

# On $\text{NS}_{\omega_1}$ being saturated

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**Definition 0.1** Let  $\delta$  be a cardinal. We say that  $\delta$  is Woodin with  $\diamond$  iff there is some sequence  $(a_\kappa: \kappa < \delta)$  such that  $a_\kappa \subset V_\kappa$  for every  $\kappa < \delta$  and for every  $A \subset V_\delta$  the set

$$\{\kappa < \delta: A \cap V_\kappa = a_\kappa \wedge \kappa \text{ is } A\text{-strong up to } \delta\}$$

is stationary in  $\delta$ .

**Lemma 0.2** Suppose  $V = L[E]$ . Every Woodin cardinal is Woodin with  $\diamond$ .

PROOF. Let us define  $((a_\kappa, c_\kappa): \kappa < \delta)$  recursively as follows. If  $((a_\kappa, c_\kappa): \kappa < \mu)$  is defined for some  $\mu < \delta$ , then we let  $(a_\mu, c_\mu)$  be the least (in the order of constructibility) pair  $(a, c)$  such that  $a \subset V_\mu$ ,  $c \subset \mu$  is club in  $\mu$ , and

$$\{\kappa < \mu: a \cap V_\kappa = a_\kappa \wedge \kappa \text{ is } a\text{-strong up to } \mu\} \cap c = \emptyset$$

(if such a pair  $(a, c)$  exists).

We claim that  $(a_\kappa: \kappa < \delta)$  is as desired. If not, then let  $(A, C)$  be least (in the order of constructibility) such that  $A \subset V_\delta$ ,  $C \subset \delta$  is club in  $\delta$ , and

$$(1) \quad \{\kappa < \delta: A \cap V_\kappa = a_\kappa \wedge \kappa \text{ is } A\text{-strong up to } \delta\} \cap C = \emptyset.$$

As the set

$$\{\kappa < \delta: \kappa \text{ is } A\text{-strong up to } \delta\}$$

is stationary in  $\delta$ , an easy Skolem hull argument together with condensation for  $L[E]$  yields some  $\kappa \in C$  which is  $A$ -strong up to  $\delta$  and  $(A \cap V_\kappa, c \cap \kappa)$  is the least (in the order of constructibility) pair  $(a, c)$  such that  $a \subset V_\kappa$ ,  $c \subset \kappa$  is club in  $\kappa$ , and

$$\{\lambda < \kappa: a \cap V_\lambda = a_\lambda \wedge \lambda \text{ is } a\text{-strong up to } \kappa\} \cap c = \emptyset.$$

But then  $(A \cap V_\kappa, c \cap \kappa) = (a_\kappa, c_\kappa)$ , which contradicts (1).  $\square$

**Lemma 0.3** Suppose that  $\delta$  is a Woodin cardinal. Then  $\delta$  is Woodin with  $\diamond$  in  $V^{\text{Col}(\delta, \delta)}$ .

PROOF. We may identify  $\text{Col}(\delta, \delta)$  with the forcing

$$\mathbb{P} = \{(a_\kappa: \kappa < \mu): \mu < \delta \wedge \forall \kappa < \mu \ a_\kappa \subset V_\kappa\},$$

ordered by end-extension. Let  $\tau, \sigma \in V^{\mathbb{P}}$ , and let  $p \in \mathbb{P}$  be such that

$$p \Vdash \tau \subset V_\delta \wedge \sigma \subset \delta \text{ is club in } \delta.$$

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We aim to find some  $q = (a_\lambda : \lambda < \mu) \leq p$  and some  $\kappa < \delta$  such that

$$q \Vdash \kappa \in \sigma \text{ is } \tau\text{-strong up to } \delta \wedge \tau \cap \kappa = a_\kappa.$$

Let us recursively construct a sequence  $(p_\kappa : \kappa < \delta) = ((a_\lambda : \lambda < \mu_\kappa))$  of stronger and stronger conditions end-extending  $p$  with the following properties.

- (a)  $\{\mu_\kappa : \kappa < \delta\}$  is club in  $\delta$ .
- (b) For all  $\kappa$  there is some  $c_\kappa \subset \mu_\kappa$  which is unbounded in  $\mu_\kappa$  such that  $p_\kappa \Vdash \sigma \cap \mu_\kappa = c_\kappa$ ; in particular,  $p_\kappa \Vdash \mu_\kappa \in \sigma$ .
- (c) For all  $\kappa$  there is some  $A_\kappa \subset V_{\mu_\kappa}$  such that  $p_\kappa \Vdash \tau \cap V_{\mu_\kappa} = A_\kappa$ .
- (d) For all  $\kappa$ ,  $a_{\mu_\kappa} = A_\kappa$ .
- (e) If  $(a_\lambda : \lambda < \mu_{\kappa+1})$  does not force  $\kappa$  to be  $\tau$ -strong up to  $\delta$ , then there is some  $\alpha < \mu_{\kappa+1}$  such that

$$p_{\kappa+1} \Vdash \kappa \text{ is not } \tau\text{-strong up to } \alpha.$$

There is no problem with this construction.

Now set  $A = \bigcup_{\kappa < \delta} A_\kappa$ , so that  $A \cap V_{\mu_\kappa} = A_\kappa$  for all  $\kappa$ . As  $\delta$  is Woodin, by (a) we may pick some  $\kappa = \mu_\kappa$  which is  $A$ -strong up to  $\delta$ . Set  $q = (a_\lambda : \lambda < \kappa + 1)$ . By (b), (c), (d) we have that

$$q \Vdash \kappa \in \sigma \wedge \tau \cap \kappa = a_\kappa.$$

If  $q$  does not force  $\kappa$  to be  $\tau$ -strong up to  $\delta$ , then by (c), (e), and the definition of  $A$ , there is some  $\alpha < \mu_{\kappa+1}$  with

$$p_{\kappa+1} \Vdash \kappa \text{ is not } A\text{-strong up to } \alpha,$$

which is nonsense.

$q$  is thus as desired. □

**Theorem 0.4 (Shelah)** *Let  $\delta$  be a Woodin cardinal. There is some semi-proper  $\mathbb{P} \subset V_\delta$  with the  $\delta$ -c.c. such that if  $G$  is  $\mathbb{P}$ -generic over  $V$ , then  $V[G] \models$  “ $\text{NS}_{\omega_1}$  is saturated.”*

PROOF. Let us assume that  $\delta$  is Woodin with  $\diamond$ . We perform an RCS iteration (cf. [1]) of length  $\delta + 1$  of semi-proper forcings each of size  $< \delta$ , where in each successor step of the iteration, we either force with the poset  $\mathbb{S}(\vec{S})$  to seal a given maximal antichain  $\vec{S} \subset (\text{NS}_{\omega_1})^+ / \text{NS}_{\omega_1}$ , provided that  $\mathbb{S}(\vec{S})$  is semi-proper, or else we force with  $\text{Col}(\omega_1, 2^{\aleph_2})$  (which is  $\omega$ -closed, hence [semi-]proper). The choice of the maximal antichain  $\vec{S}$  is according to the  $\diamond$ -Woodinness of  $\delta$  and will be left to the reader's discretion.

If  $\vec{S}$  is a (not necessarily maximal) antichain, then the sealing forcing  $\mathbb{S}(\vec{S})$  consists of all pairs  $(c, p)$  such that for some  $\beta < \omega_1$  we have that  $c: \beta + 1 \rightarrow \omega_1$ ,  $p: \beta + 1 \rightarrow \vec{S}$ ,  $\text{ran}(c)$  is a closed subset of  $\omega_1$ , and for all  $\xi \leq \beta$ ,  $c(\xi) \in \bigcup_{i < \xi} p(i)$ .

$\mathbb{S}(\vec{S})$  is ordered by end-extension. The forcing  $\mathbb{S}(\vec{S})$  is  $\omega$ -distributive and preserves all the stationary subsets of all  $S \in \vec{S}$ , so that  $\mathbb{S}(\vec{S})$  is stationary set preserving if  $\vec{S}$  is maximal.

Let us write  $\mathbb{P}$  for the entire iteration. Let us pick some  $G$  which is  $\mathbb{P}$ -generic over  $V$ . We aim to prove that in  $V[G]$ , every antichain in  $(\mathbf{NS}_{\omega_1})^+/\mathbf{NS}_{\omega_1}$  has size  $\leq \aleph_1$ .

Suppose not, and let  $\vec{S} = (S_i : i < \delta) \in V[G]$  be a maximal antichain. Let  $\vec{S} = \tau^G$ , where  $\tau \in V^{\mathbb{P}} \cap V_{\delta+1}$ . We may find some  $\kappa < \delta$  such that

- (i)  $\kappa$  is  $\mathbb{P} \oplus \tau$ -strong up to  $\delta$  in  $V$ ,
- (ii)  $\kappa = \omega_2^{V[G \upharpoonright \kappa]}$ , and
- (iii)  $\vec{S} \upharpoonright \kappa = (S_i : i < \kappa) = (\tau \cap V_\kappa)^{G \upharpoonright \kappa}$  is the maximal antichain in  $V[G \upharpoonright \kappa]$  which is picked at stage  $\kappa$ .

The forcing  $\mathbb{S}(\vec{S} \upharpoonright \kappa)$  for sealing  $\vec{S} \upharpoonright \kappa$ , as defined in  $V[G \upharpoonright \kappa]$ , cannot be semi-proper in  $V[G \upharpoonright \kappa]$ , so that there is some  $(c, p) \in \mathbb{S}(\vec{S} \upharpoonright \kappa)$  such that the set

$$\begin{aligned} \tilde{T} = \{ X \prec (H_{\kappa^+})^{V[G \upharpoonright \kappa]} : \text{Card}(X) = \aleph_0 \wedge (c, p) \in X \wedge \neg \exists Y \supset X (Y \prec (H_{\kappa^+})^{V[G \upharpoonright \kappa]} \wedge \\ \text{Card}(Y) = \aleph_0 \wedge Y \cap \omega_1 = X \cap \omega_1 \wedge \exists (d, q) \leq (c, p) \quad (d, q) \text{ is } Y\text{-generic}) \} \end{aligned}$$

is stationary in  $V[G \upharpoonright \kappa]$ , and the  $\kappa^{\text{th}}$  forcing in the iteration  $\mathbb{P}$  is  $\text{Col}(\omega_1, 2^{\aleph_2})$ . In  $V[G \upharpoonright \kappa + 1]$  there is a surjective  $f : \omega_1 \rightarrow (H_{\kappa^+})^{V[G \upharpoonright \kappa]}$ . Because  $\text{Col}(\omega_1, 2^{\aleph_2})$  is proper,  $\tilde{T}$  is still stationary in  $V[G \upharpoonright \kappa + 1]$ , and hence the set

$$T = \{ \alpha < \omega_1 : f'' \alpha \in \tilde{T} \wedge \alpha = f'' \alpha \cap \omega_1 \}$$

is stationary in  $V[G \upharpoonright \kappa + 1]$ . As the tail  $\mathbb{P}_{[\kappa+1, \delta]}$  of the iteration  $\mathbb{P}$  over  $V[G \upharpoonright \kappa + 1]$  is semi-proper,  $T$  will remain stationary in  $V[G]$ , and as  $\vec{S}$  is a maximal antichain there is some (unique)  $i_0 < \delta$  such that

$$(2) \quad T \cap S_{i_0} \text{ is stationary in } V[G].$$

(It is not hard to verify that  $i_0 \geq \kappa$ .)

Let  $\lambda < \delta$ ,  $\lambda > \max(i_0, \kappa + 1)$  be such that  $(\tau \cap V_\lambda)^{G \upharpoonright \lambda} = \vec{S} \upharpoonright \lambda$ , so that  $S_{i_0} = (\tau \cap V_\lambda)^{G \upharpoonright \lambda}(i_0)$ , the  $(i_0)^{\text{th}}$  element of  $(\tau \cap V_\lambda)^{G \upharpoonright \lambda}$ . Pick an elementary embedding

$$j : V \rightarrow M$$

such that  $\text{crit}(j) = \kappa$ ,  $M$  is transitive,  ${}^\kappa M \subset M$ ,  $V_{\lambda+\omega} \subset M$ ,  $j(\mathbb{P}) \cap V_\lambda = \mathbb{P} \cap V_\lambda$ , and  $j(\tau) \cap V_\lambda = \tau \cap V_\lambda$ .

Let  $(X_i : i < \omega_1) \in V[G \upharpoonright \kappa + 1]$  be an increasing continuous chain of countable substructures of  $(H_{j((2^\kappa)^+)})^{M[G \upharpoonright \kappa + 1]}$  with  $\{\tau \cap V_\lambda, i_0\} \subset X_0$  and such that for all  $i < \omega_1$ ,

- (a)  $i \in X_{i+1}$ ,
- (b)  $f''(X_i \cap \omega_1) \subset X_i$ , and

(c)  $j''(X_i \cap (2^\kappa)^{V[G \upharpoonright \kappa]}) \subset X_i$ .

Write  $\bar{G} = G \upharpoonright [\kappa + 2, \lambda]$ . We have that

$$\{X_i[\bar{G}] \cap \omega_1 : i < \omega_1\} \in V[G \upharpoonright \lambda]$$

is club in  $\omega_1$ , so that by (2) we may find some  $i < \omega_1$  with  $X_i[\bar{G}] \cap \omega_1 = X_i \cap \omega_1 \in T \cap S_{i_0}$ .

Write  $X = X_i$  and  $\alpha = X \cap \omega_1$ . As  $\text{Col}(\omega_1, 2^{\aleph_2})$  is  $\omega$ -closed,  $X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]} \in V[G \upharpoonright \kappa]$ . As  $\alpha \in T$ ,  $f''\alpha \in \tilde{T}$  and  $\alpha = f''\alpha \cap \omega_1$ , and hence by (b)

$$f''\alpha \subset X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]} \in V[G \upharpoonright \kappa].$$

This implies that

$$(3) \quad X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]} \in \tilde{T}.$$

As the segment  $(\mathbb{P}_{[\lambda+1, j(\kappa)]})^{M[G \upharpoonright \lambda]}$  of  $j(\mathbb{P})$  over  $M[G \upharpoonright \lambda]$  is semi-proper, we may now pick  $H$  to be generic for the segment  $(\mathbb{P}_{[\lambda+1, j(\kappa)]})^{M[G \upharpoonright \lambda]}$  of  $j(\mathbb{P})$  over  $M[G \upharpoonright \lambda]$  in such a way that

$$X[\bar{G}, H] \cap \omega_1 = X[\bar{G}] \cap \omega_1.$$

We may lift  $j: V \rightarrow M$  to an elementary embedding

$$j^*: V[G \upharpoonright \kappa] \rightarrow M[G \upharpoonright \lambda, H].$$

Notice that  $(V_{\lambda+\omega})^{M[G \upharpoonright \lambda]} = (V_{\lambda+\omega})^{V[G \upharpoonright \lambda]}$ . By (3),

$$(4) \quad j^*(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]}) \in j^*(\tilde{T}).$$

We now have that  $X[\bar{G}, H] \cap \omega_1 = X[\bar{G}] \cap \omega_1 = \alpha \in S_{i_0} = (\tau \cap V_\lambda)^{G \upharpoonright \lambda}(i_0) \in X[\bar{G}] \subset X[\bar{G}, H] \prec (H_{j((2^\kappa)^+)})^{M[G \upharpoonright \lambda, H]}$ .

But now by (c),

$$j^*(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]}) = j^{**}(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]}) \subset X[\bar{G}, H].$$

Therefore,  $X[\bar{G}, H]$  witnesses that  $j^*(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]})$  is *not* in  $j^*(\tilde{T})$ , as the condition  $j((c, p)) = (c, p) \in \mathbb{S}(\vec{S} \upharpoonright \kappa) \subset j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$  from the definition of  $\tilde{T}$  may be extended in  $j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$  to some  $X[\bar{G}, H]$ -generic condition  $(c^*, p^*) \in j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$  with  $\text{dom}(c^*) = \text{dom}(p^*) = \alpha + 1$ ,  $c^*(\alpha) = \alpha$ , and  $p^*(i) = S_{i_0} \in j^*(S \upharpoonright \kappa)$  for some  $i < \alpha$ .

This contradicts (4). □ (Theorem 0.4)

**Theorem 0.5 (Woodin)** *Suppose that  $\text{NS}_{\omega_1}$  is saturated and  $(\mathcal{P}(\omega_1))^\#$  exists. Then  $\delta_2^1 = \omega_2$ .*

PROOF SKETCH. (Cf. [4].) If  $N \cong X \prec \mathcal{M} = ((\mathcal{P}(\omega_1))^\#; \in, \text{NS}_{\omega_1})$ , where  $N$  is countable and transitive, then  $N$  is generically  $(\omega_1 + 1)$ -iterable via the preimage of  $\text{NS}_{\omega_1}$  and its images. By the Boundedness Lemma, the ordinal height of every  $(\omega_1)^{\text{th}}$  iterate of  $N$  is  $< (\omega_1^V)^{+L[z]}$ , where  $z \in \mathbb{R}$  codes  $N$ . On the other hand, if  $N_i \cong X_i = \text{Hull}^{\mathcal{M}}(X \cup \{X_j \cap \omega_1 : j < i\}) \prec \mathcal{M}$  for  $i \leq \omega_1$ , then  $(N_i : i \leq \omega_1)$ ,

together with the obvious maps, is a generic iteration of  $N$ . Hence if  $\beta \in X$ , where  $\beta < \omega_2$ ,  $\beta < (\omega_1^Y)^{+L[z]} < \delta_2^1$ .  $\square$  (Theorem 0.5)

[4] shows that if  $\mathbb{P}$  is the poset of Theorem 0.4, as defined over  $M_1$ , and if  $G$  is  $\mathbb{P}$ -generic over  $M_1$ , then  $\delta_2^1 < \omega_2$  in  $M_1[G]$ . The following Theorem gives a bit more information.

**Theorem 0.6** *Let  $\mathbb{P}$  be the poset of Theorem 0.4, as defined over  $M_1$ , and let  $G$  be  $\mathbb{P}$ -generic over  $M_1$ . Then  $(\delta_2^1)^{M_1[G]} = (\delta_2^1)^{M_1} < \omega_2^{M_1} < \omega_2^{M_1[G]}$ .*

PROOF. Deny. Let  $x \in \mathbb{R} \cap M_1[G]$  witness that  $(\delta_2^1)^{M_1[G]} > (\delta_2^1)^{M_1}$ . So if

$$(N_i, \pi_{ij} : i \leq j \leq \omega_1)$$

is the iteration of  $x^\dagger = N_0$  of length  $\omega_1 + 1$  which is obtained by hitting the bottom (total) measure of  $x^\dagger$  and its images  $\omega_1$  times, then  $(\omega_1^Y)^{+N\omega_1} > (\delta_2^1)^{M_1}$ .

As  $x^\dagger \models$  “There is no inner model with a Woodin cardinal,” we may let  $K$  denote the core model of  $x^\dagger$  of height  $\Omega$ , where  $\Omega$  is the top measurable cardinal of  $x^\dagger$ . By [3], there is a normal iteration tree  $\mathcal{T} \in x^\dagger$  on  $K$  with  $[0, \infty)_{\mathcal{T}} = \emptyset$  and last model  $K^{N_1}$  such that  $\pi_{01} = \pi_{0\infty}^{\mathcal{T}}$ . Letting  $\mathcal{T}^*$  be the concatenation of all  $\pi_{0i}(\mathcal{T})$ ,  $0 \leq i < \omega_1$ ,  $\mathcal{T}^*$  is then a (non-normal) iteration tree on  $K$  with  $[0, \infty)_{\mathcal{T}^*} = \emptyset$  and last model  $K^{N\omega_1}$  such that  $\pi_{0\omega_1} \upharpoonright K = \pi_{0\infty}^{\mathcal{T}^*}$ . By absoluteness,  $K$  is in fact iterable in  $M_1[G]$ , and  $\mathcal{T}^*$  is according to the (unique) relevant iteration strategy.

We claim that  $K$  iterates past  $M_1 \upharpoonright \omega_1$ .

Otherwise suppose that  $\alpha < \omega_1$  is such that  $M_1 \upharpoonright \alpha$  absorbs  $K$ . There is then, in  $M_1[G]$ , an iteration tree  $\mathcal{U}$  on  $M_1 \upharpoonright \alpha$  of length  $\omega_1 + 1$  such that  $\mathcal{M}_{\omega_1}^{\mathcal{U}} \cap \text{OR} \geq N_{\omega_1} \cap \text{OR} > (\delta_2^1)^{M_1}$ . (Cf. [2] for a writeup of this argument.) On the other hand, by the Boundedness Lemma, if  $z \in \mathbb{R} \cap M_1$  codes  $M_1 \upharpoonright \alpha$  and if  $\gamma$  denotes the supremum of all the ordinal heights of all  $(\omega_1)^{\text{th}}$  iterates of  $M_1 \upharpoonright \alpha$ , then

$$\gamma < (\omega_1)^{+L[z]}.$$

In particular,  $(\delta_2^1)^{M_1} > (\omega_1)^{+L[z]} > \gamma > \mathcal{M}_{\omega_1}^{\mathcal{U}} \cap \text{OR} > (\delta_2^1)^{M_1}$ .

This contradiction indeed shows that  $K$  iterates past  $M_1 \upharpoonright \omega_1$ . But then  $\omega_1$  has to be an inaccessible cardinal of  $M_1$ , which is nonsense.  $\square$  (Theorem 0.6)

**Question.** Is  $M_1[G] \models \neg \text{CH}$ ? Is  $\mathbb{R} \cap M_1[G] \subset M_1$ ?

## References

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