SACKS FORCING PRESERVES SELECTIVE ULTRAFILTERS

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Theorem 0.2 for $n = 1$ is a well-known result of (in the words of [1, p. 562]) "R. Solovay, and possibly others," and the general case is [1, Theorem 6].

Lemma 0.1. *Let U be a selective ultrafilter on ω and let $D \subset {}^{<\omega}\omega$ and $(X_s : s \in D)$ be such that*

(1) $\emptyset \in D$,
(2) for all $s \in D$, s is strictly increasing and $X_s \in U$, and
(3) for all $s \in D$ and for all $m \in X_s$, $s^\frown m \in D$.

There is then some $Y \in U$ such that if $x \colon \omega \to Y$ is the monotone enumeration of Y, then for every $k < \omega$, $x \restriction k \in D$ and $x(k) \in X_{x\restriction k}$.

Proof. This is basically [2, Problem 9.3.(b)]. There is some $f \colon \omega \to \omega$ and some $Z \in U$ such that

$$Z \setminus f(n) \subset \bigcap \{X_s \colon s \in {}^{<\omega}n \cap D\}. \tag{1}$$

(See [2, Problem 9.1].) We may assume that f is strictly increasing, and write $b_n = f^n(0)$, $n < \omega$. Let $Z' \in U$ be such that $Z' \cap [b_n, b_{n+1})$ is a singleton for each $n < \omega$, and let Z^* be either $Z' \cap \bigcup\{[b_{2n}, b_{2n+1}) : n < \omega\}$ or $Z' \cap \bigcup\{[b_{2n+1}, b_{2n+2}) : n < \omega\}$, depending on which one of these two sets is in U. Write $Y = Z \cap Z^* \cap X_\emptyset$. (Cf. [2, Problem 9.3.(a)].)

Let $x \colon \omega \to Y$ be the monotone enumeration of Y. Let $k < \omega$ be such that $x \restriction k \in D$. Say $b_n \leq x(k) < b_{n+1}$. Then $x \restriction k \in {}^{<\omega}b_{n-1}$ (or $k = 0$ and $x \restriction k = \emptyset$), so that $x(k) \in Z \setminus b_n = Z \setminus f(b_{n-1}) \subset X_{x\restriction k}$ by (1) and $x \restriction k+1 \in D$ (or by $x(0) \in X_\emptyset$). □

Theorem 0.2. *Let U be a selective ultrafilter on ω, let $1 \leq n < \omega$, and let $\mathbb{S}_n$ be the product of n Sacks forcings. Let g be $\mathbb{S}_n$-generic over V, and let $U' = \{x \in \mathscr{P}(\omega) \cap V[g] : \exists y \in U \, y \subset x\}$. Then U' is an ultrafilter in $V[g]$.*

Proof. Let $p \in \mathbb{S}_n$, $\tau \in V^{\mathbb{S}_n}$, and $p \Vdash \tau \subset \omega$. Suppose that $p \Vdash \forall x \in U \, x \cap \tau \neq \emptyset$. We aim to construct some $q \leq p$ and some $x \in U$ with $q \Vdash x \subset \tau$.

Let us construct $D \subset {}^{<\omega}\omega$, $(p_s : s \in D)$, and $(X_s : s \in D)$.

Set $p_\emptyset = p$.

Suppose that $s \in D$ and p_s has been constructed, and let $m = \mathrm{lh}(s)$.

Let us write $p_s = (T_1, \ldots, T_n)$. Let $(\vec{t_i} : i < 2^{m+n})$ enumerate all $\vec{t} = (t_1, \ldots, t_n)$ such that each t_l is an m^{th} branching node of T_l, $1 \leq l \leq n$. For each $i < 2^{m+n}$ and each l, $1 \leq l \leq n$, pick an extension $s(i,l)$ of t_l (where $\vec{t_i} = (t_1, \ldots, t_n)$) in such a way that if $i \neq i'$, then $s(i,l)$ and $s(i',l)$ are incompatible in T_l. For $i < 2^{m+n}$ write $(p_s)_i$ for $((T_1)_{s(i,1)}, \ldots, (T_n)_{s(i,n)})$.

Let $i < 2^{m+n}$. By hypothesis,

$$x^i = \{k < \omega : \exists q \leq (p_s)_i \, q \Vdash k \in \tau\} \in U,$$

as otherwise $\{k < \omega : (p_s)_i \Vdash k \notin \tau\} \in U$, but $(p_s)_i \leq p$. Let

$$X_s = \bigcap \{x^i : i < 2^{m+n}\} \in U.$$

Let $k \in X_s$ be bigger than all the natural numbers from the sequence s. Exactly in this case we will put $s^\frown k$ into D. For each $i < 2^{m+n}$ pick $q = q_i^s \leq (p_s)_i$ such that $q \Vdash k \in \tau$; writing $q_i^s = (T_1^i, \ldots, T_n^i)$, we let

$$p_{s^\frown k} = (\bigcup\{T_1^i \colon i < 2^{m+n}\}, \ldots, \bigcup\{T_n^i \colon i < 2^{m+n}\}).$$

We will have that $p_{s^\frown k} \in \mathbb{S}_n$, $p_{s^\frown k} \leq p_s$, and $\bigcup\{T_l^i \colon i < 2^{m+n}\}$ and T_l have the same m^{th} branching nodes. Also, $p_{s^\frown k} \Vdash k \in \tau$.

We have defined $D \subset {}^{<\omega}\omega$, $(p_s : s \in D)$, and $(X_s : s \in D)$.

By the above Lemma, there is some $Y \in U$ such that if $x : \omega \to Y$ is the monotone enumeration of Y, then for every $k < \omega$, $x \upharpoonright k \in D$ and $x(k) \in X_{s \upharpoonright k}$. For $k < \omega$, write $p_{x \upharpoonright k} = (T_1^k, \ldots, T_n^k)$.

Let

$$q = (\bigcap\{T_1^k : k < \omega\}, \ldots, \bigcap\{T_n^k : k < \omega\}).$$

Then $q \in \mathbb{S}_n$ and $q \leq p$. Also, $q \Vdash x(k) \in \tau$ for every $k < \omega$, in other words, $q \Vdash Y \subset \tau$. □

References

[1] Pincus, David, and Halpern, J. D. (1981), "Partitions of products", Transactions of the American Mathematical Society, 267 (2): 549568

[2] Schindler, R., *Set theory. Exploring independence and truth.*

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