I.

Recall the definition of L :

$$J_0 = \emptyset .$$

$$J_{\alpha+1} = \mathrm{rud}(J_\alpha \cup \{J_\alpha\}) , \quad \text{where}$$

$\mathrm{rud}(X)$ is the closure of X under the <u>rudimentary functions</u> which are generated by :

$$f(\vec{x}) = x_i$$

$$f(\vec{x}) = x_i \setminus x_j$$

$$f(\vec{x}) = \{x_i, x_j\}$$

$$f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_n(\vec{x}))$$

$$f(\vec{x}) = \bigcup_{y \in x_1} g(y, x_2, \ldots, x_n) .$$

$$J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha \quad \text{for } \lambda \text{ a limit} .$$

$$L = \bigcup_\alpha J_\alpha .$$

All J_α's are transitive and rud-closed.

Moreover, for all α:

$(J_\gamma : \gamma < \alpha)$ is Σ_1 over J_α, and
$(J_\gamma : \gamma < \beta) \in J_\alpha$ for all $\beta < \alpha$.

Point is:

(1) Every inner model of set theory must contain all the J_α's.

(2) (Gödel) L is an inner model of set theory.

<u>Definition.</u> Let W be a class. W is an <u>inner model</u> of set theory iff W is transitive,
W contains all the ordinals, and
$W \models ZFC$.

<u>Theorem</u> (Gödel). $L \models GCH$.

Proof: Let $X \subset \alpha$, $X \in L$. Write $\beta = \alpha^{+L}$. Need to see $X \in J_\beta$, as $L \models \overline{\overline{J_\beta}} = \overline{\overline{\beta}}$.

Say $X \in J_\gamma$. Pick, working in L,

$$\pi : M \cong Z \prec J_\gamma,$$

where $X \cup \alpha+1 \subset Z$, $\overline{\overline{Z}} = \alpha$, and M is transitive.

By the <u>Condensation Lemma</u> for L, $M = J_\delta$ for some δ. But we must have $\delta < \beta$. Moreover $X \in M$.

So $X \in M = J_\delta \subset J_\beta$.

$\dashv$

Condensation Lemma for L.

Let α be arbitrary, and let

$$\pi : M \longrightarrow_{\Sigma_1} J_\alpha ,$$

where M is transitive. There is then some $\bar{\alpha} \leq \alpha$ s.t. $M = J_{\bar{\alpha}}$.

Another application:

Theorem (Jensen). $L \models \forall \kappa \; \Diamond_{\kappa^+}$.

In fact, more is true.

To show stronger results, though, we need Jensen's fine structure theory.

<u>Definition.</u> $\Box_\kappa$ is the statement :

There is some $(C_\alpha : \alpha < \kappa^+)$ s.t.

for all limit ordinals $\alpha < \kappa^+$:

$C_\alpha \subset \alpha$, in fact

C_α is a club subset of α of

order type $\leq \kappa$, and

whenever $\beta < \alpha$ is a limit point of C_α,

then $C_\alpha \cap \beta = C_\beta$.

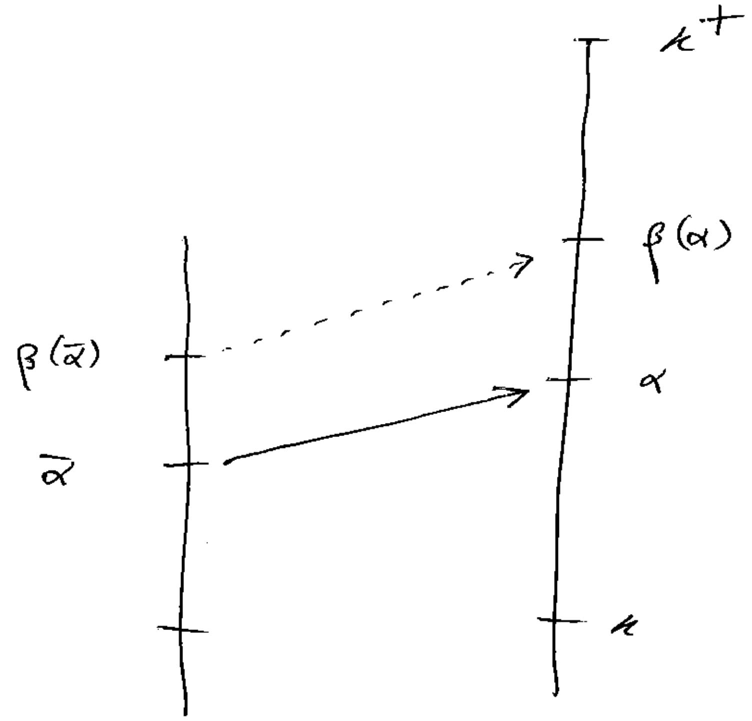
<u>Theorem.</u> (Jensen) $L \models \forall \kappa \, \Box_\kappa$.

Proof : Let, working in L,

$C = \{ \alpha < \kappa^+ : J_\alpha \prec J_{\kappa^+} \}$.

C contains a club. It suffices to

construct $(C_\alpha : \alpha \in C)$.



For $\alpha \in C$ let $\beta(\alpha) \geq \alpha$ the least $\beta \geq \alpha$ s.t. $J_{\beta+1}$ has a new subset of κ, i.e.,

(*) $\quad \mathcal{P}(\kappa) \cap J_{\beta+1} \not\subset J_\alpha$.

The plan is to have $\bar\alpha \in C_\alpha$ iff $\bar\alpha < \alpha$ and there is a "canonical" map

$$\pi_{\bar\alpha\alpha} : J_{\beta(\bar\alpha)} \longrightarrow J_{\beta(\alpha)} .$$

If (*) holds, then $\mathcal{P}(\kappa) \cap \underset{\sim}{\Sigma}_\omega^{J_\beta} \not\subset J_\alpha$.

The map $\pi_{\bar\alpha\alpha}$ thus cannot be fully elementary, and one has to be careful.

Suppose $\mathcal{P}(\kappa) \cap \underset{\sim}{\Sigma}_1^{J_\beta} \not\subset J_\alpha$, and say p is the least finite subset of $\omega\beta$ s.t. there is some X in

$$(\mathcal{P}(\kappa) \cap \Sigma_1^{J_\beta}(\{p\})) \setminus J_\alpha .$$

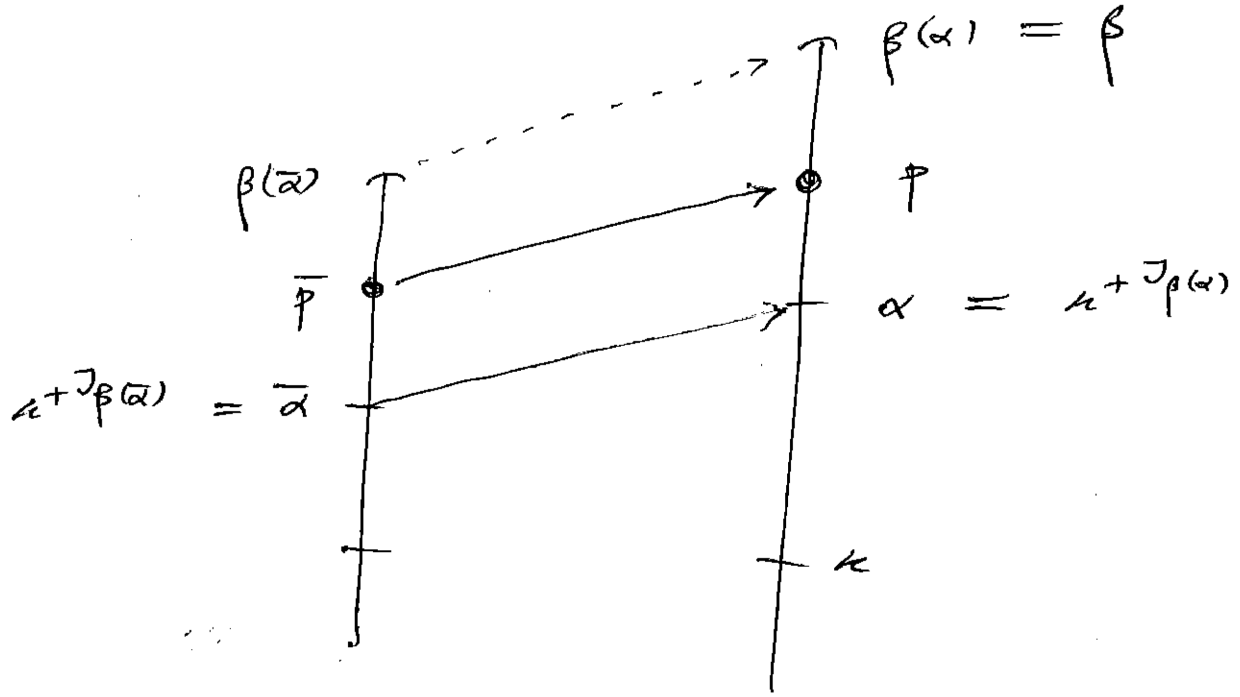
p is then the <u>standard parameter</u> of J_β, and κ is the (<u>first</u>) <u>projectum</u> of J_β.

We define $\bar{\alpha} \in C'_\alpha$ iff there is a (unique) embedding

$$\pi_{\bar{\alpha}\alpha} : J_{\beta(\bar{\alpha})} \longrightarrow_{\Sigma_0} J_{\beta(\alpha)}$$

s.t. there is some finite $\bar{p} \subset \omega\beta(\bar{\alpha})$ with $(\mathcal{P}(\kappa) \cap \Sigma_1^{J_{\beta(\bar{\alpha})}}(\{\bar{p}\})) \setminus J_{\bar{\alpha}} \neq \emptyset$, and if $\bar{p}$ is least such, then

$$\pi_{\bar{\alpha}\alpha} \restriction \bar{\alpha} = \mathrm{id} \quad \text{and} \quad \pi_{\bar{\alpha}\alpha}(\bar{\alpha}, \bar{p}) = \alpha, p .$$



The case

$$\mathcal{P}(\kappa) \cap \underset{\sim}{\Sigma}_{n+1}^{J_\beta} \not\subset J_\alpha, \qquad \text{but}$$

$$\mathcal{P}(\kappa) \cap \underset{\sim}{\Sigma}_{n}^{J_\beta} \subset J_\alpha,$$

where $n > 0$ will be reduced to the case that $n = 0$, by considering "reducts" of J_β.

Finally, we obtain C_α by thinning out C'_α a little bit. $\dashv$

The fine structure theory is also needed in the proof of the <u>Covering Theorem</u> for L :

<u>Theorem</u> (Jensen). TFAE.

(1) $0^{\#}$ does not exist.

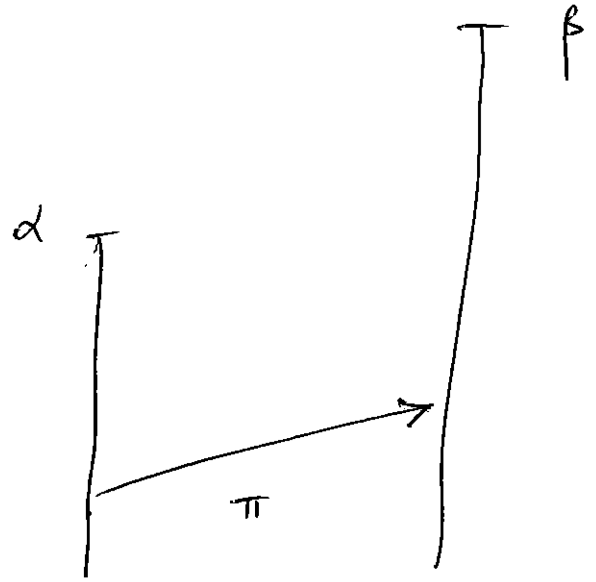
(2) For every unctble. set X of ordinals there is a set $Y \in L$ of the same size as X s.t. $Y \supset X$.

Proof of (1) $\Longrightarrow$ (2) : Pick

$$\pi : J_{\bar{\alpha}} \longrightarrow J_{\beta} ,$$

where $\beta > \sup(X)$, $X \subset \operatorname{ran}(\pi)$, and $\overline{\overline{J_{\bar{\alpha}}}} = \overline{\overline{\bar{\alpha}}} = \overline{\overline{X}}$.

We'll use : $0^{\#}$ exists iff there is a nontrivial $\sigma : L \longrightarrow L$.



<u>Case</u> 1. α is a cardinal in L.

We may then assume π was picked in such a way that π can be extended to $\tilde{\pi} : L \longrightarrow L$.

If π has a critical point, then $0^{\#}$ ex.

<u>Case</u> 2. α is not a cardinal in L.

Let $\gamma(\alpha) \geq \alpha$ be the least $\gamma \geq \alpha$ s.t.

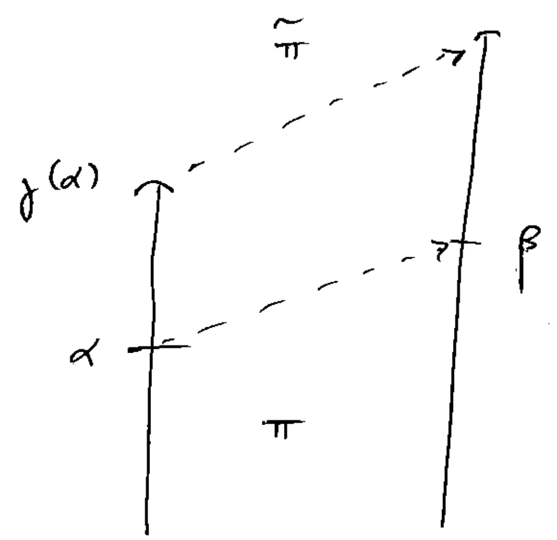
$$\mathcal{P}(\xi) \cap J_{\gamma+1} \not\subset J_\alpha \text{ for some } \xi < \alpha .$$

$$\shortparallel$$

$$\mathcal{P}(\xi) \cap \underset{\sim}{\Sigma}_\omega^{J_\gamma}$$

Let us assume that

$$\mathcal{P}(\xi) \cap \underset{\sim}{\Sigma}_1^{J_{\gamma(\alpha)}} \not\subset J_\alpha , \quad \text{some } \xi < \alpha$$



We may take the "ultrapower" of $J_{\gamma(\alpha)}$ by π, getting

$$\tilde{\pi} : J_{\gamma(\alpha)} \longrightarrow_{\Sigma_0} M \quad \text{cofinal}.$$

We may assume π was picked in such a way that M is transitive, so that in fact $M = J_\delta$, some δ.

$X \subset \operatorname{ran}(\pi)$ may then be covered using the canonical Σ_1 Skolem function of J_δ.

Let $p \subset \omega\gamma(\alpha)$ be least s.t. there is some $\xi < \alpha$ with $\mathcal{P}(\xi) \cap \Sigma_1^{J_{\gamma(\alpha)}}(\{p\})$, and let ξ_0 be the least such ξ.

Then

$$J_{\gamma(\alpha)} = \mathrm{Hull}_{\Sigma_1}^{J_{\gamma(\alpha)}}(\xi_0 \cup \{p\}).$$

[Proof: If $\sigma : J_{\bar{\gamma}} \cong \mathrm{Hull}_{\Sigma_1}^{J_{\gamma(\alpha)}}(\xi_0 \cup \{p\})$, then $\bar{\gamma} = \gamma(\alpha)$, as the new subset of ξ_0 is definable over $J_{\bar{\gamma}}$. Then $\sigma^{-1}(p) = p$ by the choice of p. Then $\sigma = \mathrm{id}$.]

Therefore,

$$X \subset \mathrm{ran}(\pi) \subset \mathrm{Hull}_{\Sigma_1}^{J_\delta}(\pi''\xi_0 \cup \{\pi(p)\}).$$

Inductively, $\pi''\xi_0 \in L$, and hence we may set $Y = \mathrm{Hull}_{\Sigma_1}^{J_\delta}(\pi''\xi_0 \cup \{\pi(p)\})$.

$\dashv$

Applications :

(1) Let κ be singular, and assume that $\square_\kappa$ fails. Then $0^\#$ exists.
In particular, PFA (the Proper Forcing Axiom) implies that $0^\#$ exists.

(2) Suppose SCH (the Singular Cardinals Hypothesis) to be false. Then $0^\#$ exists.

(3) The Axiom of Determinacy (in fact just Π^1_1 determinacy) implies that $0^\#$ exists (Harrington).

(4) If (ZF +) every uncountable cardinal is singular, then $0^\#$ exists.

(5) If there is a precipitous / saturated ideal (say on ω_1), then $0^\#$ exists.

We now aim to produce generalizations of (1), (2), (4), (5), etc.

I.e., exploiting a given hypothesis φ, we aim to produce not only $0^\#$ but also models with measurable cardinals, etc.

"$0^\#$ exists" states that L is <u>not</u> <u>rigid</u>.

The idea is to produce a model, K, which is "saturated" with respect to the non-rigidity of its proper inner models.

This model should then itself be rigid and reflect the large cardinal situation of V.

It turns out that the model needs to be constructed in <u>two</u> <u>steps</u>.

1st step: Construct K^c, the "certified core model." K^c is a preliminary version of K, which, however, sometimes is useful on its own.

2nd step: Isolate K from K^c.

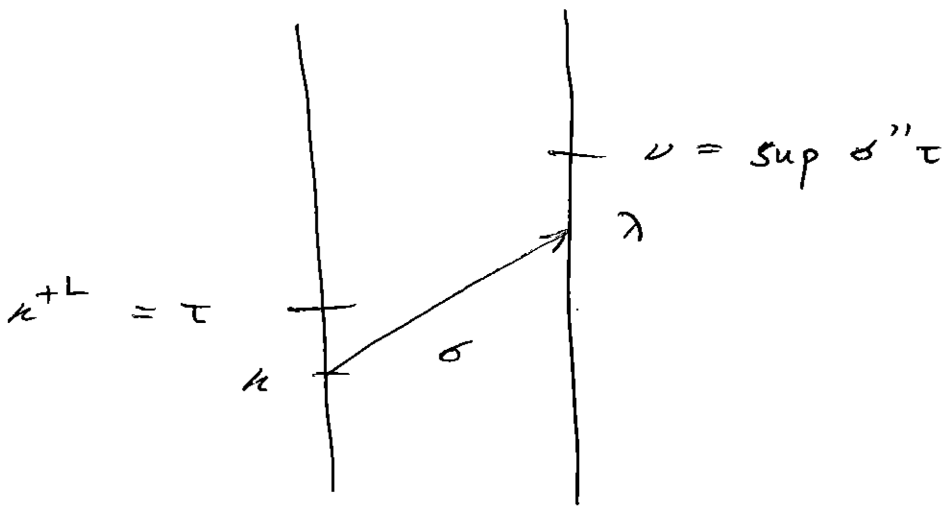
Let us first discuss the construction of K^c. K^c results from L by adding (codes for) elementary embeddings.

This has to be done recursively and carefully. We aim to produce a model which we can analyze in great detail.

$0^\#$, revisited.

"$0^\#$ exists" iff "there is a nontrivial $\sigma : L \longrightarrow L$."

Fix $\sigma : L \longrightarrow L$ with critical point κ, say.



Set $\tau = \kappa^{+L}$, $\lambda = \sigma(\kappa)$, $\nu = \sup \sigma''\tau$. Consider

$$\sigma \restriction J_\tau : J_\tau \longrightarrow_{\Sigma_0} J_\nu \quad \text{cofinally.}$$

$\sigma \restriction J_\tau$ codes a nontrivial $L \longrightarrow L$, as $\sigma \restriction J_\tau$ may be extended to such an embedding.

One can show $\nu = \lambda^{+L}$.

Moreover,

$$(J_\nu ; \in, \sigma \restriction J_\tau)$$

is a nice structure :

For instance, $(J_\nu; \in, \sigma \restriction J_\tau)$ is amenable:

Let $x \in J_\nu$. We need to see

$$x \cap (\sigma \restriction J_\tau) \in J_\nu .$$

Pick $\xi \in \mathrm{ran}(\sigma)$, $\xi < \nu$, s.t. $x \in J_\xi$. Say $\xi = \sigma(\bar\xi)$. Pick $f : \kappa \leftrightarrow \bar\xi$, $f \in J_\tau$. Then

$$(\sigma \restriction J_{\bar\xi})(X) = Y \quad \text{iff}$$

$$\text{there is } i < \kappa \text{ s.t. } X = f(i) \text{ and } Y = \sigma(f)(i).$$

$f, \sigma(f) \in J_\nu$, so $\sigma \restriction J_{\bar\xi} \in J_\nu$, so

$$x \cap (\sigma \restriction J_\tau) = x \cap (\sigma \restriction J_{\bar\xi}) \in J_\nu .$$

If λ is least such that the above situation is realized, then the structure

$$(J_\nu ; \in, \sigma \restriction J_\tau)$$

is a <u>mouse</u> which we shall call $0^{\#}$.

The map $\sigma \restriction J_\tau$ is also called an <u>extender</u>.

Mice can be iterated. If

$$0^\# = (J_\nu ; \in, \sigma \upharpoonright J_\tau)$$

is constructed as above, then we may take iterated ultrapowers of $0^\#$ as follows.

Set $\mathcal{M}_0 = 0^\#$, $\pi_{00} = \mathrm{id} \upharpoonright \mathcal{M}_0$.

Now let α be an arbitrary ordinal, and suppose $\mathcal{M}_\beta$, $\pi_{\gamma\beta}$ have been defined for all $\gamma \leq \beta < \alpha$.

If α is a limit ordinal, then we define

$$(\mathcal{M}_\alpha, (\pi_{\beta\alpha} : \beta < \alpha))$$

as the direct limit of

$$(\mathcal{M}_\beta, \pi_{\gamma\beta} : \gamma \leq \beta < \alpha).$$

Now suppose α is a successor ordinal.

Say $\alpha = \beta + 1$.

We'll have

$$\pi_{0\beta} : 0^{\#} \longrightarrow_{\Sigma_0} \mathcal{M}_\beta \quad \text{cofinally.}$$

$\pi_{0\beta}$ is elementary enough so that

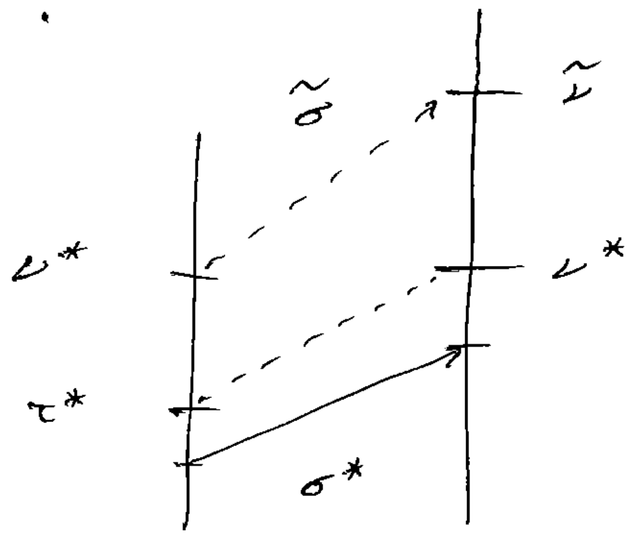
$$\mathcal{M}_\beta = (J_{\nu^*} ; \in, \sigma^*)$$

for some ν^* and $\sigma^* : J_{\tau^*} \longrightarrow_{\Sigma_0} J_{\nu^*}$ cofinally, some τ^*; also $\tau^* = \mathrm{crit}(\sigma^*)^{+J_{\nu^*}}$,

and σ^* may be extended to

$$\tilde{\sigma}^* : J_{\nu^*} \longrightarrow_{\Sigma_0} J_{\tilde{\nu}} \quad \text{cofinally,}$$

some $\tilde{\nu}$.



We may also define

$$\tilde{\sigma} = \tilde{\sigma}^*(\sigma^*)$$
$$= \bigcup_{x \in \sigma^*} \tilde{\sigma}^*(x).$$

(This uses amenability.)

We set

$$\mathcal{M}_\alpha = (J_{\tilde{\iota}} \, ; \, \in, \tilde{\sigma}) \, ,$$

$$\pi_{\beta\alpha} = \tilde{\sigma}^* , \quad \text{and}$$

$$\pi_{\alpha\alpha} = id \upharpoonright \mathcal{M}_\alpha \, .$$

As a matter of fact, every $\mathcal{M}_\alpha$ will be well-founded (and may hence be identified with its transitive collapse).

$0^\#$ is hence (<u>fully</u>) <u>iterable</u> !

If $\mathcal{M}$, $\mathcal{N}$ are mice which both "look like $0^\#$," then $\mathcal{M}$ is an iterate of $\mathcal{N}$, i.e., there is an iteration

$$(\mathcal{M}_\beta, \pi_{\gamma\beta} : \gamma \le \beta \le \alpha)$$

with $\mathcal{M}_0 = \mathcal{M}$ and $\mathcal{M}_\alpha = \mathcal{N}$, or vice versa.
I.e., any two mice can be compared.

There are mice which are much more complicated than $0^\#$.

We recursively define mice $\mathcal{M}_\xi$ and $\mathcal{N}_\xi$, $\xi \in OR$. Those mice should converge to K^c.
We keep adding extenders which come from taking hulls of V.

The K^c construction.

The models $\mathcal{M}_\xi$, $\mathcal{N}_\xi$ from the K^c construction will be of the form

$$(\mathcal{M}; \in, \vec{E}, F).$$

Here, $\mathcal{M} = J_\alpha[\vec{E}]$, some α, where $\vec{E}$ is a sequence of extenders of $\mathcal{M}$ which are actually elements of $\mathcal{M}$, and F is the top extender of $\mathcal{M}$.

They thus look like $0^\#$ except that there may be more extenders around.

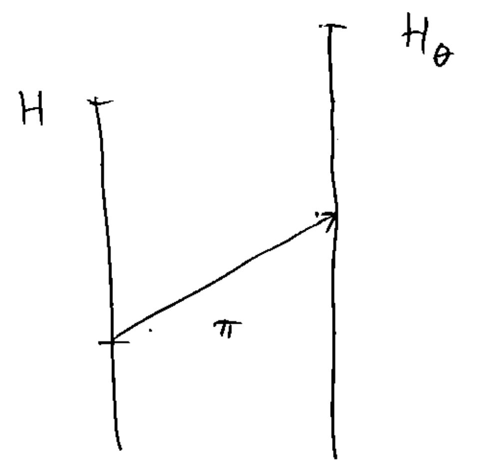
(But possibly $F = \emptyset$. $\vec{E} = \emptyset$ only until we reach $0^\#$.)

Set $\mathcal{M}_0 = \mathcal{N}_0 = (V_\omega ; \in, \emptyset, \emptyset)$.

Suppose $\mathcal{M}_\xi, \mathcal{N}_\xi$ have been constructed.

Assume $\mathcal{M}_\xi = (|\mathcal{M}_\xi|; \in, \vec{E}, \emptyset)$, some $\vec{E}$.

Suppose there is some $\pi : H \longrightarrow H_\theta$,

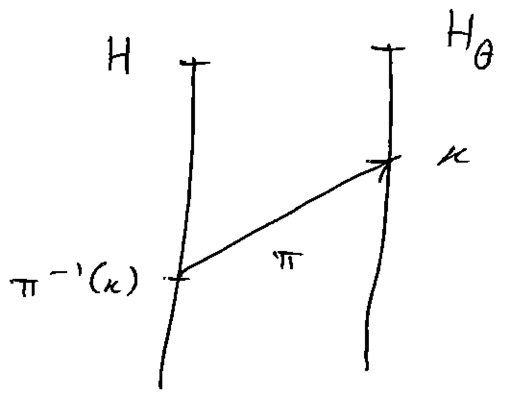


H transitive, ${}^\omega H \subset H$, such that the extender F derived from π gives an amenable structure in much the same way as $\sigma : L \to L$ produced $0^\#$ as a structure. F is called <u>certified</u> (<u>by a collapse</u>).

We then set $\mathcal{N}_{\xi+1} = (|\mathcal{M}_\xi|; \in, \vec{E}, F)$.

Example.

Suppose that $0^{\#}$ exists. We may then take hulls $\pi : H \longrightarrow H_\theta$



where H is transitive, ${}^{\omega}H \subset H$, and the critical point of π gets mapped to a regular cardinal κ, and also

$$\mathcal{P}(\pi^{-1}(\kappa)) \cap L \subset H.$$

Therefore, at some stage ξ,

$$W_{\xi+1} = (\text{an iterate of})\ 0^{\#}.$$

If $0^{\#}$ does not exist, hulls as above also don't exist.

If there is no such certified extender, then we just construct one step further, i.e., we let $\mathcal{N}_{\xi+1}$ be the $\mathrm{rud}_{\vec{E}}$ closure of $\mathcal{M}_\xi \cup \{\mathcal{M}_\xi\}$, where in addition to the rud functions $\mathrm{rud}_{\vec{E}}$ also has

$$x \longmapsto x \cap \vec{E} .$$

Assume now that $\mathcal{M}_\xi = (|\mathcal{M}_\xi| ; \in, \vec{E}, F)$, where $F \neq \emptyset$. Then we let $\mathcal{N}_{\xi+1}$ be the $\mathrm{rud}_{\vec{E}^\frown F}$ closure of $\mathcal{M}_\xi \cup \{\mathcal{M}_\xi\}$, and we set $\mathcal{N}_{\xi+1} = (|\mathcal{N}_{\xi+1}|; \in, \vec{E}^\frown F, \emptyset)$.

We also let, in both cases,

$$\mathcal{M}_{\xi+1} = \mathrm{Core}\,(\mathcal{N}_{\xi+1}) .$$

Here, the core $\mathrm{core}(\mathcal{N}_{\xi+1})$ of $\mathcal{N}_{\xi+1}$ is the transitive collapse of a (fine structural!) hull of $\mathcal{N}_{\xi+1}$.

Now suppose that all $\mathcal{M}_\xi$, $\mathcal{N}_\xi$, $\xi < \lambda$, have been constructed, where λ is a limit ordinal.

For any ordinal α, we let $\mathcal{N}_\lambda$ up thru α be the eventual value of $\mathcal{M}_\xi$ up thru α as $\xi \to \lambda$, if this value exists.
(O.w. $\mathcal{N}_\lambda$ has height $< \alpha$.)

We also set $\mathcal{M}_\lambda = \mathrm{core}(\mathcal{N}_\lambda)$.

We now face a bunch of problems!

Problems.

(1) In the case where we add the next certified F, is F unique?

(2) $\xi \longmapsto \mathcal{M}_\xi \cap \mathrm{OR}$ is not monotone. Given α, is there some ξ_α s.t. $\mathcal{M}_\xi$ agrees with $\mathcal{M}_{\xi_\alpha}$ up thru α for all $\xi \geq \xi_\alpha$?

(3) Can we prove statements about the models $\mathcal{M}_\xi$ as we did for L (for instance GCH, $\square_\kappa$, etc.) ?

(4) Can we prove covering ?

Solutions to (1) — (3) :

The models W_ξ from the K^c- construction are countably iterable.

Solution to (4) :

More delicate. Sometimes there is a solution, sometimes not.

This will lead to the "stacking of mice" ~~tech~~ technique and to the core model induction.