Week 2. No. 1

recap: Fix a model M, ($\leftarrow$ (ctble) transitive model of ZFC)

a poset $\mathbb{P} \in M$,

a M-generic $g \subseteq \mathbb{P}$

Last time, we defined $M[g] := \{x^g \mid x \in M\}$

$x^g := \{y^g \mid (p, y) \in x \text{ for some } p \in g\}$

showed $M[g]$ is transitive.

$M[g] \cap OR = M \cap OR$

$M \cup \{g\} \subseteq M[g]$.

$\boxed{M \subsetneq M[g] \text{ if } \mathbb{P} \text{ is atomless}}$

$M[g] \models$ extensionality, foundation, infinity, pairing, union.

To show $M[g] \models ZFC$,

we are missing: Power set axiom,

Separation,

Replacement, and

AC (the Axiom of Choice)

All of these statement ~~[illegible]~~ are shown to be true in $M[g]$ via the forcing language

def M, $\mathbb{P}$ fixed. Let φ be a formula in the language of set theory, ~~and~~ let $\tau_1, \dots, \tau_k \in M$, and let $p \in \mathbb{P}$.

We say p **forces** $\varphi(\tau_1 \cdots \tau_k)$ (in $\mathbb{P}$ over M), written $p \Vdash^{M}_{\mathbb{P}} \varphi(\tau_1 \cdots \tau_k)$ (or $p \Vdash \varphi(\tau_1 \cdots \tau_k)$ or $p \Vdash^{\mathbb{P}}_{M} \varphi(\tau_1 \cdots \tau_k)$)

iff for every $g \subseteq \mathbb{P}$, ~~and~~ an M-generic filter, s.t. $p \in g$, $M[g] \models \varphi(\tau_1^g, \dots, \tau_k^g)$

- The relation $p \Vdash \varphi(\tau_1 \cdots \tau_k)$ is called forcing relation
- It is non-trivial that the relation $p \Vdash \varphi(\tau_1 \cdots \tau_k)$ is definable inside of M

Note Fix φ.

$$\left\{(p,\tau_1,\dots,\tau_k) \;\middle|\; \begin{array}{l} p\in\mathbb{P}, \tau_1\cdots\tau_k\in M, \\ p\Vdash\varphi(\tau_1\cdots\tau_k) \end{array}\right\} \subseteq M^{k+1}$$

We will show that this relation is definable over M.

Lemma Fix a formula φ.

The relation

$$\left\{(p,\tau_1,\dots,\tau_k) \;\middle|\; p\in\mathbb{P}, \tau_1,\dots\tau_k\in M, p\Vdash^{\mathbb{P}}_{M}\varphi(\tau_1\cdots\tau_k)\right\}$$

is definable over M

def Let $p\in\mathbb{P}$, $D\subseteq\mathbb{P}$.

We say D is dense below p if

$\forall q\leq p\ \exists r\in D\ (r\leq q)$

WWU MÜNSTER

MM Mathematics Münster Cluster of Excellence

No. 4

def

① Let $\tau_1, \tau_2 \in M$, let $p \in M$

$p \Vdash \tau_1 = \tau_2$ if $\forall (s_1, \pi_1) \in \tau_1$

$\{ q \leq p \mid q \leq s_1 \rightarrow \exists (s_2, \pi_2) \in \tau_2 \left(q \leq s_2 \wedge q \Vdash \pi_1 = \pi_2 \right) \}$

is dense below p, and

$\forall (s_2, \pi_2) \in \tau_2$

$\{ q \leq p \mid q \leq s_2 \rightarrow \exists (s_1, \pi_1) \in \tau_1 \left(q \leq s_1 \wedge q \Vdash \pi_1 = \pi_2 \right) \}$

is ~~dense~~ dense below p.

② Let $\tau_1, \tau_2 \in M$, let $p \in M$

$p \Vdash \tau_1 \in \tau_2$ if $\{ q \leq p : \exists (s, \pi) \in \tau_2 \left(q \leq s \wedge q \Vdash \tau_1 = \pi \right) \}$

is dense below p.

③ Let $\tau_1, \dots, \tau_k \in M$, let $\sigma_1, \dots, \sigma_\ell \in M$,
φ, ψ : fmlae, $p \in \mathbb{P}$

$p \Vdash \varphi(\tau_1 \cdots \tau_k) \wedge \psi(\sigma_1 \cdots \sigma_\ell)$

if $p \Vdash \varphi(\tau_1 \cdots \tau_k)$ and $p \Vdash \psi(\sigma_1 \cdots \sigma_\ell)$

④ Let $\tau_1 \cdots \tau_k \in M$, φ a fml, $p \in \mathbb{P}$

$p \Vdash \neg\varphi(\tau_1 \cdots \tau_k)$ if for all $q \leq p$

$q \not\Vdash \varphi(\tau_1, \dots, \tau_k)$

⑤ Let $\tau_1 \cdots \tau_k \in M$, φ a fml, $p \in \mathbb{P}$

$p \Vdash \exists x\, \varphi(\tau_1, \dots, \tau_k)$ if

$\{q \leq p \mid \exists \sigma \in M \;\; q \Vdash \varphi(\sigma, \tau_1 \cdots \tau_k)\}$ is dense below p

This finishes the def.

So for each φ,

$$\{(p, \tau_1 \dots \tau_k) : p \in \mathbb{P}, \tau_1 \dots \tau_k \in M, p \Vdash \varphi(\tau_1 \dots \tau_k)\}$$

is definable over M

Easy lemma

For a fml φ, $\tau_1 \dots \tau_k \in M$,
the following are equivalent:

(i) $p \Vdash \varphi(\tau_1 \dots \tau_k)$,

(ii) $\forall q \leq p \, (q \Vdash \varphi(\tau_1 \dots \tau_k))$,

(iii) $\{q \leq p \mid q \Vdash \varphi(\tau_1 \dots \tau_k)\}$ is dense below p.

(just looking at the definition)

Forcing theorem

Fix M, a ctble transitive model of ZFC,

$\mathbb{P} \in M$ a poset, $g \subseteq \mathbb{P}$ M-generic

Fix $\varphi, \tau_1 \cdots \tau_k \in M$.

① if $p \Vdash \varphi(\tau_1 \cdots \tau_k)$ and $p \in g$,

then $M[g] \models \varphi(\tau_1^g, \cdots \tau_k^g)$

② if $M[g] \models \varphi(\tau_1^g, \cdots \tau_k^g)$, then there is a

$p \in g$ with $p \Vdash \varphi(\tau_1 \cdots \tau_k)$

(Proof) By induction, we simult~~o~~aneous prove

(1) + (2).

No, 8

① for φ being an identity statement:

let $p \in g$, $p \Vdash \tau_1 = \tau_2$.

We show $\tau_1^g \subseteq \tau_2^g$ (by symmetry, also get $\tau_2^g \subseteq \tau_1^g$)

Let $x \in \tau_1^g$, say $x = \pi_1^g$, where $(s_1, \pi_1) \in \tau_1$, $s_1 \in g$.

(Need to see: $x \in \tau_2^g$)

Pick $r \in g$ with $r \leq p, s_1$

We have $r \Vdash \tau_1 = \tau_2$, since $p \Vdash \tau_1 = \tau_2$.

Hence by def, we have

$$D = \{q \leq r \mid q \leq s_1 \to \exists (s_2, \pi_2) \in \tau_2 (q \leq s_2 \wedge q \Vdash \pi_1 = \pi_2)\}$$

is dense below $r \in g$.

Look at $D' = D \cup \{s \in \mathbb{P} \mid \text{there is no } t \in \mathbb{P} \ t \leq s, r\}$

Then D' is dense in $\mathbb{P}$, $D' \in M$.

Since g is M-generic, $D' \cap g \neq \emptyset$.

Let $s \in D' \cap g$. Since $r \in g$, $s \in D$.

Since $S \leq S_1$, there is $(S_2, \pi_2) \in \tau_2$ s.t.

$S \leq S_2$ and $S \Vdash \pi_1 = \pi_2$

Then by induction, $\pi_1^g = \pi_2^g$.

Therefore $x = \pi_1^g = \pi_2^g \in \tau_2^g$ (as $S_2 \in g$).

We proved $\pi_1^g \subseteq \pi_2^g$.

By symmetry, $\pi_1^g = \pi_2^g$ //

② for φ being an identity:

$\neg \exists t\,(t \leq q, r)$

q, r are incompatible

Let $\tau_1^g = \tau_2^g$. Consider:

$$\psi_1(r) \equiv \exists (s_1, \pi_1) \in \tau_1 \Big(r \leq s_1 \wedge \forall (s_2, \pi_2) \in \tau_2 \,\forall q \,\big(q \leq s_2 \wedge q \Vdash \pi_1 = \pi_2 \rightarrow q \perp r \big)\Big)$$

Exercise

① if $r \in g$, then $\neg \psi_1(r)$ ② if $r \in g$, then $\neg \psi_2(r)$

$$\psi_2(r) \equiv \exists (s_2, \pi_2) \in \tau_2 \Big(r \leq s_2 \wedge \forall (s_1, \pi_1) \in \tau_1 \,\forall q \,\big(q \leq s_1 \wedge q \Vdash \pi_1 = \pi_2 \rightarrow q \perp r \big)\Big)$$

Proof of exercise We only check ①.
Assume we had $\varphi_1(r)$ for some $r \in g$.
Let $(s_1, \pi_1) \in r$ witness this fact.
Let $(s_2, \pi_2) \in T_2$ be s.t. $s_2 \in g$ and $\pi_1^g = \pi_2^g$.
~~Let $q \in g$ be s.t. $q \leq$~~ By induction, there is $q_0 \in g$
s.t. $q_0 \vdash \pi_1 = \pi_2$. Let $q \in g$ be s.t. $q \leq q_0, s_2$.
Then we must have $q \perp r$.
But $q, r \in g$. This is a contradiction !!
//

Let $D = \{ r \in \mathbb{P} \mid \varphi_1(r) \vee \varphi_2(r) \vee r \vdash T_1 = T_2 \}$

ex
$\vdash$ D is dense

proof of ex

Given any $p \in \mathbb{P}$. Assume $p \not\Vdash \tau_1 = \tau_2$

Then there is some $(\check{\pi_1}, s_1) \in \tau_1$ s.t.

$D = \{ q \leq p \mid q \leq s_1 \longrightarrow \exists (\check{\pi_2}, s_2) \in \tau_2 \ (q \leq s_2 \wedge q \Vdash \pi_1 = \pi_2) \}$

is not dense below p. (By symmetry, we may assume)

Then there is $q \leq p$ s.t. $\forall r \leq q \ (r \notin D)$

Then $\psi_1(q)$ holds //

(and $q \leq p$)

Therefore, $p \in g \cap D$.

By ex, we cannot have $\psi_1(p)$, neither $\psi_2(p)$

Hence $p \Vdash \tau_1 = \tau_2$ //

① for $\tau_1 \in \tau_2$: assume $g \ni p \vdash \tau_1 \in \tau_2$.

Hence there is $q \in g$ s.t.

$q \leq p \wedge \exists (s, \pi) \in \tau_2 \, (q \leq s \wedge q \vdash \tau_1 = \pi)$

Say $(s, \pi) \in \tau_2$ witnesses that.

Then by induction, $\tau_1^g = \pi^g$.

Since $q \leq s$ and $q \in g$, $s \in g$.

Hence $\tau_1^g = \pi^g \in \tau_2^g$ //

② for $\tau_1 \in \tau_2$: Say $\tau_1^g \in \tau_2^g$.

Then there is $(s, \pi) \in \tau_2$ s.t.

$s \in g$ and $\tau_1^g = \pi_2^g$.

Then we have $g \ni r \vdash \tau_1 = \pi_2$.

Let $p \in g$, $p \leq s, r$

Then $p \vdash \tau_1 \in \tau_2$ by def. //

①, ② for $\psi \wedge \varphi$: Easy.

① for negation;

$p \in \mathcal{G},\ p \Vdash \neg\varphi(\tau_1 \cdots \tau_k)$

Suppose $M[\mathcal{G}] \models \varphi(\tau_1^{\mathcal{G}}, \cdots, \tau_k^{\mathcal{G}})$,

Then by induction, $\exists q \in \mathcal{G}$ s.t.

$$q \Vdash \varphi(\tau_1 \cdots \tau_k).$$

Let $r \leq q, p$ with $r \in \mathcal{G}$.

~~Then $r \Vdash \varphi(\tau_1 \cdots \tau_k) \wedge \neg\varphi$~~

Then $r \Vdash \varphi(\tau_1 \cdots \tau_k)$ and $r \leq p$.

This is a contradiction !! //

② for negation :

Suppose $M[\mathcal{G}] \models \neg\varphi(\tau_1^{\mathcal{G}} \cdots \tau_k^{\mathcal{G}})$

$D = \{ p \in \mathbb{P} \mid p \Vdash \varphi(\tau_1 \dots \tau_k) \text{ or } p \Vdash \neg\varphi(\tau_1 \dots \tau_k) \}$

is dense. (by def)

Let $p \in D \cap g$

Then we must have $p \Vdash \neg\varphi(\tau_1 \dots \tau_k)$ /

① for existential statements :

$g \ni p \Vdash \exists x\, \varphi(x, \tau_1 \dots \tau_k)$

By def, we have $q \in g$ s.t.

$q \leq p \wedge \exists \sigma \in M \; q \Vdash \varphi(\sigma, \tau_1 \dots \tau_k)$

Then by induction

$M[g] \models \varphi(\sigma^g, \tau_1^g \dots \tau_k^g)$ //

$(\rightarrow M[g] \models \exists y\, \varphi(y, \tau_1^g \dots \tau_k^g))$

② for existen~~tional~~tial statements :

Assume $M[g] \models \exists x \; \varphi(x, \tau_1^g \cdots \tau_k^g)$

$\sigma \in M$ s.t.

$$M[g] \models \varphi(\sigma^g, \tau_1^g \cdots \tau_k^g)$$

By induction, we have $p \in g$ s.t.

$$p \Vdash \varphi(\sigma, \tau_1 \cdots \tau_k). \qquad \text{ok} //$$

$$(\Rightarrow p \Vdash \exists x \; \varphi(x, \tau_1 \cdots \tau_k))$$

This finishes the proof of thm $\blacksquare$

We keep fixing a ctble transitive model M of ZFC,
a poset $\mathbb{P} \in M$,
(sometimes) a M-generic filter $g \subseteq \mathbb{P}$.

<u>def</u>

$$\underset{\substack{\uparrow \\ \mathbb{P}}}{p} \Vdash^{\mathbb{P}}_{M} \varphi(\underbrace{\tau_1 \cdots \tau_k}_{\substack{\uparrow \\ M}})$$ iff for all $h \subseteq \mathbb{P}$ M-generic with $p \in h$,
$$M[h] \models \varphi(\tau_1^h \cdots \tau_k^h)$$

We also defined

$p \vdash \varphi(\tau_1 \cdots \tau_k)$ (syntactical version)

We showed:

<u>Forcing theorem, part 1</u>

① if $p \in g$, $p \vdash \varphi(\tau_1 \cdots \tau_k)$,
then $M[g] \models \varphi(\tau_1^g \cdots \tau_k^g)$

② if $M[g] \models \varphi(\tau_1^g, \dots, \tau_k^g)$, then
there is some $p \in g$ s.t. $p \vdash \varphi(\tau_1 \cdots \tau_k)$

2^{nd} part of the Forcing theorem

Let φ fml, $\tau_1 \cdots \tau_k \in M$

① for all $p \in \mathbb{P}$

$$p \Vdash^{M}_{\mathbb{P}} \varphi(\tau_1 \cdots \tau_k) \text{ iff } p \vdash \varphi(\tau_1 \cdots \tau_k)$$

② If $M[g] \models \varphi(\tau_1^g \cdots \tau_k^g)$, then there is $p \in g$ s.t. $p \Vdash^{\mathbb{P}}_{M} \varphi(\tau_1 \cdots \tau_k)$

(proof)

① $(\Longrightarrow)$

Suppose $p \Vdash \varphi(\tau_1 \cdots \tau_k)$.

Suppose that

$p \not\vdash \varphi(\tau_1 \cdots \tau_k)$

Let $q \leq p$ be s.t.

for all $r \leq q$

$r \not\vdash \varphi(\tau_1 \cdots \tau_k)$

Then ~~we~~ by def,

We have $q \vdash \neg\varphi(\tau_1 \cdots \tau_k)$

Let $h \subseteq \mathbb{P}$ be M-generic with $q \in h$. Then $M[h] \models \neg\varphi(\tau_1^h \cdots \tau_k^h)$

But since $p \in h$, we have $M[h] \models \varphi(\tau_1^h \cdots \tau_k^h)$.

This is a contradiction !!

$\dashv$ ①

$(\Longleftarrow)$ Given h M-generic with $p \in h$.

Then by the Forcing theorem part 1. ①,

We have

$M[\cancel{g}] \models \varphi(\tau_1^h \cdots \tau_k^h)$
h

$\rightsquigarrow$ We have

$p \Vdash \varphi(\tau_1 \cdots \tau_k)$ //

② : clear //   ⊣ thm $\blacksquare$

Theorem

$M[g] \models ZFC$

(proof)

We checked

- Infinity
- Pairing.
- Foundation
- Extensionality.
- Union

Separation

We need to see :

in $M[g]$: if $x, z_1 \cdots z_k \in M[g]$ and

φ is a fml, then

$\{ y \in x \mid \varphi(y, z_1 \cdots z_k) \}$ exists

Say $\tau^g = x$, $\tau_1^g = z_1, \ldots, \tau_k^g = z_k$.

Let

$$\pi = \left\{ (\rho, p) : \exists q \geq p \left[(q, \rho) \in \tau,\ p \Vdash^{\mathbb{P}}_{M} \varphi(\rho, \tau_1 \cdots \tau_k) \right] \right\}$$

$\in M$ by the Forcing theorem ($p \Vdash \varphi(\rho, \tau_1 \cdots \tau_k)$ is definable over M)

<u>Want</u> $\quad \pi^g = \{ y \in x \mid \varphi(y, z_1 \cdots z_k) \}$.

"$\subseteq$" Let $(\rho, p) \in \pi$ with $p \in g$.

Then by def of π,

$\rho^g \in \tau^g = x$, since for some $q \geq p$. $(q, \rho) \in \tau$,

and $M[g] \models \varphi(\rho^g, \tau_1^g \cdots \tau_k^g)$, since

$p \Vdash \varphi(\rho, \tau_1 \cdots \tau_k)$.

ok //

$M[g] \models$

"$\supseteq$" Given $y \in x$ w/ $\varphi(y, z_1 \cdots z_k)$. ~~Say~~

~~Let $q \in g$ be s.t.~~

Let $(q, \rho) \in \tau$ be s.t. $q \in g$ and $\rho^g = y$.

By the Forcing theorem, there is $p' \in g$ s.t.

$p' \Vdash \varphi(\rho, \tau_1 \cdots \tau_k)$.

Let $p \in g$ s.t. $p \leq q, p'$. (g : filter)

Then by def of π, $(p, p) \in \pi$.

Hence $p^g \in \pi^g$, since $p \in g$ // ok

This finishes the proof of Separation.

Replacement

Assume $M[g] \models \forall x \in \tau^g \exists y \; \varphi(x, y, \tau_1^g, \cdots \tau_k^g)$

We need to see $\exists z \in M[g]$ s.t.

$$M[g] \models \forall x \in \tau^g \exists y \in z \; \varphi(x, y, \tau_1^g, \cdots, \tau_k^g).$$

Let $\pi = \{ (p, \sigma) : \exists q \geq p \; \exists \bar{\sigma} \; ((q, \bar{\sigma}) \in \tau \wedge$
$p \Vdash \varphi(\bar{\sigma}, \sigma, \tau_1 \cdots \tau_k)) \}$

> π seems to work, but π is not a set
> → We need to restrict σ

Redefine, fix suff large $\alpha \in \mathrm{OR} \cap M$; $= \mathrm{ON} \cap M$

$$\pi = \{ (p, \sigma) : \exists q \geq p \; \exists \bar{\sigma} \; ((q, \bar{\sigma}) \in \tau \wedge p \Vdash \varphi(\bar{\sigma}, \sigma, \tau_1 \cdots \tau_k) \wedge \sigma \in V_\alpha^M) \}$$

Then π is a set in M !!

How to find such an α?

Use Replacement in M!

Consider the function.

$F : (p, \sigma) \longmapsto$ the least $\alpha \in ON \cap M$ s.t.

$$\exists q \geq p \; \exists \bar{\tau} \text{ s.t. } (q, \bar{\tau}) \in \tau \;\wedge$$
$$p \Vdash \varphi(\bar{\tau}, \sigma, \tau_1 \cdots \tau_k) \;\wedge$$
$$\sigma \in V_\alpha^M$$

Then by replacement in M, $\mathrm{ran}(F)$ is a set in M.

Let $\alpha = \sup \mathrm{ran}(F) \in ON \cap M$

Then α works //

And π^g is a desired set // (Same argument as for separation.)

Power set Axiom

Exercise! (Same)

$$V_\alpha = \{ x : rk_\epsilon(x) < \alpha \}$$
$$= \bigcup \{ P(V_\beta) \mid \beta < \alpha \}$$

$$V_0 = \phi$$
$$V_{\alpha+1} = P(V_\alpha)$$
$$V_\alpha = \bigcup_{\beta < \alpha} V_\beta \text{, if } \alpha \text{ is limit}$$

Axiom of Choice

We want to see that if $x \in M[g]$,

then

$$M[g] \models \exists \alpha \; \exists \text{ injection } f : x \hookrightarrow \alpha$$

Fix $x = \tau^g \in M[g]$.

In M, there is some α and some bijection $\bar{f} : \tau \longrightarrow \alpha$

We may inside M assign to every pair σ, π some $\ulcorner \sigma, \pi \urcorner$ s.t. $\ulcorner \sigma, \pi \urcorner^g = (\sigma^g, \pi^g)$

Let

$$\Pi = \left\{ (p, \ulcorner p, \check{\xi} \urcorner) \;\middle|\; (p, \rho) \in \tau \wedge \bar{f}((p, \rho)) = \xi \right\}$$

Π^g is 1-to-many relation.

So we can make Π^g to an injective function by separation in $M[g]$ //

(just minimize the range)

We showed $M[g] \models ZFC$ //

We developed the theory of forcing

We showed $M[g] \models ZFC$

Next : Vary Cohen forcing to produce a

$$M[g] \models 2^{\aleph_0} = \aleph_2$$

Before doing this, we'll produce a

$$M[g] \models \underbrace{2^{\aleph_0} = \aleph_1}_{CH}$$