $\mathcal{M}||\beta = (J_\beta[E \restriction \beta]; \in, E \restriction \beta, E_\beta)$, and we write $\mathcal{M}|\beta$ for $(J_\beta[E \restriction \beta]; \in, E \restriction \beta, \emptyset)$. If $F = E_\gamma^{\mathcal{M}} \neq \emptyset$ is an extender on the sequence of $\mathcal{M}$, then the *index* γ of F is equal to

$$F(\mathrm{crit}(F))^{+\mathrm{Ult}(\mathcal{M}|\gamma; F)}.$$

(This approach to indexing is called *Jensen indexing.*)

We propose the following use of the word "mouse."

Definition 1.1 *Let $\mathcal{M}$ be a premouse. We call $\mathcal{M}$ a* mouse *if and only if the following holds true. For every $n < \omega$, if*

$$\pi : \mathcal{N} \to \mathfrak{C}_n(\mathcal{M})$$

is a weak n-embedding (cf. [9, p. 52ff.], [19, Definition 4.1]), where $\mathcal{N}$ is a countable premouse, then $\mathcal{N}$ is $(n, \omega_1, \omega_1 + 1)$ iterable (cf. [19, Definition 4.4]).

Let $\mathcal{M}$ be a premouse. In particular, $\mathcal{M}$ is an amenable J-structure; the reducts $\mathcal{M}^n$ for $n < \omega$ and the rest of the fine structural concepts may then be defined as in [16]. All reducts $\mathcal{M}^n$, $n < \omega$, are amenable, and we may take fine ultrapowers "by the Dodd-Jensen procedure of coding $\mathcal{M}^0$ onto $\rho_n(\mathcal{M})$, taking a Σ_0 ultrapower of the coded structure, and then decoding" (cf. [9, p. 40], cf. also [16, §8]).

If $\mathcal{P}$ is an amenable J-structure, then we shall write $S_\alpha^{\mathcal{P}}$ for the α^{th} level of the S-hierarchy which produces $\mathcal{P}$. In particular, $S_{\mathcal{P} \cap \mathrm{OR}}^{\mathcal{P}} = \mathcal{P}$. We shall need the following well-known fact.

Lemma 1.2 *Let $\mathcal{M}$ be a premouse. Let κ be a cardinal of $\mathcal{M}$, let $\mathcal{M}$ be sound above κ, and let $p_{\mathcal{M}} \restriction (n+1)$ be solid and universal. Suppose that $\rho_{n+1}(\mathcal{M}) \leq \kappa < \rho_n(\mathcal{M})$. Then* $\mathrm{cf}^V(\rho_{n+1}(\mathcal{M})^{+\mathcal{M}}) = \mathrm{cf}^V(\kappa^{+\mathcal{M}}) = \mathrm{cf}^V(\rho_n(\mathcal{M}))$.

Proof. Write $\eta = \mathrm{cf}(\kappa^{+\mathcal{M}})$. Let us first show that $\eta = \mathrm{cf}(\rho_n(\mathcal{M}))$. By hypothesis, $\rho_n(\mathcal{M}) = \mathcal{M}^n \cap \mathrm{OR}$, so that we need to see that $\mathrm{cf}(\mathcal{M}^n \cap \mathrm{OR}) = \eta$. Again by hypothesis,

$$\mathcal{M}^n = \mathrm{Hull}_1^{\mathcal{M}^n}(\kappa \cup \{p_{n+1}(\mathcal{M})\}).$$

Let $(\xi_i : i < \eta) \in V$ be cofinal in $\kappa^{+\mathcal{M}}$. For each $i < \eta$, let $\alpha_i < \mathcal{M}^n \cap \mathrm{OR}$ be the least α such that

$$\mathrm{Hull}_1^{S_\alpha^{\mathcal{M}_n}}(\kappa \cup \{p_{n+1}(\mathcal{M})\}) \cap (\xi_i, \kappa^{+\mathcal{M}}) \neq \emptyset.$$

We must have that $(\alpha_i : i < \eta)$ is cofinal in $\mathcal{M}^n \cap \mathrm{OR}$, and hence $\mathrm{cf}(\mathcal{M}^n \cap \mathrm{OR}) \leq \eta$.

On the other hand, let $(\alpha_i : i < \mathrm{cf}(\mathcal{M}^n \cap \mathrm{OR})) \in V$ be cofinal in $\mathcal{M}^n \cap \mathrm{OR}$. For each $i < \mathrm{cf}(\mathcal{M}^n \cap \mathrm{OR})$, let

$$\xi_i = \sup(\mathrm{Hull}_1^{S_{\alpha_i}^{\mathcal{M}_n}}(\kappa \cup \{p_{n+1}(\mathcal{M})\}) \cap \kappa^{+\mathcal{M}}).$$

Then $(\xi_i : i < \mathrm{cf}(\mathcal{M}^n \cap \mathrm{OR}))$ is cofinal in $\kappa^{+\mathcal{M}}$, and thus $\eta \leq \mathrm{cf}(\mathcal{M}^n \cap \mathrm{OR})$.

Now let us verify that $\mathrm{cf}(\rho_{n+1}(\mathcal{M})^{+\mathcal{M}}) = \eta$. Let

$$\bar{\mathcal{M}} = \mathfrak{C}_{n+1}(\mathcal{M}) = \mathrm{Hull}_n^{\mathcal{M}}(\rho_{n+1}(\mathcal{M}) \cup \{p(\mathcal{M})\}),$$

and let

$$\pi : \bar{\mathcal{M}} \to_{r\Sigma_{n+1}} \mathcal{M}$$

be the core embedding. By hypothesis, $\rho_{n+1}(\mathcal{M})^{+\bar{\mathcal{M}}} = \rho_{n+1}(\mathcal{M})^{+\mathcal{M}}$. Also, π is cofinal at $\bar{\mathcal{M}}^n$. Moreover, $\bar{\mathcal{M}}$ is sound above $\rho_{n+1}(\mathcal{M})$, so that by what we proved so far (applied to $\bar{\mathcal{M}}$ rather than $\mathcal{M}$), $\mathrm{cf}(\bar{\mathcal{M}}^n \cap \mathrm{OR}) = \mathrm{cf}(\rho_{n+1}(\mathcal{M})^{+\bar{\mathcal{M}}})$. Putting these things together yields

$$\mathrm{cf}((\rho_{n+1}(\mathcal{M})^{+\mathcal{M}})) = \mathrm{cf}((\rho_{n+1}(\mathcal{M})^{+\bar{\mathcal{M}}})) = \mathrm{cf}(\bar{\mathcal{M}}^n \cap \mathrm{OR}) = \mathrm{cf}(\mathcal{M}^n \cap \mathrm{OR}) = \eta.$$

$\square$ (Lemma 1.2)

We now state the Condensation Lemma (cf. [5, §8, Lemma 4]).

Lemma 1.3 *Let $\mathcal{M}$ be a mouse which does not have a superstrong extender, and let $\mathcal{N}$ be a premouse. Let*

$$\pi : \mathcal{N} \to_{\Sigma_0} \mathcal{M}$$

be such that $\pi \neq \mathrm{id}$, and set $\kappa = \mathrm{crit}(\pi)$. Suppose $n < \omega$ is such that $\rho_{n+1}(\mathcal{N}) \leq \kappa < \rho_n(\mathcal{N})$. Suppose further that $\mathcal{N}$ is sound above κ and in fact π is weakly $r\Sigma_{n+1}$ elementary (cf. [16, Definition 5.12]).[5]

Then $\mathcal{N}$ is a mouse and one of the following holds true.

(a) $\mathcal{N}$ is the κ-core of $\mathcal{M}$ and π is the core map,[6]

(b) $\mathcal{N} \triangleleft \mathcal{M}$,

(c) $\mathcal{N} = \mathrm{Ult}_k(\mathcal{M}||\eta; E_\gamma^{\mathcal{M}})$, where $\mathcal{M}|\kappa = \mathcal{N}|\kappa$ has a largest cardinal, say μ, $E_\gamma^{\mathcal{M}} \neq \emptyset$, $\mu = \mathrm{crit}(E_\gamma^{\mathcal{M}}) < \kappa < \gamma \leq \eta < \mathcal{M} \cap \mathrm{OR}$, $\kappa = \mu^{+\mathcal{M}||\gamma}$, η is the least $\bar{\eta} \geq \gamma$ such that $\rho_\omega(\mathcal{M}||\bar{\eta}) \leq \kappa$, $k < \omega$ is least such that $\rho_{k+1}(\mathcal{M}||\eta) \leq \kappa$, and in fact $E_\gamma^{\mathcal{M}}$ is generated by $\{\mu\}$.

[5] A weakly $r\Sigma_{n+1}$ elementary map is $\Sigma_0^{(n)}$ elementary in the language of [5].

[6] I.e., $\mathcal{N}$ is the transitive collapse of the appropriate fine structural hull of $\kappa \cup \{p_{n+1}(M)\}$ taken over $\mathcal{M}$, and π is the inverse of the transitive collapse which may also obtained by coiterating $(\mathcal{M}, \mathcal{N}, \kappa)$ with $\mathcal{M}$.

The following is a trivial consequence of the Condensation Lemma 1.3.

Lemma 1.4 *Let $\mathcal{M}$ be a mouse which does not have a superstrong extender, and let $\mathcal{N}$ be a premouse. Let*

$$\pi \colon \mathcal{N} \rightarrow_{\Sigma_\omega} \mathcal{M}$$

be such that $\pi \neq \mathrm{id}$, and set $\kappa = \mathrm{crit}(\pi)$. Suppose that $\rho_\omega(\mathcal{M}) = \pi(\kappa)$ and $\mathcal{M}$ is sound.

Then $\mathcal{N} \lhd \mathcal{M}$ (in particular, $\mathcal{N}$ is a sound mouse).

PROOF. Notice that $\rho_\omega(\mathcal{N}) = \kappa$ and $\mathcal{N}$ is sound by the full elementarity of π. But then (a) of Lemma 1.3 is ruled out because otherwise $\rho_\omega(\mathcal{M}) = \rho_\omega(\mathcal{N})$, and (c) of Lemma 1.3 is ruled out because otherwise $\mathcal{N}$ would not be sound. Therefore $\mathcal{N} \lhd \mathcal{M}$ by Lemma 1.3.

□ (Lemma 1.4)

2 K^c constructions

We need a K^c construction which is an amalgamation of [2] and [8].

Definition 2.1 *A K^c* construction *(also called an* array*) is a sequence*

$$(\mathcal{N}_\xi, \mathcal{M}_\xi \colon \xi < \theta)$$

of mice,[7] *where $\theta \leq \mathrm{OR} + 1$, such that for all $\xi < \theta$,*

(a) $\mathcal{M}_\xi$ is the core of $\mathcal{N}_\xi$,

(b) if $\mathcal{N}_\xi$ is active, then $\xi = \bar{\xi} + 1$ for some $\bar{\xi}$, and setting $\alpha = \mathcal{N}_\xi \cap \mathrm{OR}$, $\mathcal{N}_\xi | \alpha = \mathcal{M}_{\bar{\xi}}$, i.e., $\mathcal{N}_\xi$ results from $\mathcal{M}_{\bar{\xi}}$ by adding a top extender,

(c) if $\mathcal{N}_\xi$ is passive and $\xi = \bar{\xi} + 1$ for some $\bar{\xi}$, then setting $\alpha = \mathcal{M}_{\bar{\xi}} \cap \mathrm{OR}$, $\mathcal{N}_\xi || \alpha = \mathcal{M}_{\bar{\xi}}$ and $\mathcal{N}_\xi \cap \mathrm{OR} = \alpha + \omega$, i.e., $\mathcal{N}_\xi$ results from $\mathcal{M}_{\bar{\xi}}$ by constructing one step further, and

(d) if $\mathcal{N}_\xi$ is passive and ξ is a limit ordinal, then $\mathcal{N}_\xi$ is the "lim inf" of the $\mathcal{M}_{\bar{\xi}}$ for $\bar{\xi} < \xi$, i.e., for all $\mathcal{N}$, $\mathcal{N} \lhd \mathcal{N}_\xi$ iff there is some $\bar{\xi} < \xi$ such that whenever $\bar{\xi} \leq i < \xi$, $\mathcal{M}_i || (\mathcal{N} \cap \mathrm{OR}) = \mathcal{N}$.

A K^c construction is determined by a criterion for which extender to add at a given stage of the construction. A classical K^c construction is the one which is presented in the last section of [9]. More liberal K^c constructions are the ones of [2, Section 2], [8, §2], and [6, §1]. Our criterion for constructing K^c will be "being certified by a collapse" which is a strengthening of [8, Definition 1.6] for Jensen premice as well as a strengthening of [6, §1, p. 5].

A cardinal γ is called *countably closed* (or, *ω-closed*) iff $\eta^{\aleph_0} < \gamma$ for every $\eta < \gamma$.

[7] We shall not be interested in arrays which contain premice which are not mice.

Definition 2.2 *Let $\mathcal{M}$ be a premouse with no top extender, say $\mathcal{M} = (J_\alpha[E]; \in, E)$, and let F be an extender with $\kappa = \mathrm{crit}(F)$ and $\lambda = F(\kappa)$ such that $(J_\alpha[E]; \in, E, F)$ is a premouse. We say that F is* certified by a collapse *iff for some regular ω-closed cardinal $\gamma \geq \lambda$ with $2^{<\gamma} = \gamma$ there is some elementary embedding*

$$\pi : H \to H_{\gamma^+}$$

such that (the universe of) H is transitive, ${}^\omega H \subset H$, $\gamma = \pi(\kappa)$, $E \restriction \kappa \in H$, and

$$F = (\pi \restriction (\mathcal{P}(\kappa) \cap J_\alpha[E]))|\lambda,$$

i.e., F is derived from π. In this situation, we also say that $\pi \restriction (\mathcal{P}(\kappa) \cap J_\alpha[E])$ is certified by a collapse*.*

A deficiency here is that ZFC does not prove the existence of a regular ω-closed cardinal γ with $2^{<\gamma} = \gamma$. However, if γ is regular and ω-closed (for instance, $\gamma = (\mu^{\aleph_0})^+$ for some μ), then in $V^{\mathrm{Col}(\gamma,\gamma)}$ we shall have that γ is regular and ω-closed and $2^{<\gamma} = \gamma$. This will suffice for our purposes.

Let us now first verify that being "certified by a collapse" is essentially stronger than the notion of being "certified" from [8, Definition 1.6].

In order to define being "certified," let us assume that $V = L[A]$, where $A \subset \mathrm{OR}$. We may assume that $\mathcal{P}(\kappa) \subset L_{2^\kappa}[A \cap 2^\kappa]$ and ${}^\omega\kappa \subset L_{\kappa^{\aleph_0}}[A \cap \kappa^{\aleph_0}]$ for all infinite cardinals κ. If α is an ordinal, then we write $\bar{H}_\alpha$ for the structure $(L_\alpha[A \cap \alpha]; \in, A \cap \alpha)$. If κ is an infinite cardinal with $2^{<\kappa} = \kappa$, then (the universe of) $\bar{H}_\kappa$ is H_κ, i.e., the collection of sets which are hereditarily smaller than κ.

The class of Σ_{1+} formulae is defined in [8, Definition 1.3]; it is a class which is strictly between Σ_1 and Σ_2. A formula is said to be Σ_{1+} (cf. [8, Definition 1.3]) iff it is of the form

$$\exists v_0 \, \exists v_1 \, \exists v_2 ({}^\omega v_0 \subset v_0 \wedge v_2 = A \cap v_3 \wedge \varphi(v_0, v_1, v_2, v_3, v_4)),$$

where φ is Σ_0 (cf. [8, Definition 1.3]).

If F is an extender with $\kappa = \mathrm{crit}(F)$ and $\lambda = F(\kappa)$, and if U is a countable set, then F is called *countably complete with respect to U* iff there is a map τ such that $U \cap \lambda \subset \mathrm{dom}(\tau)$, $\tau \restriction U \cap \lambda : U \cap \lambda \to \kappa$ is order-preserving, and for all $\xi \in U \cap \lambda$ and for every $X \in \mathrm{dom}(F) \cap U$ we have that if $\xi \in F(X)$, then $\tau(\xi) \in X$ (cf. [8, Definition 1.1]).

The following is a reformulation of [8, Definition 1.6] to the context of Jensen premice.

Definition 2.3 *Let F be an extender with $\kappa = \mathrm{crit}(F)$ and $\lambda = F(\kappa)$. We say that F is* certified *iff, letting δ be the least regular cardinal such that $\delta \geq (\mathrm{Card}(\lambda)^{\aleph_0})^+$, $2^{<\delta} = \delta$, and δ is countably closed, we have that for all countable $U \prec_{\Sigma_{1+}} \bar{H}_\delta$ there is some $\tau : U \to_{\Sigma_{1+}} \bar{H}_\kappa$ witnessing that F is countably complete with respect to U.*

We emphasize that if there is no regular countably closed cardinal $\delta > \lambda$ such that $2^{<\delta} = \delta$, then F cannot be certified. We also emphasize that whether a given extender is certified may depend on the choice of A.

Lemma 2.4 *Let F be an extender with $\kappa = \mathrm{crit}(F)$ and $\lambda = F(\kappa)$. If F is certified by a collapse, witnessed by $\pi\colon H \to H_{\gamma^+}$, where $A \cap \pi(\kappa) \in \mathrm{ran}(\pi)$, then F is certified in the sense of Definition 2.3.*

Proof. This is by the proof of [8, Lemma 3.6]. Let

$$\pi\colon H \to H_{\gamma^+}$$

witness that F is certified by a collapse, where $A \cap \pi(\kappa) \in \mathrm{ran}(\pi)$. Notice that if δ is as in Definition 2.3, then $\lambda < \pi(\kappa) = \gamma$ yields that in fact $\delta \leq \gamma$. Also, $\bar{H}_\delta \prec_{\Sigma_{1+}} \bar{H}_\gamma \prec_{\Sigma_{1+}} V$ (cf. [8, Lemma 1.5]). Let $U \prec_{\Sigma_{1+}} \bar{H}_\delta$ be countable and let $\sigma\colon \bar{U} \cong U$, where $\bar{U}$ transitive. Let $(a_n, X_n\colon n < \omega)$ be a list of all pairs (a, X) such that $a \in [U \cap \lambda]^{<\omega}$, $X \in \mathcal{P}([\kappa]^{\mathrm{Card}(a)}) \cap H \cap U$, and $a \in \pi(X)$. Let $\bar{a}_n = \sigma^{-1}(a_n) = \sigma^{-1}\text{"}a_n$ for $n < \omega$. Notice that $(\bar{a}_n, X_n\colon n < \omega) \in H$.

Now σ witnesses that in H_{γ^+} there is some $\varphi\colon \bar{U} \to_{\Sigma_{1+}} \bar{H}_{\pi(\kappa)}$ such that $\varphi\text{"}\bar{a}_n \in \pi(X_n)$ for all $n < \omega$. By elementarity of π, there is hence some $\varphi \in H$, $\varphi\colon \bar{U} \to_{\Sigma_{1+}} \bar{H}_\kappa$, such that $\varphi\text{"}\bar{a}_n \in X_n$ for all $n < \omega$. Let $\varphi_0 \in H$ be a witness, and set $\tau = \varphi_0 \circ \sigma^{-1}$. Then $\tau\colon U \to_{\Sigma_{1+}} \bar{H}_\kappa$, and moreover $\tau(a_n) \in X_n$ for all $n < \omega$, i.e., τ witnesses that F is countably complete with respect to U. Hence the map τ is as desired.

$\square$ (Lemma 2.4)

Without the hypothesis that $A \cap \pi(\kappa) \in \mathrm{ran}(\pi)$ we wouldn't get that $\bar{H}_{\pi(\kappa)} \in \mathrm{ran}(\pi)$ in the proof of Lemma 2.4, so that we couldn't pull the existence of the map φ back to H.

We may now use a similar argument to show that being "certified by a collapse" is stronger than being "robust." In order to define "robustness," we need the Chang model.

If B is any set, then we recursively define $C_0(B) = \mathrm{TC}(\{B\})$, $C_{\alpha+1}(B) = \mathrm{Def}(C_\alpha(B)) \cup [\alpha]^\omega$, where $\mathrm{Def}(C_\alpha(B))$ is the set of all subsets of $C_\alpha(B)$ which are definable over $C_\alpha(B)$ with parameters from $C_\alpha(B)$, and if λ is a limit ordinal, then $C_\lambda(B) = \bigcup\{C_\alpha(B)\colon \alpha < \lambda\}$. If $J_\beta[E]$ is a J-model, and if $\eta \leq \beta$ and μ are ordinals, then we write $\bar{C}^E_{\eta,\mu}$ for

$$C_\mu((J_\eta[E], E \restriction \eta))$$

and $C^E_{\eta,\mu}$ for the structure

$$(\bar{C}^E_{\eta,\mu}; \in, (\bar{C}^E_{\eta,\bar{\mu}}\colon \bar{\mu} < \mu)).$$

Notice that "$v = C^E_{\eta,\mu}$" is Σ_{1+} in the parameters $E \restriction \eta$ and μ.

Definition 2.5 *Let $\mathcal{M}$ be a potential premouse with top extender F such that $\kappa = \mathrm{crit}(F)$ and $\lambda = F(\kappa)$. Then F is called* robust *iff for all $U \subset \lambda$ and $W \subset \mathcal{P}(\kappa) \cap \mathcal{M}$ which are both countable, there is some order preserving $\tau\colon U \to \kappa$ which witnesses that F is countably complete with respect to $U \cup W$ and such that for all $U' \subset U$, setting $\beta = \sup(U')$ and $\bar{\beta} = \sup(\tau\text{"}U')$, if φ is a Σ_1 formula, then*

$$C^E_{\bar{\beta},\kappa} \models \varphi(\tau\text{"}U', \tau\text{"}U) \Longleftrightarrow C^E_{\beta,\infty} \models \varphi(U', U).$$

Lemma 2.6 *Let F be an extender with $\kappa = \mathrm{crit}(F)$ and $\lambda = F(\kappa)$. If F is certified by a collapse, then F is robust.*

PROOF. This is by the proof of [6, §1, Lemma 4.1]. Let

$$\pi: H \to H_{\gamma^+}$$

witness that F is certified by a collapse. Let $U \subset \lambda$ and $W \subset \mathcal{P}(\kappa) \cap \mathcal{M}$ both be countable, let $g: \omega \to U$ be bijective, let $\vec{a} = (a_\xi : \xi < 2^{\aleph_0})$ be an enumeration of $\mathcal{P}(\omega)$, let

$$T = \{(\varphi, \xi): \varphi \text{ is } \Sigma_1, \xi < 2^{\aleph_0}, \text{ and } C^E_{\sup(g\text{"}a_\xi),\infty} \models \varphi(g\text{"}a_\xi, U)\},$$

and let $U^* \prec_{\Sigma_3} H_{\gamma^+}$ be countable and such that $U \cup W \cup \{U, W, g, \vec{a}, T\} \subset U^*$. In much the same way as in the proof of Lemma 2.4 we may construct a map $\tau: U^* \to_{\Sigma_3} H$ which witnesses that F is countably complete with respect to U^* such that $\tau(\vec{a}) = \vec{a}$, $\tau(T) = T$, $\tau(\gamma) = \kappa$, $\tau(A \cap \gamma) = A \cap \kappa$, and $\tau(E \restriction \gamma) = E \restriction \kappa$.

The point now is that "$v = C^E_{\eta,\mu}$" is Σ_{1+} in the parameters $E \restriction \eta$ and μ (and A is not needed). Therefore,

$$\forall \varphi \in \Sigma_1 \; \forall \xi < 2^{\aleph_0} \; ((\varphi, \xi) \in T \iff C^E_{\sup(g\text{"}a_\xi),\infty} \models \varphi(g\text{"}a_\xi, U))$$

is a true Π_3 statement, and because $\bar{H}_\gamma \prec_{\Sigma_{1+}} V$ and by the choice of U^* and τ, we get

$$\forall \varphi \in \Sigma_1 \; \forall \xi < 2^{\aleph_0} \; ((\varphi, \xi) \in T \iff C^E_{\sup(\tau(g)\text{"}a_\xi),\kappa} \models \varphi(\tau(g)\text{"}a_\xi, \tau(U)))$$

to hold true.

Let $U' \subset U$ and write $\beta = \sup(U')$ and $\bar{\beta} = \sup(\tau\text{"}U')$. If $U' = g\text{"}a_\xi$, where $\xi < 2^{\aleph_0}$, then $\tau(g)\text{"}a_\xi = \tau\text{"}U'$. Also, $\tau(U) = \tau\text{"}U$. We therefore get that

$$C^E_{\beta,\infty} \models \varphi(U', U)) \iff (\varphi, \xi) \in T \iff C^E_{\bar{\beta},\kappa} \models \varphi(\tau\text{"}U', \tau\text{"}U)),$$

as desired.

□ (Lemma 2.6)

The maximal certified K^c construction will now be defined via the concept of extenders which are "certified by a collapse."

Definition 2.7 *The* maximal certified K^c construction *is the unique K^c construction*

$$(\mathcal{N}_\xi, \mathcal{M}_\xi : \xi < \theta)$$

such that

(a) for all $\xi < \theta$, $\mathcal{N}_\xi$ is active with top extender F if and only if there is some $\bar{\xi}$ such that $\xi = \bar{\xi} + 1$ and F is the unique extender G such that $(\mathcal{M}_{\bar{\xi}}; \in, E^{\mathcal{M}_{\bar{\xi}}}, G)$ is a premouse and G is certified by a collapse, and

(b) θ is largest such that such a K^c construction exists.

If $\theta = \mathrm{OR} + 1$, and if for every $\bar{\xi}$, if there is an extender G such that $(\mathcal{M}_{\bar{\xi}}; \in, E^{\mathcal{M}_{\bar{\xi}}}, G)$ is a premouse and G is certified by a collapse, then there is a unique such G, then we write K^c for $\mathcal{M}_{\mathrm{OR}}$ and say that the certified K^c exists, *or simply:* K^c exists.

The following theorem is a version of [17, Theorem 9.14], which produced such a theorem for the first time. (Cf. also [2, Theorem 2.28] and [6, §1, Theorem 1].)

Theorem 2.8 *Let $(\mathcal{N}_\xi, \mathcal{M}_\xi : \xi < \theta)$ be the certified K^c construction, and let $\xi < \theta$. Let $n < \omega$, and let*

$$\pi: \mathcal{P} \to \mathcal{N}_\xi$$

be a weak n-embedding, where $\mathcal{P}$ is countable. Let $\mathcal{T}$ be a countable putative n-bounded normal iteration tree on $\mathcal{P}$. Let β be the length of $\mathcal{T}$. Then exactly one of the following holds.

(a) $\beta = \bar{\beta} + 1$ for some $\bar{\beta}$ and for some $\bar{\xi} \leq \xi$ and some $k < \omega$ there is a weak k-embedding

$$\sigma: \mathcal{M}^{\mathcal{T}}_{\bar{\beta}} \to \mathcal{N}_{\bar{\xi}}.$$

Moreover, if there is no drop along $[0, \bar{\beta}]_{\mathcal{T}}$, then $\bar{\xi} = \xi$, $k \leq n$, and $\pi = \sigma \circ \pi^{\mathcal{T}}_{0\bar{\beta}}$.

(b) there is a maximal *branch b through $\mathcal{T}$ such that for some $\bar{\xi} \leq \xi$ and some $k < \omega$, there is a weak k-embedding*

$$\sigma: \mathcal{M}^{\mathcal{T}}_b \to \mathcal{N}_{\bar{\xi}}.$$

Moreover, if there is no drop along b, then $\bar{\xi} = \xi$, $k \leq n$, and $\pi = \sigma \circ \pi^{\mathcal{T}}_b$.

Proof. By Lemma 2.6, every certified extender is robust. The theorem therefore follows immediately from [6, §1, Theorem 1], which shows what we aim to see from the hypothesis that $(\mathcal{N}_\xi, \mathcal{M}_\xi : \xi < \theta)$ is a "robust K^c construction." □ (Theorem 2.8)

[2, Theorem 2.28] states a more detailed version of what may be shown along these lines.

We want to stress that we could not have used [8, §2] in the proof of Theorem 2.8 because (as we observed after the proof of Lemma 2.4) we need $A \cap \pi(\kappa)$ as a hypothesis in Lemma 2.4. Because of this, our use of [6] rather than [8] avoids problems in arguments later in the paper. However we show in the last section that these problems can be surmounted in such a way that the main results of this paper can be based on [8] after all.

In order to show now that the certified K^c exists, we need an anti large cardinal hypothesis. The following definition is from [2, Definition 3.1].

Definition 2.9 *Let $\mathcal{M}$ be a premouse. Then $\mathcal{M}$ is called* domestic *iff there is no $\alpha \leq \mathcal{M} \cap \mathrm{OR}$ such that*

(a) $\mathcal{M}||\alpha$ is active, and if $\kappa = \mathrm{crit}(F)$, then

(b) κ is a limit of ordinals δ such that $\mathcal{M}|\alpha \models$ "δ is a Woodin cardinal," and

(c) κ is a limit of ordinals μ such that $\mathcal{M}|\kappa \models$ "μ is a strong cardinal."

The following theorem is the main result of [2], cf. [2, Theorem 3.2]. (Cf. [2] on the concepts which are used in this statement.)

Theorem 2.10 *Let* $(\mathcal{N}_\xi, \mathcal{M}_\xi : \xi < \theta)$ *be the maximal certified* K^c *construction, and let* $\xi < \theta$. *Let* $n < \omega$, *and let*

$$\pi : \mathcal{P} \to \mathcal{N}_\xi$$

be a weak n*-embedding, where* $\mathcal{P}$ *is countable. Assume* ξ *to be the least* $\bar{\xi}$ *such that there is some weak* n*-embedding* $\bar{\pi} : \mathcal{P} \to \mathcal{N}_{\bar{\xi}}$, *and let* π *be the leftmost* $\bar{\pi}$ *such that* $\bar{\pi} : \mathcal{P} \to \mathcal{N}_\xi$ *is a weak* n*-embedding. Let* $\mathcal{T}$ *be a countable* n*-bounded normal iteration tree on* $\mathcal{P}$ *of limit length.*

If $\mathcal{N}_\xi$ *is domestic, then there is at most one cofinal branch* b *through* $\mathcal{T}$ *which is super-realizable.*

As explained in [2] (cf. [2, Corollary 3.3]), Theorem 2.8 and Theorem 2.10 show the following.

Corollary 2.11 *If there is no non-domestic premouse, then the certified* K^c *exists and is a mouse.*

We shall need below that there is no "sharp" for K^c in the sense of the following lemma which was shown in [5].

Lemma 2.12 *Suppose that the certified* K^c *exists, but there is no superstrong cardinal in* K^c. *Let* $\kappa < \lambda$ *be cardinals of* K^c, *and let* $\tau = \kappa^{+K^c}$ *and* $\eta = \lambda^{+K^c}$.

There is no

$$\pi : K^c|\tau \to_{\Sigma_0} K^c|\eta$$

such that $\kappa = \mathrm{crit}(\pi)$, $\pi(\kappa) = \lambda$, *and* $\pi \restriction (\mathcal{P}(\kappa) \cap K^c)$ *is certified by a collapse.*

Moreover, if λ *is regular in* V, *then there is no mouse* $\mathcal{S} \rhd K^c|\lambda$ *such that there is some*

$$\pi : K^c|\tau \to_{\Sigma_0} \mathcal{S}$$

such that $\kappa = \mathrm{crit}(\pi)$, $\pi(\kappa) = \lambda$, *and* $\pi \restriction (\mathcal{P}(\kappa) \cap K^c)$ *is certified by a collapse.*

PROOF. Let us first prove the first statement. Assume that there is some such π. Let F be the extender on K^c derived from π, i.e., $F = \pi \restriction (\mathcal{P}(\kappa) \cap K^c)$. Set $\tilde{\eta} = \sup(\pi''\tau) \leq \eta$. Notice that $\pi : K^c|\tau \to_{\Sigma_1} K^c|\tilde{\eta}$. We may consider the potential premouse $\mathcal{M} = (K|\tilde{\eta}, F)$ which results from $K|\tilde{\eta}$ by adding F as its top extender.

Let $\alpha < \lambda$ be the least cutpoint of F (cf. [24], i.e., if $f \in {}^\kappa\kappa \cap K^c$ and $\xi < \alpha$, then $\pi(f)(\xi) < \alpha$) such that $F|\alpha \notin K^c|\tilde{\eta}$, or $\alpha = \lambda$ if there is no such cutpoint. We may factor π as

$$K^c|\tau \to^{\bar{\pi}} \mathcal{N} = \mathrm{Ult}_0(K^c|\tau, F|\alpha) \to^k K^c|\tilde{\eta},$$

where $\mathrm{crit}(k) = \alpha$ and $k(\alpha) = \lambda$. By the Condensation Lemma 1.3, $\mathcal{N} \lhd K^c|\tilde{\eta}$ and of course α is a cardinal of K^c. Set $\theta = \mathcal{N} \cap \mathrm{OR} = \sup(\bar{\pi}''\tau)$.

Notice that $(K^c|\theta, F|\alpha)$ is now a premouse (by the choice of α), and of course $F|\alpha$ is certified by a collapse.

Let $(\gamma_i : i < \theta)$ be increasing and cofinal in θ such that for all $i < \theta$, $\rho_\omega(K^c||\gamma_i) = \alpha$. For each $i < \theta$, let ξ_i be least such that $\rho_\omega(\mathcal{M}_\xi) \geq \alpha$ as well as $K^c||\gamma_i \trianglelefteq \mathcal{M}_\xi$ for all $\xi \geq \xi_i$. (Here and in what follows, $(\mathcal{N}_\xi, \mathcal{M}_\xi : \xi \leq \mathrm{OR})$ is the certified K^c construction). By thinning out the sequence $(\gamma_i : i < \theta)$ if necessary, we may and shall also assume that for each $i < \theta$, $(\mathcal{P}(\alpha) \cap K^c||(\gamma_{i+1} + 1)) \setminus K^c||(\gamma_i + 1) \neq \emptyset$, so that $K^c||\gamma_{i+1}$ is *not* an initial segment of $\mathcal{M}_{\xi_i}$. Setting $\xi^* = \sup(\{\xi_i : i < \theta\})$, ξ^* is a limit ordinal and $K^c|\theta \trianglelefteq \mathcal{N}_{\xi^*}$. But we cannot have that $K^c|\theta \triangleleft \mathcal{N}_{\xi^*}$, as otherwise $K^c|\theta \trianglelefteq \mathcal{N}_\xi$ for all sufficiently large $\xi < \xi^*$, and hence for all sufficiently large $i < \theta$, $K^c||\gamma_j \trianglelefteq \mathcal{M}_{\xi_i}$ for all $j < \theta$.

Therefore, $K^c|\theta = \mathcal{N}_{\xi^*} = \mathcal{M}_{\xi^*}$. Because $F|\alpha$ is certified by a collapse, this means that $\mathcal{N}_{\xi^*+1} = (\mathcal{M}_{\xi^*}, F|\alpha)$, i.e., $\mathcal{N}_{\xi^*+1}$ results from $\mathcal{M}_{\xi^*}$ by adding $F|\alpha$ as its top extender. But we must now in fact have $\mathcal{N}_{\xi^*+1} \triangleleft K^c$.

However, $\rho_1((\mathcal{M}_{\xi^*}, F|\alpha)) < \alpha$, because $F|\alpha$ is not superstrong. Thus α is not a cardinal in K^c. Contradiction!

The second statement is shown in exactly the same way. Notice that if λ is regular in V, then α, the least cutpoint of F, must actually be strictly smaller than λ, so that the proof still goes through.

$\square$ (Lemma 2.12)

3 Stacking mice

We now turn to the key ingredient for the proofs of Theorems 0.1 and 0.3, a "covering lemma" for stacks of mice.

Throughout this section, we work under the hypothesis that the certified K^c exists and that there is no premouse with a superstrong extender. The aim is now to stack mice over $K^c||\kappa$, where κ is a regular cardinal (in V).

Lemma 3.1 *Assume that K^c exists and that there is no premouse with a superstrong extender. Let κ be an uncountable regular cardinal. For $h \in \{0, 1\}$, let $\mathcal{M}^h$ be a sound mouse such that $K^c||\kappa \trianglelefteq \mathcal{M}^h$ and $\rho_\omega(\mathcal{M}^h) = \kappa$.*[8] *Then $\mathcal{M}^0 \trianglelefteq \mathcal{M}^1$ or $\mathcal{M}^1 \trianglelefteq \mathcal{M}^0$.*

Proof. This is an immediate consequence of Lemma 1.4. Let

$$\pi : H \to H_\theta,$$

where $\theta > \kappa$ is regular, H is transitive, $\{\kappa, \mathcal{M}^0, \mathcal{M}^1\} \subset \mathrm{ran}(\pi)$, and $\mathrm{crit}(\pi) = \pi^{-1}(\kappa)$. Set $\bar{\kappa} = \pi^{-1}(\kappa)$, $\bar{\mathcal{M}}^0 = \pi^{-1}(\mathcal{M}^0)$, and $\bar{\mathcal{M}}^1 = \pi^{-1}(\mathcal{M}^1)$. By Lemma 1.4, for $h \in \{0, 1\}$, $\bar{\mathcal{M}}^h \triangleleft \mathcal{M}^h$, so that in fact $\bar{\mathcal{M}}^h \triangleleft K^c||\kappa$, as $\rho_\omega(\bar{\mathcal{M}}^h) = \bar{\kappa} < \kappa$. Therefore, $\bar{\mathcal{M}}^0 \trianglelefteq \bar{\mathcal{M}}^1$ or $\bar{\mathcal{M}}^1 \trianglelefteq \bar{\mathcal{M}}^0$, so that $\mathcal{M}^0 \trianglelefteq \mathcal{M}^1$ or $\mathcal{M}^1 \trianglelefteq \mathcal{M}^0$ by elementarity.

$\square$ (Lemma 3.1)

[8] Notice that we in fact require $\rho_\omega(\mathcal{M}^h) = \kappa$ rather than $\rho_\omega(\mathcal{M}^h) \leq \kappa$. On the other hand, we allow $\mathcal{M}^h$ to have extenders $E_\nu^{\mathcal{M}^h}$ on its sequence which "overlap" κ, i.e., such that $\mathrm{crit}(E_\nu^{\mathcal{M}^h}) \leq \kappa$ and $\nu > \kappa$.

In the light of Lemma 3.1, we may let $\mathcal{S}$ denote the "stack" of sound mice $\mathcal{M} \trianglerighteq K^c||\kappa$ with $\rho_\omega(\mathcal{M}) = \kappa$.

Definition 3.2 *Assume that K^c exists and that there is no premouse with a superstrong extender. Let κ be an uncountable regular cardinal. Let $\mathcal{S} = \mathcal{S}(\kappa)$ denote the unique premouse such that $\mathcal{N} \trianglelefteq \mathcal{S}$ iff there is some sound mouse $\mathcal{M} \trianglerighteq K^c||\kappa$ with $\rho_\omega(\mathcal{M}) = \kappa$ such that $\mathcal{N} \trianglelefteq \mathcal{M}$.*

In the situation of Definition 3.2, $K^c|\kappa^{+K^c} \trianglelefteq \mathcal{S}$. However, $K^c|\kappa^{+K^c} \triangleleft \mathcal{S}$ seems possible. We are now going to show that $\mathcal{S}$ does not have a "last mouse" and that it is in fact itself a mouse:

Lemma 3.3 *Assume that K^c exists and that there is no premouse with a superstrong extender. Let κ be an uncountable regular cardinal, and let $\mathcal{S} = \mathcal{S}(\kappa)$. For all $\mathcal{M} \trianglelefteq \mathcal{S}$ with $\rho_\omega(\mathcal{M}) = \kappa$ there is some $\mathcal{N} \triangleleft \mathcal{S}$ such that $\mathcal{N} \triangleright \mathcal{M}$. In particular, $\mathcal{S} \models \mathsf{ZFC}^-$ and κ is the largest cardinal of $\mathcal{S}$. Moreover, $\mathcal{S}$ is a mouse.*

PROOF. Suppose first that $\mathcal{S} = \mathcal{M}$, where $\mathcal{M}$ is a sound mouse with $\mathcal{M} \trianglerighteq K^c||\kappa$ and $\rho_\omega(\mathcal{M}) = \kappa$. Let $\beta > \mathcal{M} \cap \mathrm{OR}$ be least such that $\rho_\omega(J_\beta[\mathcal{M}]) \leq \kappa$.[9] (In fact, $\beta = (\mathcal{M} \cap \mathrm{OR}) + \omega$.)

Let us suppose that $\rho_\omega(J_\beta[\mathcal{M}]) = \kappa$. Then $J_\beta[\mathcal{M}]$ cannot be a mouse, as otherwise $J_\beta[\mathcal{M}] \trianglelefteq \mathcal{S}$. Pick a countable $\mathcal{N}$ and some $k: \mathcal{N} \to J_\beta[\mathcal{M}]$ such that $\mathcal{N}$ is not $\omega_1 + 1$ iterable. Pick a fully elementary $\pi: J_{\bar{\beta}}[\bar{\mathcal{M}}] \to J_\beta[\mathcal{M}]$ such that $\mathrm{crit}(\pi) = \mathrm{ran}(\pi) \cap \kappa$ and $\mathrm{ran}(\pi) \supset \mathrm{ran}(k)$. Then $\bar{\mathcal{M}} \triangleleft \mathcal{M}$ by Lemma 1.4, and therefore in fact $J_{\bar{\beta}}[\bar{\mathcal{M}}] \triangleleft K^c||\kappa$, so that $\mathcal{N}$ is $\omega_1 + 1$ iterable after all. Contradiction!

Therefore, $\rho_\omega(J_\beta[\mathcal{M}]) < \kappa$. An application of the Condensation Lemma 1.3 then gives a contradiction as follows.

Let

$$\pi: H \to H_\theta,$$

where $\theta > \kappa$ is regular, H is transitive, $\{\kappa, \mathcal{M}, \beta\} \subset \mathrm{ran}(\pi)$, and $\mathrm{crit}(\pi) = \pi^{-1}(\kappa) > \rho_\omega(J_\beta[\mathcal{M}])$. Set $\bar{\kappa} = \pi^{-1}(\kappa)$, $\bar{\mathcal{M}} = \pi^{-1}(\mathcal{M})$, and $\bar{\beta} = \pi^{-1}(\beta)$. By Lemma 1.4, $\bar{\mathcal{M}} \triangleleft K^c||\kappa$, so that $J_{\bar{\beta}}[\bar{\mathcal{M}}]$ cannot be the $\mathrm{crit}(\pi)$-core of $J_\beta[\mathcal{M}]$ (using the fact that there are no extenders above $\bar{\mathcal{M}}$ on the sequence of $J_{\bar{\beta}}[\bar{\mathcal{M}}]$); similarily, $J_{\bar{\beta}}[\bar{\mathcal{M}}]$ cannot be an ultrapower of an initial segment of $J_\beta[\mathcal{M}]$. We also certainly cannot have that $J_{\bar{\beta}}[\bar{\mathcal{M}}] \triangleleft J_\beta[\mathcal{M}]$, as otherwise the witness to $\rho_\omega(J_\beta[\mathcal{M}]) = \rho_\omega(J_{\bar{\beta}}[\bar{\mathcal{M}}])$ would be an element of $J_\beta[\mathcal{M}]$. This gives a contradiction with Lemma 1.3.

We have shown that for all $\mathcal{M} \trianglelefteq \mathcal{S}$ with $\rho_\omega(\mathcal{M}) = \kappa$ there is some $\mathcal{N} \triangleleft \mathcal{S}$ with $\mathcal{M} \triangleleft \mathcal{N}$. We are left with having to verify that $\mathcal{S}$ is a mouse.

Well, if not, then there is some countable $\mathcal{N}$ and some $k: \mathcal{N} \to \mathcal{S}$ such that $\mathcal{N}$ is not $\omega_1 + 1$ iterable. Pick a fully elementary $\pi: \bar{\mathcal{S}} \to \mathcal{S}$ such that $\mathrm{crit}(\pi) = \mathrm{ran}(\pi) \cap \kappa$ and

[9] We here use the following notation. If $\mathcal{N} = (J_\delta[E]; \in, E, E_\delta)$ is a premouse, and if $\delta' > \delta$, then $J_{\delta'}[\mathcal{N}] = (J_{\delta'}[E^\frown E_\delta]; \in, E^\frown E_\delta)$.

$\mathrm{ran}(\pi) \supset \mathrm{ran}(k)$. By Lemma 1.3 applied to cofinally many initial segments of $\bar{\mathcal{S}}$ we get that $\bar{\mathcal{S}} \lhd \mathcal{S}$ and in fact $\bar{\mathcal{S}} \lhd K^c||\kappa$. Therefore, $\mathcal{N}$ is $\omega_1 + 1$ iterable after all. Contradiction! $\square$ (Lemma 3.3)

The above argument in fact shows that in the situation of Lemma 3.3, $L[\mathcal{S}]$ is a mouse and $\mathcal{S} \cap \mathrm{OR}$ is the cardinal successor of κ in $L[\mathcal{S}]$.

The following "weak covering lemma" for stacks is the key fact. It is due to the first author. The fact that κ is regular is used heavily in its proof.

Theorem 3.4 *Assume that K^c exists and that there is no premouse with a superstrong extender. Let $\kappa \geq \aleph_3$ be an ω-closed regular cardinal with $2^{<\kappa} = \kappa$, and let $\mathcal{S} = \mathcal{S}(\kappa)$. Suppose that κ is a limit cardinal in K^c. Then $\mathrm{cf}^V(\mathcal{S} \cap \mathrm{OR}) \geq \kappa$.*

Proof. Let us write $\eta = \mathrm{cf}^V(\mathcal{S} \cap \mathrm{OR})$. Let $(\mathcal{M}_i : i < \eta)$ be such that for every $i < \eta$, $\mathcal{M}_i$ is a sound mouse with $\rho_1(\mathcal{M}_i) = \kappa$ (in particular, $\mathcal{M}_i \lhd \mathcal{S}$), $\mathcal{M}_i \lhd \mathcal{M}_{i+1}$, and $(\mathcal{M}_i : i < \eta)$ is cofinal in $\mathcal{S}$, i.e, for every $\mathcal{N} \lhd \mathcal{S}$ there is some $i < \eta$ with $\mathcal{N} \trianglelefteq \mathcal{M}_i$. (Such a sequence exists by Lemma 3.3.)

Let us now suppose that $\eta < \kappa$. Let $\theta >> \kappa$. We may then pick a continuous chain $(X_\alpha : \alpha < \kappa)$ of elementary substructures of H_θ of size $< \kappa$ such that $\{\mathcal{S}, \kappa\} \cup \{\mathcal{M}_i : i < \eta\} \subset X_0$ and for all $\alpha < \kappa$, $X_\alpha \cap \kappa \in \kappa$, $X_\alpha \in X_{\alpha+1}$, and ${}^\omega X_{\alpha+1} \subset X_{\alpha+1}$. Set $\kappa_\alpha = X_\alpha \cap \kappa$, and let

$$\pi_\alpha : \mathcal{S}_\alpha \to \mathcal{S}$$

be the inverse of the transitive collapse of X_α, restricted to the preimage of $\mathcal{S}$. In particular, π_α has critical point κ_α. By Lemma 1.4, for every α, $\mathcal{S}_\alpha \trianglelefteq K^c||\kappa_\alpha^{+\mathcal{S}}$, i.e., if we let $\lambda_\alpha = \mathcal{S}_\alpha \cap \mathrm{OR}$, then $\mathcal{S}_\alpha = \mathcal{S}||\lambda_\alpha = K^c||\lambda_\alpha$ or $\mathcal{S}_\alpha = \mathcal{S}|\lambda_\alpha = K^c|\lambda_\alpha$ and $\lambda_\alpha \leq \kappa_\alpha^{+\mathcal{S}} = \kappa_\alpha^{+K^c}$. Of course, $\mathrm{cf}^V(\lambda_\alpha) = \eta$, as being witnessed by $(\pi_\alpha^{-1}(\mathcal{M}_i) : i < \eta)$.

Let E_0 be the set of all $\alpha < \kappa$ such that α is a successor ordinal or a limit ordinal of uncountable cofinality. We must in fact have $\lambda_\alpha < \kappa_\alpha^{+\mathcal{S}} = \kappa_\alpha^{+K^c}$ whenever $\alpha \in E_0$. This is because if $\alpha \in E_0$ and $\lambda_\alpha = \kappa_\alpha^{+K^c}$, then because X_α is countably closed, κ is ω-closed, and $2^{<\kappa} = \kappa$, then the extender derived from π_α is certified by a collapse. This contradicts Lemma 2.12.

For $\alpha \in E_0$, let $\mathcal{P}_\alpha$ be the least $\mathcal{P}$ such that $K^c||\lambda_\alpha \trianglelefteq \mathcal{P} \lhd K^c$ and $\rho_\omega(\mathcal{P}) \leq \kappa_\alpha$. In particular, $\lambda_\alpha = \kappa_\alpha^{+\mathcal{P}_\alpha}$. Because κ is a limit cardinal in K^c, there must be a club $C \subset \kappa$ such that for all $\alpha \in C$, $\kappa_\alpha = \alpha$ and α is a cardinal in K^c. In particular, $\rho_\omega(\mathcal{P}_\alpha) = \kappa_\alpha$ (rather than $\rho_\omega(\mathcal{P}_\alpha) < \kappa_\alpha$) whenever $\alpha \in E_0 \cap C$.

Write $E = E_0 \cap C$. Notice that E is stationary, and in fact E is closed at points of uncountable cofinality. Now let, for $\alpha \in E$,

$$\tilde{\pi}_\alpha : \mathcal{P}_\alpha \to \mathcal{Q}_\alpha = \mathrm{ult}_n(\mathcal{P}_\alpha, \pi_\alpha),$$

where n is least such that $\rho_{n+1}(\mathcal{P}_\alpha) = \kappa_\alpha < \rho_n(\mathcal{P}_\alpha)$. Notice that $\mathcal{Q}_\alpha$ is a mouse because π_α is certified (by a collapse).[10]

In order to get a contradiction, it suffices to see that there is some $\alpha \in E$ such that $\rho_\omega(\mathcal{Q}_\alpha) = \kappa$, because then $\mathcal{S} \lhd \mathcal{Q}_\alpha$ would contradict the definition of $\mathcal{S}$. Let us assume that for all $\alpha \in E$ we have that $\rho_\omega(\mathcal{Q}_\alpha) < \kappa$, and work towards a contradiction.

For $\alpha \leq \beta < \kappa$ we may set $\pi_{\alpha\beta} = \pi_\beta^{-1} \circ \pi_\alpha$. Let

$$\tilde{\pi}_{\alpha\beta}: \mathcal{P}_\alpha \to \mathcal{Q}_\alpha^\beta = \mathrm{ult}_n(\mathcal{P}_\alpha, \pi_{\alpha\beta}),$$

where n is least such that $\rho_{n+1}(\mathcal{P}_\alpha) \leq \kappa_\alpha < \rho_n(\mathcal{P}_\alpha)$. Notice that $\mathcal{Q}_\alpha^\beta$ is a mouse due to the existence of the canonical factor map $k: \mathcal{Q}_\alpha^\beta \to \mathcal{Q}_\alpha$ which sends $[a, f]_{\pi_{\alpha\beta}}$ to $[\sigma_\beta(a), f]_{\pi_\alpha}$, where $a \in [\lambda_\beta]^{<\omega}$, $f: [\delta]^{\mathrm{Card}(a)} \to \mathcal{P}_\alpha$ for some δ such that $\pi_{\alpha\beta}(\delta) > \max(a)$, and f comes from a level n Skolem term over $\mathcal{P}_\alpha$ (cf. [9, p. 34]).

Fix $\alpha \in E$ for a while. Let $(Y_\beta : \beta < \kappa)$ be a continuous tower of elementary substructures of H_θ of size $< \kappa$ such that $\mathcal{P}_\alpha \cup \{\mathcal{P}_\alpha, \mathcal{S}, \mathcal{Q}_\alpha, \tilde{\pi}_\alpha\} \in Y_0$ and for all $\beta < \kappa$, $Y_\beta \cap \mathrm{OR} \in \kappa$ and $Y_\beta \in Y_{\beta+1}$. There is a club $C_\alpha \subset C$ such that for all $\beta \in C_\alpha$, $\mathrm{ran}(\pi_\beta) \cap \mathcal{S} = Y_\beta \cap \mathcal{S}$. For $\beta \in C_\alpha$, let $\sigma_\beta: \bar{H}_\beta \to H_\theta$ be the inverse of the transitive collapse of Y_β. Let $\beta \in C_\alpha$. We may define

$$\varphi: \mathcal{Q}_\alpha^\beta \to \sigma_\beta^{-1}(\mathcal{Q}_\alpha)$$

by setting

$$\varphi(\tilde{\pi}_{\alpha\beta}(f)(a)) = \sigma_\beta^{-1} \circ \tilde{\pi}_\alpha(f)(a)$$

for $a \in [\lambda_\beta]^{<\omega}$, $f: [\delta]^{\mathrm{Card}(a)} \to \mathcal{P}_\alpha$ for some δ such that $\pi_{\alpha\beta}(\delta) > \max(a)$, and f comes from a level n Skolem term over $\mathcal{P}_\alpha$. This is well-defined by the following reasoning. Let a, f, ... be as just described, and let ψ be $r\Sigma_n$. Then we have that

$$\mathcal{Q}_\alpha^\beta \models \psi(\tilde{\pi}_{\alpha\beta}(f)(a), \ldots) \text{ iff}$$

$$a \in \pi_{\alpha\beta}(\{(u, \ldots): \mathcal{P}_\alpha \models \psi(f(u), \ldots)\}) \text{ iff}$$

$$\sigma_\beta(a) = \pi_\beta(a) \in \pi_\alpha(\{(u, \ldots): \mathcal{P}_\alpha \models \psi(f(u), \cdots)\}) \text{ iff}$$

$$\mathcal{Q}_\alpha \models \psi(\tilde{\pi}_\alpha(f)(\sigma_\beta(a)), \ldots) \text{ iff}$$

$$\sigma^{-1}(\mathcal{Q}_\alpha) \models \psi(\sigma^{-1} \circ \tilde{\pi}_\alpha(f)(a), \ldots).$$

But φ is easily seen to be surjective: we have that $\mathcal{Q}_\alpha =$ the set of all $\tilde{\pi}_\alpha(f)(a)$, where $a \in [\mathcal{S} \cap \mathrm{OR}]^{<\omega}$, $f: [\delta]^{\mathrm{Card}(a)} \to \mathcal{P}_\alpha$ for some δ such that $\pi_{\alpha\beta}(\delta) > \max(a)$, and f comes from a level n Skolem term over $\mathcal{P}_\alpha$, so that $\sigma^{-1}(\mathcal{Q}_\alpha) =$ the set of all $\sigma^{-1} \circ \tilde{\pi}_\alpha(f)(a)$,

[10] $\mathcal{Q}_\alpha$ is a premouse and not a protomouse. For this, we must show that $F^{\mathcal{Q}_\alpha}$ is a total extender over $\mathcal{Q}_\alpha$. Suppose otherwise. Let $\mu = \mathrm{crit}(\mathrm{F}^{\mathcal{P}_\alpha})$. Then $\tilde{\pi}$ is discontinuous at $(\mu^+)^{\mathcal{P}_\alpha}$. It follows that $(\mu^+)^{\mathcal{P}_\alpha} = \kappa_\alpha$. Via the elementarity of π, this his leads to the contradiction that κ is a successor cardinal in K^c.

where $a \in [\lambda_\beta]^{<\omega}$, $f: [\delta]^{\mathrm{Card}(a)} \to \mathcal{P}_\alpha$ for some δ such that $\pi_{\alpha\beta}(\delta) > \max(a)$, and f comes from a level n Skolem term over $\mathcal{P}_\alpha$. We have shown that

$$\mathcal{Q}^\beta_\alpha = \sigma_\beta^{-1}(\mathcal{Q}_\alpha).$$

In particular, we now have that for all $\beta \in C_\alpha$, $\rho_\omega(\mathcal{Q}^\beta_\alpha) < \kappa_\beta = \beta$.

Now pick $\beta \in E \cap \triangle_{\alpha<\kappa} C_\alpha$ such that $\mathrm{cf}^V(\beta) \neq \eta$. As $\kappa \geq \aleph_3$, this choice is possible. We have that $\beta \in C_\alpha$ for each $\alpha \in E \cap \beta$, so that $\rho_\omega(\mathcal{Q}^\beta_\alpha) < \kappa_\beta = \beta$ for each $\alpha \in E \cap \beta$.

We now claim that there is some $\alpha \in E \cap \beta$ such that

$$\mathcal{Q}^\beta_\alpha = \mathcal{P}_\beta.$$

As $\rho_\omega(\mathcal{P}_\beta) = \kappa_\beta$, this gives a contradiction.

Let $n < \omega$ be such that $\rho_{n+1}(\mathcal{P}_\beta) = \kappa_\beta = \beta < \rho_n(\mathcal{P}_\beta)$. By Lemma 1.2, $\eta = \mathrm{cf}^V(\beta^{+\mathcal{P}_\beta}) = \mathrm{cf}^V(\lambda_\beta) = \mathrm{cf}^V(\rho_n(\mathcal{P}_\beta))$. Let us pick a sequence $(\delta_i : i < \eta)$ of ordinals cofinal in $\rho_n(\mathcal{P}_\beta) = \mathcal{P}^n_\beta \cap \mathrm{OR}$. Let us write $\mathcal{P} = \mathcal{P}^n_\beta$.

For $i < \eta$, let

$$\bar{\sigma}_i : \bar{\mathcal{N}_i} \cong \mathrm{Hull}_1^{S^{\mathcal{P}}_{\delta_i}}(\beta \cup \{p_{n+1}(\mathcal{P}_\beta)\}),$$

where $\bar{\mathcal{N}_i}$ is transitive. We may construe $\bar{\sigma}_i$ as a Σ_0-elementary map from $\bar{\mathcal{N}_i}$ to $\mathcal{P}$. So by the Downward Extension of Embeddings Lemma (cf. [16, §§3 and 5]), there is some transitive $\mathcal{N}_i$ such that $\bar{\mathcal{N}_i} = \mathcal{N}_i^n$ and there is a weakly $r\Sigma_{n+1}$ elementary embedding

$$\sigma_i : \mathcal{N}_i \to \mathcal{P}_\beta$$

with $\sigma_i \supset \bar{\sigma}_i$. By the Condensation Lemma 1.3, $\mathcal{N}_i \lhd \mathcal{P}_\beta | \beta^{+\mathcal{P}_\beta}$.

Let us write $\mathcal{M}^\alpha_i = \pi_\alpha^{-1}(\mathcal{M}_i)$, for every $\alpha < \kappa$ and $i < \eta$. Because $\mathrm{cf}^V(\beta) \neq \eta$, there is some $\alpha < \beta$ and sets T, $T' \subset \eta$ which are both cofinal in η such that

$$i \in T \Longrightarrow \mathcal{M}^\beta_i,\, p_{\mathcal{M}^\beta_i} \in \mathrm{Hull}_1^{\mathcal{P}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\})$$

and

$$i \in T' \Longrightarrow \mathcal{N}_i,\, \sigma_i^{-1}(p_{n+1}(\mathcal{P}_\beta)) \in \mathrm{ran}(\pi_{\alpha\beta}).$$

We claim that

(1) $$\mathrm{Hull}_1^{\mathcal{P}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\}) \cap \lambda_\beta = \mathrm{ran}(\pi_{\alpha\beta}) \cap \lambda_\beta.$$

Well, first let $\xi \in \mathrm{ran}(\pi_{\alpha\beta}) \cap \lambda_\beta$. Let $\pi_{\alpha\beta}(\bar{\xi}) = \xi$. Then $\bar{\xi} \in \mathrm{Hull}_1^{\mathcal{M}^\alpha_i}(\alpha \cup \{p_{\mathcal{M}^\alpha_i}\})$ for some $i \in T$. But then

$$\xi \in \mathrm{Hull}_1^{\mathcal{M}^\beta_i}(\alpha \cup \{p_{\mathcal{M}^\beta_i}\}) \subset \mathrm{Hull}_1^{\mathcal{P}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\}).$$

Now let $\xi \in \mathrm{Hull}_1^{\mathcal{P}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\}) \cap \lambda_\beta$. We must then have $\xi \in \mathrm{Hull}_1^{S^{\mathcal{P}}_{\delta_i}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\})$ for some $i \in T'$. Fix such $i \in T'$, and pick a Σ_1 Skolem term τ and a parameter $\vec{\epsilon} \in [\alpha]^{<\omega}$

such that $\xi = \tau^{S^{\mathcal{P}}_{\delta_i}}(\vec{\epsilon}, p_{n+1}(\mathcal{P}_\beta))$. We have that $\beta \in \mathrm{Hull}_1^{\mathcal{P}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\})$ by (1), "$\supset$." We may therefore assume $i \in T'$ to be such that $\beta \in \mathrm{Hull}_1^{S^{\mathcal{P}}_{\delta_i}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\})$. But then $\mathrm{Hull}_1^{S^{\mathcal{P}}_{\delta_i}}(\beta \cup \{p_{n+1}(\mathcal{P}_\beta)\}) \models$ "There is a surjection from β onto ξ," and therefore we must have that $\xi + 1 \subset \mathrm{Hull}_1^{S^{\mathcal{P}}_{\delta_i}}(\beta \cup \{p_{n+1}(\mathcal{P}_\beta)\})$. This implies that $\xi \in \mathrm{Hull}_1^{\bar{\mathcal{N}}_i}(\beta \cup \{\sigma_i^{-1}(p_{n+1}(\mathcal{P}_\beta))\})$, and in fact that $\xi = \tau^{\bar{\mathcal{N}}_i}(\vec{\epsilon}, \sigma_i^{-1}(p_{n+1}(\mathcal{P}_\beta))$. We therefore also get that

$$\xi \in \mathrm{Hull}_1^{\bar{\mathcal{N}}_i}(\alpha \cup \{\sigma_i^{-1}(p_{n+1}(\mathcal{P}_\beta))\}) \subset \mathrm{ran}(\pi_{\alpha\beta}).$$

We have shown (1). Now let

$$\bar{\sigma} : \bar{\mathcal{P}} \cong \mathrm{Hull}_1^{\mathcal{P}}(\alpha \cup \{p_{n+1}(\mathcal{P}_\beta)\}).$$

By the Downward Extension of Embeddings Lemma (cf. [16, §§3 and 5]), there is some transitive $\mathcal{P}^*$ such that $\bar{\mathcal{P}} = (\mathcal{P}^*)^n$ and there is a weakly $r\Sigma_{n+1}$ elementary embedding

$$\sigma : \mathcal{P}^* \to \mathcal{P}_\beta$$

with $\sigma \supset \bar{\sigma}$. By the Condensation Lemma 1.3, $\mathcal{P}^* \lhd \mathcal{P}_\beta$. By (1), $\bar{\mathcal{P}}|\alpha^{+\bar{\mathcal{P}}} = \mathcal{S}_\alpha = \mathcal{P}_\alpha|\alpha^{+\mathcal{P}_\alpha}$, so that in fact $\mathcal{P}^* = \mathcal{P}_\alpha$. But then again by (1), we must have that

$$\mathcal{Q}_\alpha^\beta = \mathrm{ult}_n(\mathcal{P}_\alpha; \pi_{\alpha\beta}) = \mathrm{ult}_n(\mathcal{P}^*; \bar{\sigma} \restriction \mathcal{P}^*|\lambda_\alpha) = \mathcal{P}_\beta.$$

Contradiction!

□ (Theorem 3.4)

In the situation of Theorem 3.4, there can be no mouse $\mathcal{Q} \trianglerighteq \mathcal{S}$ with $\rho_\omega(\mathcal{Q}) \leq \kappa$, by the definition of $\mathcal{S}$ and by Lemma 1.2. We do not know, though, if there can be some mouse $\mathcal{Q} \rhd K^c||\kappa$ such that $\rho_\omega(\mathcal{Q}) < \kappa$.

Corollary 3.5 *Assume that K^c exists and that there is no premouse with a superstrong extender. Let κ be an ω-closed regular cardinal with $2^{<\kappa} = \kappa$, and let $\mathcal{S} = \mathcal{S}(\kappa)$. Suppose that κ is a limit cardinal in K^c. Then there is no mouse $\mathcal{M} \rhd \mathcal{S}$ such that $\rho_\omega(\mathcal{M}) < \kappa$ and $\mathcal{M}$ is sound above κ.*

Proof. Suppose that there were such a mouse $\mathcal{M}$. We may and shall assume that $\mathcal{M}$ is a least counterexample, so that $\mathcal{S} \cap \mathrm{OR} = \kappa^{+\mathcal{M}}$. Let $n < \omega$ be least such that $\rho = \rho_{n+1}(\mathcal{M}) < \kappa \leq \rho_n(\mathcal{M})$. If $\rho_n(\mathcal{M}) = \kappa$, then $\mathrm{cf}(\rho_{n+1}(\mathcal{M})^{+\mathcal{M}}) = \kappa$ by Lemma 1.2, and thus in fact $\rho_{n+1}(\mathcal{M})^{+K^c} = \rho_{n+1}(\mathcal{M})^{+\mathcal{M}} = \kappa$. If $\rho_n(\mathcal{M}) > \kappa$, then $\mathrm{cf}(\rho_{n+1}(\mathcal{M})^{+\mathcal{M}}) = \mathrm{cf}(\kappa^{+\mathcal{M}}) = \kappa$ by Lemma 1.2, and thus again $\rho_{n+1}(\mathcal{M})^{+K^c} = \rho_{n+1}(\mathcal{M})^{+\mathcal{M}} = \kappa$. Hence in both cases κ is a successor cardinal in K^c. Contradiction!

□ (Corollary 3.5)

The proof of Theorem 3.4 also shows the following.

Theorem 3.6 *Assume that K^c exists and that there is no premouse with a superstrong extender. Assume* CH*, and let $\mathcal{S} = \mathcal{S}(\aleph_2)$. Suppose that $\aleph_2$ is a limit cardinal in K^c. Then* $\mathrm{cf}^V(\mathcal{S} \cap \mathrm{OR}) > \omega$.

PROOF. Otherwise we may pick β with $\mathrm{cf}^V(\beta) + \omega_1 \neq \omega = \eta$ in the proof of Theorem 3.4.

$\square$ (Theorem 3.6)

4 K^c and $\square(\kappa)$

A sequence $(C_\nu : \nu < \alpha)$ is coherent iff for all limit ordinals $\nu < \alpha$, $C_\nu \subset \nu$ is club in ν and $C_{\bar{\nu}} = C_\nu \cap \bar{\nu}$ whenever $\bar{\nu}$ is a limit point of C_ν. Here, α is allowed to be a successor ordinal, say $\alpha = \lambda + 1$, where λ is a limit ordinal, in which case C_λ is called a thread through $(C_\nu : \nu < \lambda)$. We say that $\square(\lambda)$ holds iff there is some coherent sequence $(C_\nu : \nu < \lambda)$ without a thread through it. It is easy to see that $\square_\kappa$ implies $\square(\kappa^+)$.

Our proofs of Theorems 0.1 and 0.3 will need a result of Todorcevic (cf. [21]) which says that if PFA holds, then for all κ with $\mathrm{cf}(\kappa) \geq \omega_2$, $\square(\kappa)$ fails. Another ingredient for the proofs of Theorems 0.1 and 0.3 is a result of Zeman and the second author (cf. [14]) according to which if $\mathcal{M}$ is a mouse, then in $\mathcal{M}$, $\square_\kappa$ holds for all cardinals κ which are not subcompact.

Theorem 4.1 *Suppose there is no non-domestic premouse, or just suppose that K^c exists and there is no subcompact cardinal in K^c. Let $\kappa \geq \aleph_3$ be regular and countably closed. If $2^{<\kappa} > \kappa$, then let us also suppose that the K^c of $V^{\mathrm{Col}(\kappa,\kappa)}$ exists and there is no subcompact cardinal in the K^c of $V^{\mathrm{Col}(\kappa,\kappa)}$. Then one of the following is true:*

(a) $\square(\kappa)$,

(b) $\square_\kappa$.

PROOF. Let us first prove this under the additional hypothesis that $2^{<\kappa} = \kappa$.

If κ is a successor cardinal in K^c, say $\kappa = \nu^{+K^c}$, then $\square_\nu$ and hence also $\square(\kappa)$ holds true in V by [14]. Let us thus assume κ to be a limit cardinal in K^c, and let us also assume that $\square_\kappa$ fails. Let $\mathcal{S} = \mathcal{S}(\kappa)$. Since $\square_\kappa$ holds in $\mathcal{S}$ by [14], we shall have that $\kappa^{+\mathcal{S}} < \kappa^{+V}$. In the light of Theorem 3.4, we must then have that $\kappa = \mathrm{cf}(\mathcal{S} \cap \mathrm{OR})$.

By Corollary 3.5, there is no mouse $\mathcal{M} \rhd \mathcal{S}$ such that $\rho_\omega(\mathcal{M}) < \kappa$ and $\mathcal{M}$ is sound above κ.

Claim. The $\square_\kappa$-sequence of $\mathcal{S}$, as defined by Zeman and the second author cannot be threaded.

PROOF. Suppose otherwise. Say D threads the canonical $\square_\kappa$-sequence of $\mathcal{S}$. Let $\lambda = (\kappa^+)^{\mathcal{S}}$. (I.e., λ is the set of ordinals of $\mathcal{S}$.) Then D is club in λ. By [11, Lemma 4.6], there is a unique premouse $\mathcal{Q}$ such that $\mathcal{Q}$ extends $\mathcal{S}$ and collapses λ. For this, we use that λ has uncountable cofinality. As written, [11, Lemma 4.7] applies to K not $\mathcal{S}$ but its proof shows that $\mathcal{Q}$ is iterable. In that proof, substitute $\mathcal{S}$ for K and our Theorem 4.4 for the weak covering theorem for K, and stop at line 19 on page 110. By the definition of $\mathcal{S}$ and as there is no mouse $\mathcal{M} \triangleright \mathcal{S}$ such that $\rho_\omega(\mathcal{M}) < \kappa$ and $\mathcal{M}$ is sound above κ, $\mathcal{Q}$ is a proper initial segment of $\mathcal{S}$. This is a contradiction.

$\square$ (Claim)

This shows Theorem 4.1 under the additional hypothesis that $2^{<\kappa} = \kappa$.

Let us now drop the hypothesis that $2^{<\kappa} = \kappa$, so that we may no longer directly apply Theorem 3.4. However, inside $V^{\mathrm{Col}(\kappa,\kappa)}$ we do have that κ is regular, countably closed, and $2^{<\kappa} = \kappa$. We may thus run the above argument with the K^c and the $S(\kappa)$ of $V^{\mathrm{Col}(\kappa,\kappa)}$. Let us write

$$(K^c)^* = (K^c)^{V^{\mathrm{Col}(\kappa,\kappa)}}$$

and

$$\mathcal{S}^* = (\mathcal{S}(\kappa))^{V^{\mathrm{Col}(\kappa,\kappa)}}.$$

So $\mathcal{S}^*$ is the stack over $(K^c)^*|\kappa$ produced inside $V^{\mathrm{Col}(\kappa,\kappa)}$. Notice that $\mathcal{S}^* \in V$ by the homogeneity of $\mathrm{Col}(\kappa,\kappa)$.

We may now argue as above to get either $\square(\kappa)$ or else $\square_\kappa$. Notice that the $\square$-sequences of $\mathcal{S}^*$ are in V by $\mathcal{S}^* \in V$, that $\mathcal{S}^* \cap \mathrm{OR} \leq \kappa^{+V}$, that $\mathcal{S}^* \cap \mathrm{OR} < \kappa^{+V}$ implies $\mathrm{cf}^V(\mathcal{S}^* \cap \mathrm{OR}) = \kappa$, and that the unthreadability of the $\square_\kappa$-sequence of $\mathcal{S}^*$ in $V^{\mathrm{Col}(\kappa,\kappa)}$ trivially implies the unthreadability of the $\square_\kappa$-sequence of $\mathcal{S}^*$ in V.

$\square$ (Theorem 4.1)

Proofs of Theorems 0.1 and 0.3 and of Corollary 0.2. Theorem 0.1 is immediate. To show Corollary 0.2, suppose PFA to hold. This implies $\aleph_2^{\aleph_0} = \aleph_2$, so that if the conclusion of Corollary 0.2 were to fail, Theorem 4.1 would give $\square(\aleph_3)$ or else $\square_{\aleph_3}$ (which implies $\square(\aleph_4)$). On the other hand, by [21], both $\square(\aleph_3)$ as well as $\square(\aleph_4)$ fail under PFA. Contradiction! Theorem 0.3 is also immediate. $\square$ (Theorems 0.1, 0.2, and 0.3)

5 Weak covering at weakly compact cardinals

In this section, we prove Theorems 0.4 and 0.5.

The following Lemma is due to the third author.

Lemma 5.1 *Assume that K^c exists and that there is no premouse with a superstrong extender. Let κ be a weakly compact cardinal, and let $\mathcal{S} = \mathcal{S}(\kappa)$ (cf. Definition 3.2). Then $\mathcal{S} \cap \mathrm{OR} = \kappa^{+V}$.*

Proof. Set $\eta = \mathcal{S} \cap \mathrm{OR}$. Suppose that $\eta < \kappa^+$. We aim to derive a contradiction.

Let $\theta > \kappa$ be a $<\kappa$-closed regular cardinal. Let

$$\sigma\colon M \to_{\Sigma_{100}} V$$

be such that M is transitive, $\mathrm{Card}(M) = \kappa$, $\mathcal{S} \cup \{\theta\} \subset \mathrm{ran}(\sigma)$, and ${}^{<\kappa}M \subset M$. Inside M, there is some

$$\chi\colon P \to_{\Sigma_\omega} H^M_{\sigma^{-1}(\theta)}$$

such that P is transitive, $\mathrm{Card}(P) = \kappa$ in M, $\mathcal{S} \subset \mathrm{ran}(\chi)$, and ${}^{<\kappa}P \subset P$ (in M, and therefore also in V).

Because κ is weakly compact, there is some

$$\pi\colon M \to_{\Sigma_\omega} N,$$

where N is transitive, ${}^{<\kappa}N \subset N$, and $\mathrm{crit}(\pi) = \kappa$. Let us write $W = (K^c)^N$, so that $\pi(\mathcal{S}) = (\mathcal{S}(\pi(\kappa)))^N$ is the stack of sound mice end-extending $W||\pi(\kappa)$ and projecting to $\pi(\kappa)$ from the point of view of N. Let us also write $\lambda = \pi(\kappa)$.

Notice that $\pi(\mathcal{S})$ is a mouse, as π is countably complete.

Claim 1. $\mathcal{S} = W|\kappa^{+W}$.

Proof. By the Condensation Lemma 1.3, $\mathcal{S} \triangleleft \pi(\mathcal{S})$, and therefore $\mathcal{S} \trianglelefteq W|\kappa^{+W}$. If $\mathcal{S} \triangleleft W|\kappa^{+W}$, then there is some $\mathcal{M} \triangleright \mathcal{S}$ such that $\mathcal{M} \triangleleft W|\kappa^{+W}$ and $\rho_\omega(\mathcal{M}) = \kappa$. But because any such $\mathcal{M}$ is a sound mouse, this contradicts the definition of $\mathcal{S}$. Hence $\mathcal{S} = W|\kappa^{+W}$.

$\square$ (Claim 1)

Claim 2. $\pi \restriction P \in N$.

Proof. This is by Kunen's old argument. As $\mathcal{P}(\kappa) \cap M \subset N$, every set in M which is hereditarily of size $\leq \kappa$ in M is also an element of N. In particular, $P \in N$, and if $f\colon \kappa \to P$ is bijective, $f \in M$, then $f \in N$. For $x \in P$, say $x = f(\xi)$, we have that $\pi(x) = \pi(f(\xi)) = \pi(f)(\xi)$, so that $\pi \restriction P$ may be computed inside N from f, $\pi(f)$.

$\square$ (Claim 2)

Let us define an extender F by $F = \pi \restriction \mathcal{S}$. Of course, $F = (\pi \restriction P) \restriction \mathcal{S}$, so that $F \in N$ by Claims 1 and 2.

Claim 3. $N \models$ "F is certified by a collapse."

Proof. Set $k = \pi(\chi) \circ (\pi \restriction P)$. By Claim 2, $k \in N$. We have that

$$k\colon P \to_{\Sigma_\omega} \pi(H^M_{\sigma^{-1}(\theta)}) = H^N_{\pi\circ\sigma^{-1}(\theta)},$$

where $\pi \circ \sigma^{-1}(\theta)$ is a $< \kappa$-closed cardinal in N and ${}^{<\kappa}P \subset P$ in N (as well as in V). Because $F = k \restriction \mathcal{S}$, k witnesses that F is certified by a collapse inside N (cf. [8]).

$\square$ (Claim 3)

Let us consider the potential premouse $\mathcal{S}^* = (\pi(\mathcal{S}); F)$ which results from $\pi(\mathcal{S})$ by adding F as its top extender. For all we know, $\mathcal{S}^*$ need not satisfy the Initial Segment Condition (cf. [24, p. 283]), though. Let $\alpha \leq \lambda$ be the least cutpoint of $\mathcal{S}^*$, i.e., α is least such that if $f \in {}^\kappa\kappa \cap \mathcal{S}$ and $\xi < \alpha$, then $i_F(f)(\xi) < \alpha$. We then have that

$$\mathcal{S}^{**} = (\pi(\mathcal{S})|\alpha^{+\pi(\mathcal{S})}; F|\alpha)$$

does satisfy the Initial Segment Condition and is hence a premouse. Notice that $\mathcal{S}^{**} \in N$.

Case 1. $\alpha < \lambda$.

Then $\pi(\mathcal{S})|\alpha^{+\pi(\mathcal{S})} = W|\alpha^{+\pi(\mathcal{S})}$. However, $F|\alpha$ is certified by a collapse inside N by Claim 3, so that we may apply Lemma 2.12 inside N to get a contradiction.

Case 2. $\alpha = \lambda$, i.e., $\mathcal{S}^{**} = \mathcal{S}^*$.

Notice that the generators of F must be unbounded in λ, as λ is an inaccessible cardinal of N. Therefore, $\mathcal{S}^*$ is a premouse with a superstrong extender. Using [3], $\mathcal{S}^*$ can in fact easily be verified to be a mouse. Contradiction!

$\square$ (Lemma 5.1)

Proofs of Theorems 0.4 and 0.5.[11] Suppose one of the conclusions of Theorems 0.4 or 0.5 to fail. Let $\mathcal{S}$ denote $\mathcal{S}(\kappa)$ as constructed in $V^{\mathrm{Col}(\kappa,\kappa^+)}$. We may apply Lemma 5.1 inside $V^{\mathrm{Col}(\kappa,\kappa^+)}$ to see that κ^{+V} has size κ in $\mathcal{S}$. However, $\mathcal{S} \in V$ by the homogeneity of $\mathrm{Col}(\kappa, \kappa^+)$. Contradiction!

$\square$ (Theorems 0.4 and 0.5)

6 An amendment

In the proofs of our main Theorems, we cannot directly work with the K^c construction of [8], as the definition of the K^c of [8] makes reference to some $A \subset \mathrm{OR}$ such that $V = L[A]$. If $2^{<\kappa} > \kappa$ in the situation of the proof of Theorem 4.1, then for the K^c of $V^{\mathrm{Col}(\kappa,\kappa)}$ as [8] would define it, $K^c||\kappa$ will be defined by way of some $A \cap \kappa \subset \kappa$ which in $V^{\mathrm{Col}(\kappa,\kappa)}$ codes all of H_κ and can therefore not exist in V, so that there is no reason for $K^c||\kappa$ to be in V. A similar problem arises in the proof of Theorems 0.4 and 0.5. We also refer the reader to the discussion right after the proof of Theorem 2.8.

In this final section, we describe how to manage using the K^c construction of [8] (and thereby avoid having to cite [6]) in order to arrive at proofs of our main theorems.

Let $\kappa \geq \aleph_3$ be regular and countably closed, but possibly $2^{<\kappa} > \kappa$. The goal is to isolate some $A \subset \kappa$ and use it to locally define a $K^c||\kappa$, which we shall denote by $K^{c,A}||\kappa$,

[11] The third author thanks Gunter Fuchs for pointing out to him that in order to prove these theorems one would just need to prove weak covering at a weakly compact cardinal for a hereditarily ordinal definable inner model.

in a fashion as in [8] such that even in $V^{\mathrm{Col}(\kappa,\kappa)}$ and also in $V^{\mathrm{Col}(\kappa,\kappa^+)}$, $K^{c,A}||\kappa$ will have the key "universality" properties which are needed so as to arrive at proofs at our main theorems.

To commence, we need a localization of the concept of being "certified" from [8].

Let us from now on fix a regular and countably closed cardinal $\kappa \geq \aleph_3$.

Definition 6.1 *Let $A \subset \kappa$. Let $F \in H_\kappa$ be an extender with $\mu = \mathrm{crit}(F)$ and $\lambda = F(\kappa)$. We say that F is A-*certified *iff for all countable $u \subset \lambda$ and for all countable $Y \subset \mathrm{dom}(F)$, there is some order-preserving $\tau: u \to \mu$ such that for all $\alpha \in u$ and $X \in Y$, $\alpha \in F(X)$ iff $\tau(\alpha) \in X$, and*

$$(L_\kappa[A]; \in, (\alpha: \alpha \in u)) \equiv_{\Sigma_{1+}} (L_\mu[A]; \in, (\tau(\alpha): \alpha \in u)).$$

Cf. [8, Lemma 1.8] for a formulation of "being certified" which also uses types as does Definition 6.1.

Definition 6.2 *Let $A \subset \kappa$ The* maximal A-certified $K^c||\kappa$ construction *is the unique K^c construction*

$$(\mathcal{N}_\xi, \mathcal{M}_\xi: \xi < \theta)$$

such that

(a) for all $\xi < \theta$, $\mathcal{N}_\xi$ is active with top extender F if and only if there is some $\bar{\xi}$ such that $\xi = \bar{\xi} + 1$ and F is the unique extender $G \in L_\kappa[A]$ such that $(\mathcal{M}_{\bar{\xi}}; \in, E^{\mathcal{M}_{\bar{\xi}}}, G)$ is a premouse and G is certified by a collapse, and

(b) $\theta \leq \kappa + 1$ is largest such that such a K^c construction exists.

If $\theta = \kappa + 1$, then we write $K^c||\kappa$ for $\mathcal{M}_\kappa$ and say that the A-certified $K^c||\kappa$ exists.

The arguments of [8] show the following. (Compare with Corollary 2.11.)

Lemma 6.3 *If there is no non-dometic premouse, then for every $A \subset \kappa$ the A-certified $K^c||\kappa$ exists and is a mouse.*

We now want to pick an $A \subset \kappa$ so that we have the appropriate version of Lemma 2.12 for the A-certified $K^c||\kappa$. Let us assume that for every $A \subset \kappa$ the A-certified $K^c||\kappa$ exists.

In order to find an A as desired, let us construct a sequence

$$((A_\xi: \xi < \kappa), (\gamma_\xi: \xi < \kappa), (\mathcal{N}_\xi, \mathcal{M}_\xi: \xi < \kappa))$$

such that the following hold true for every $\xi < \kappa$.

1. $A_\xi \subset \gamma_\xi$ and if $\bar{\xi} \leq \xi$, then $A_{\bar{\xi}} = A_\xi \cap \gamma_{\bar{\xi}}$.

2. For every $A \subset \kappa$ with $A_\xi = A \cap \gamma_\xi$, $L_{\gamma_\xi}[A_\xi] \prec_{\Sigma_{1+}} L_\kappa[A]$.

3. $L_{\gamma_\xi}[A_\xi] \models$ "$(\mathcal{N}_i, \mathcal{M}_i: i \leq \xi)$ is the sequence consisting of the first $\xi + 1$ models from the maximal A_ξ-certified $K^c||\kappa$ construction."

4. If $\xi = \bar{\xi} + 1$, where $\mathcal{M}_{\bar{\xi}}$ does not have a top extender, and if there are $\gamma \geq \gamma_{\bar{\xi}}$, $B \subset \gamma$, and $F \in H_\kappa$ such that $A_{\bar{\xi}} = B \cap \gamma_{\bar{\xi}}$, $L_\gamma[B] \prec_{\Sigma_{1+}} L_\kappa[A]$ for every $A \subset \kappa$ with $B = A \cap \gamma$, and setting $\mathcal{M} = (\mathcal{M}_{\bar{\xi}}; \in, E^{\mathcal{M}_{\bar{\xi}}}, F)$ and $\mathcal{N}$ = the core of $\mathcal{M}$, $L_\gamma[B] \models$ "$(\mathcal{N}_i, \mathcal{M}_i : i < \xi)^\frown(\mathcal{M}, \mathcal{N})$ is the sequence consisting of the first $\xi + 1$ models from the maximal B-certified $K^c||\kappa$ construction," then there is an $F \in L_{\gamma_\xi}[A_\xi]$ such that $\mathcal{M}_\xi = (\mathcal{M}_{\bar{\xi}}; \in, E^{\mathcal{M}_{\bar{\xi}}}, F)$ and $\mathcal{N}_\xi$ = the core of $\mathcal{M}_\xi$.

There is no problem with this construction. The second item can be arranged by having $\gamma_\xi = \sup\{\gamma_\xi^i | i < \omega_1\}$, where each $L_{\gamma_\xi^i}[A_\xi \cap \gamma_\xi^i]$ is closed under ω-sequences (here we use $\mathrm{Card}(\alpha)^{\aleph_0} < \kappa$) and each $L_{\gamma_\xi^{i+1}}[A_\xi \cap \gamma_\xi^{i+1}]$ contains witnesses to all Σ_{1+} statements with parameters in $L_{\gamma_\xi^i}[A_\xi \cap \gamma_\xi^i]$ which are true in some $L_\kappa[A]$, where $A \subset \kappa$ is such that $A \cap \gamma_\xi^i = A_\xi \cap \gamma_\xi^i$.

Let $A \subset \kappa$ be given by $(A_\xi : \xi < \kappa)$, i.e.,

$$A = \bigcup_{\xi < \kappa} A_\xi.$$

Let us write $K^{c,A}||\kappa$ for the premouse of height κ which is produced by $(\mathcal{N}_\xi, \mathcal{M}_\xi : \xi < \kappa)$.

We shall now prove the following version of Lemma 2.12.

Lemma 6.4 *Let A and $K^{c,A}||\kappa$ be defined as above, and suppose that there is no inner model with a superstrong cardinal. Let $\mathcal{S}$ denote the unique premouse such that $\mathcal{N} \trianglelefteq \mathcal{S}$ iff there is some sound mouse $\mathcal{M} \trianglerighteq K^{c,A}||\kappa$ with $\rho_\omega(\mathcal{M}) = \kappa$ and $\mathcal{N} \trianglelefteq \mathcal{M}$. There is then no elementary embedding*

$$\pi : H \to H_{\kappa^{++}}$$

such that H is transitive and ${}^\omega H \subset H$, $\mu = \mathrm{crit}(\pi) < \kappa = \pi(\mu)$, $\{A, \mathcal{S}\} \subset \mathrm{ran}(\pi)$, and $\mathcal{P}(\mu) \cap K^{c,A}||\kappa \subset H$.

Proof. We imitate the proof of Lemma 2.12. Suppose there were some such embedding π. Let $F = \pi \upharpoonright \mathcal{P}(\mu) \cap K^{c,A}||\kappa$. As in the proof of Lemma 2.12, there is then some $\alpha < \kappa$ and some $\xi^* < \kappa$ such that $(\mathcal{M}_{\xi^*}, F|\alpha)$ would be a premouse. The proof of Lemma 2.4 shows that $F|\alpha$ is A-certified in the sense of Definition 6.1.

We claim that with $\bar{\xi} = \xi^*$, $F|\alpha$ witnesses that the hypothesis in the last item of the above recursive definition of

$$((A_\xi : \xi < \kappa), (\gamma_\xi : \xi < \kappa), (\mathcal{N}_\xi, \mathcal{M}_\xi : \xi < \kappa))$$

is satisfied. This will finish the proof of Lemma 6.4, because there will then be some $G \in L_\kappa[A]$ such that $\mathcal{M}_{\xi^*+1}$ results from $\mathcal{M}_{\xi^*}$ by adding G as a top extender, which – as in the proof of Lemma 2.12 – contradicts the fact that α must be a cardinal in $K^{c,A}||\kappa$.

Let $((u_k, Y_k) : k < \mathrm{Card}(\alpha)^{\aleph_0})$ be a list of all pairs (u, Y) such that $u \subset \alpha$ is countable and $Y \subset \mathcal{P}(\mu) \cap K^{c,A}||\kappa$ is countable. As $L_{\gamma_{\xi^*}}[A \cap \gamma_{\xi^*}] \prec_{\Sigma_{1+}} L_\kappa[A]$ and because $F|\alpha$

is A-certified, we may let $(\tau_k : k < \mathrm{Card}(\alpha)^{\aleph_0})$ be such that for all $\beta \in u_k$ and $X \in Y_k$, $\beta \in F|\alpha(X)$ iff $\tau_k(\beta) \in X$, and

$$(L_{\gamma_{\xi^*}}[A \cap \gamma_{\xi^*}]; \in, (\beta : \beta \in u_k)) \equiv_{\Sigma_{1+}} (L_\mu[A \cap \mu]; \in, (\tau(\beta) : \beta \in u_k)).$$

We may then pick $\gamma \geq \gamma_{\xi^*}$ and $B \subset \gamma$ with $B \cap \gamma_{\xi^*} = A_{\xi^*}$ such that $L_\gamma[B] \prec_{\Sigma_{1+}} L_\kappa[A^*]$ for every $A^* \subset \kappa$ with $B = A^* \cap \gamma$ and

$$((u_k, Y_k, \tau_k) : k < \mathrm{Card}(\alpha)^{\aleph_0}) \in L_\gamma[B].$$

(Again, notice that $\mathrm{Card}(\alpha)^{\aleph_0} < \kappa$.) This is easily seen to show that $F|\alpha$ indeed witnesses that the hypothesis in the last item of the above recursive definition is satisfied. $\square$ (Lemma 6.4)

The reader will happily verify that Lemma 6.4 will still be true in any forcing extension of V which does not add any bounded subsets of κ. We could therefore have used $K^{c,A}||\kappa$ to run the arguments in the preceeding sections to arrive at proofs of our main theorems.

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