Last times: examples $Col(\omega, \mathbb{R})$ ($2^{\aleph_0}$ = $\aleph_1$)

$\rightsquigarrow$ forces CH (the Continuum Hypothesis)

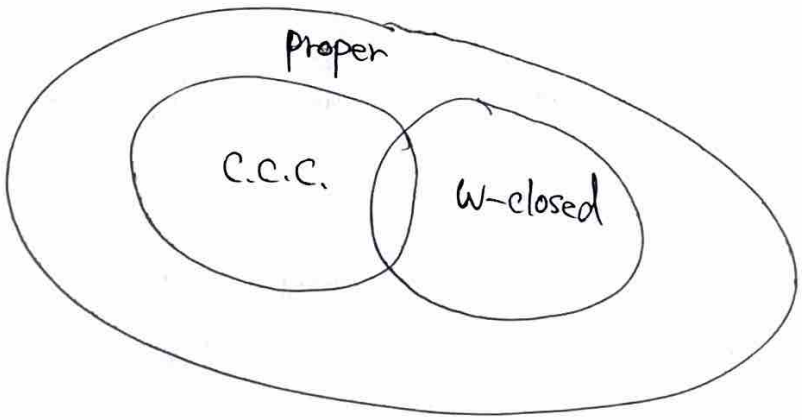
$\mathbb{C}(\alpha)$

$\rightarrow$ forces $\neg$CH

- $\mathbb{C}(\omega_2)$ forces $2^{\aleph_0} = \omega_2$

(and preserves all cardinal)

- $\mathbb{C}(\alpha)$ has the c.c.c.



$$\mathbb{C}(\alpha) = \left\{ p \;\middle|\; \begin{array}{l} p \text{ is a function, } dom(p) = \alpha \\ supp(p) \text{ is finite} \end{array} \right\} \qquad p(\xi) \in \mathbb{C} \text{ for } \xi < \alpha$$

$$supp(p) = \left\{ \xi < \alpha \mid p(\xi) \neq \emptyset \right\}$$

Why do we need finite support here?

$\mathbb{P} = \mathbb{C}^{\omega} = \{ p \mid p \text{ is a function, } dom(p) = \omega, \ p(n) \in \mathbb{C} \text{ for } n \in \omega \}$

: full support product of Cohen forcings

or

$\mathbb{C}^{ctble(\alpha)} = \{ p \mid p \text{ is a function, } p(\xi) \in \mathbb{C}, \ dom(p) = \alpha, \ supp(p) \text{ is ctble} \}$

Obs

Let g be M-generic for $\mathbb{C}^{ctble(\omega)} = \mathbb{C}^{\omega}$.

Then in $M[g]$, $\mathbb{R}^M$ is ctble

(proof) sketch

$f_n = \bigcup \{ p(n) \mid p \in g \} : \omega \to \omega$

claim

For each $f : \omega \to \omega \subseteq M$

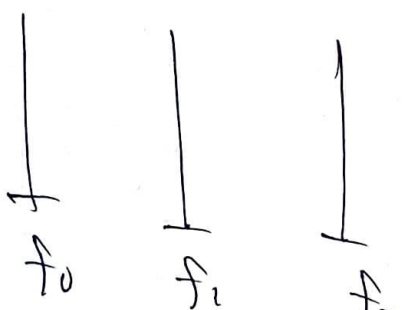
$\exists n_0 \in \omega$ s.t. $n_0 = n$

$f_0(n_0) = (n, m_1)$

$f_1(n_1) = (n_2, m_2)$

$\vdots$

$f = (m_1, m_2, \dots)$



density argument.

Product forcing

Let $\mathbb{P} = (P, \leq_{\mathbb{P}})$. $\mathbb{Q} = (Q, \leq_{\mathbb{Q}})$ be posets.

Define

$$\mathbb{P} \times \mathbb{Q} = (P \times Q, \leq_{\mathbb{P}\times\mathbb{Q}}) \text{ by :}$$

$(p, q), (p', q') \in \mathbb{P} \times \mathbb{Q}$:

$$(p, q) \leq_{\mathbb{P}\times\mathbb{Q}} (p', q')$$

$$:\Leftrightarrow \begin{cases} p \leq_{\mathbb{P}} p' \\ q \leq_{\mathbb{Q}} q' \end{cases}$$

Product lemma

Let $\mathbb{P}, \mathbb{Q} \in M$

- If G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over $M[G]$,
  then $G \times H$ is $\mathbb{P}\times\mathbb{Q}$-generic over M
- Conversely, if K is $\mathbb{P}\times\mathbb{Q}$-generic over M, then $G = \{p \mid \exists q\ (p, q) \in K\}$ is $\mathbb{P}$-generic over M, and $H = \{q \mid \exists p\ (p, q) \in K\}$ is $\mathbb{Q}$-generic over $M[G]$

(And hence also H is $\mathbb{Q}$-generic over M and
G is $\mathbb{P}$-generic over $M[H]$.
G, H are called mutually generic.)

Proof exercise!! ∎

Lemma $\mathbb{P}, \mathbb{Q} \in M$. $G \times H$ be $\mathbb{P} \times \mathbb{Q}$-generic over M
Then $M[g] \cap M[h] = M$.

(proof) Exersice.

ctble trans model of ZFC

Lemma Let $x \in {}^{\omega}\omega$. there are g_0, g_1 both $\mathbb{C}$-generic over M
s.t. $x \in M[g_0][g_1]$

proof
ex! ∎

Why iterated forcing?

def

$\diamondsuit$ is the statement that

there is a seq $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ s.t.

for any $A \subseteq \omega_1$,

$\{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary

def

$$\mathbb{D} = \left\{ a : \begin{array}{l} a \text{ is a function. } \mathrm{dom}(a) < \omega_1 \\ a(\xi) \subseteq \xi \text{ for all } \xi \in \mathrm{dom}(a) \end{array} \right\}$$

ordered by end-extension.

Lemma

Let g be $\mathbb{D}^M$-generic over M.

then $M[g] \models \diamondsuit$

Thm.

(proof) Let $(A_\xi \mid \xi < \omega_1^M) = \bigcup g$.

By density, $\bigcup g$ is a seq of length ω_1^M.

Since $\mathbb{D}$ is ω-closed, $\omega_1^M = \omega_1^{M[g]}$.

claim

$M[g] \models {}^{\forall} A \subseteq \omega_1$

$\{\alpha < \omega_1 \mid A \cap \alpha = A_\alpha\}$ is stationary

Let $A \subseteq \omega_1^M$, $A \in M[g]$.

$C \subseteq \omega_1$, $C \in M[g]$: a club in $M[g]$.

We aim to show : $\exists \alpha < \omega_1$ s.t. $\begin{cases} \alpha \in C \\ A \cap \alpha = A_\alpha \end{cases}$

Let $A = \dot{\tau}^g$.

$C = \dot{\sigma}^g$. $\tau, \sigma \in M$

Let $p \in g$ be

s.t. $p \Vdash \dot{\tau} \subseteq \check{\omega}_1$ and $\dot{\sigma} \subseteq \check{\omega}_1$ is club

Let $q \leq p$

Working in M.

Define $(q_n \mid n \in \omega)$ as follows:

$q_0 = q$

Given q_n :

first find some $r \leq q_n$ and some $\gamma = \gamma_n \geq \mathrm{dom}(q_n)$ $\gamma < \omega_1^M$ s.t.

$$r \Vdash \check{\gamma} \in \dot{\sigma}$$

find some $S \leq r$ s.t. for all $\xi < \mathrm{dom}(r)$ $S \Vdash \xi \in \dot{\tau}$ or $S \Vdash \xi \notin \dot{\tau}$

(this is possible, since $\mathbb{P}$ is δ-closed ω-closed)

$q_{n+1} := S$.

We may and shall assume

$$\mathrm{dom}(q_{n+1}) > \mathrm{dom}(q_n).$$

for all $n \in \omega$.

Write $\alpha = \sup \{\mathrm{dom}(q_n) \mid n \in \omega\}$.

Let $q^* = (\bigcup q_n) \cup \{(\alpha, A_\alpha)\}$

where $A_\alpha = \{\xi < \alpha \mid (\bigcup q_n) \Vdash \check{\xi} \in \dot{\tau}\}$.

Then $q^* \Vdash \check{\alpha} \in \dot{\sigma}$ and $\dot{A}_{\check{\alpha}} = \dot{\tau} \cap \alpha$

$q^* \Vdash \check{\alpha} \in \dot{\sigma}$ and $\dot{A}_{\check{\alpha}} = \dot{\tau} \cap \alpha$

By density, $(A_\alpha \mid \alpha < \omega_1)$ is $\diamondsuit$-seq.