On NS_{ω_1} being saturated

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Definition 0.1 *Let δ be a cardinal. We say that δ is* Woodin with $\diamondsuit$ *iff there is some sequence $(a_\kappa \colon \kappa < \delta)$ such that $a_\kappa \subset V_\kappa$ for every $\kappa < \delta$ and for every $A \subset V_\delta$ the set*

$$\{\kappa < \delta \colon A \cap V_\kappa = a_\kappa \wedge \kappa \text{ is } A\text{–strong up to } \delta\}$$

is stationary in δ.

Lemma 0.2 *Suppose $V = L[E]$. Every Woodin cardinal is Woodin with $\diamondsuit$.*

Proof. Let us define $((a_\kappa, c_\kappa) \colon \kappa < \delta)$ recursively as follows. If $((a_\kappa, c_\kappa) \colon \kappa < \mu)$ is defined for some $\mu < \delta$, then we let (a_μ, c_μ) be the least (in the order of constructibility) pair (a, c) such that $a \subset V_\mu$, $c \subset \mu$ is club in μ, and

$$\{\kappa < \mu \colon a \cap V_\kappa = a_\kappa \wedge \kappa \text{ is } a\text{–strong up to } \mu\} \cap c = \emptyset$$

(if such a pair (a, c) exists).

We claim that $(a_\kappa \colon \kappa < \delta)$ is as desired. If not, then let (A, C) be least (in the order of constructibility) such that $A \subset V_\delta$, $C \subset \delta$ is club in δ, and

$$\{\kappa < \delta \colon A \cap V_\kappa = a_\kappa \wedge \kappa \text{ is } A\text{–strong up to } \delta\} \cap C = \emptyset. \tag{1}$$

As the set

$$\{\kappa < \delta \colon \kappa \text{ is } A\text{–strong up to } \delta\}$$

is stationary in δ, an easy Skolem hull argument together with condensation for $L[E]$ yields some $\kappa \in C$ which is A–strong up to δ and $(A \cap V_\kappa, c \cap \kappa)$ is the least (in the order of constructibility) pair (a, c) such that $a \subset V_\kappa$, $c \subset \kappa$ is club in κ, and

$$\{\lambda < \kappa \colon a \cap V_\lambda = a_\lambda \wedge \lambda \text{ is } a\text{–strong up to } \kappa\} \cap c = \emptyset.$$

But then $(A \cap V_\kappa, c \cap \kappa) = (a_\kappa, c_\kappa)$, which contradicts (1). □

Lemma 0.3 *Suppose that δ is a Woodin cardinal. Then δ is Woodin with $\diamondsuit$ in $V^{\mathrm{Col}(\delta,\delta)}$.*

Proof. We may identify $\mathrm{Col}(\delta, \delta)$ with the forcing

$$\mathbb{P} = \{(a_\kappa \colon \kappa < \mu) \colon \mu < \delta \wedge \forall \kappa < \mu \; a_\kappa \subset V_\kappa\},$$

ordered by end–extension. Let τ, $\sigma \in V^{\mathbb{P}}$, and let $p \in \mathbb{P}$ be such that

$$p \Vdash \tau \subset V_\delta \wedge \sigma \subset \delta \text{ is club in } \delta.$$

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We aim to find some $q = (a_\lambda\colon \lambda < \mu) \leq p$ and some $\kappa < \delta$ such that

$$q \Vdash \kappa \in \sigma \text{ is } \tau\text{–strong up to } \delta \wedge \tau \cap \kappa = a_\kappa.$$

Let us recursively construct a sequence $(p_\kappa\colon \kappa < \delta) = ((a_\lambda\colon \lambda < \mu_\kappa)$ of stronger and stronger conditions end–extending p with the following properties.

(a) $\{\mu_\kappa\colon \kappa < \delta\}$ is club in δ.

(b) For all κ there is some $c_\kappa \subset \mu_\kappa$ which is unbounded in μ_κ such that $p_\kappa \Vdash \sigma \cap \mu_\kappa = c_\kappa$; in particular, $p_\kappa \Vdash \mu_\kappa \in \sigma$.

(c) For all κ there is some $A_\kappa \subset V_{\mu_\kappa}$ such that $p_\kappa \Vdash \tau \cap V_{\mu_\kappa} = A_\kappa$.

(d) For all κ, $a_{\mu_\kappa} = A_\kappa$.

(e) If $(a_\lambda\colon \lambda < \mu_{\kappa+1})$ does not force κ be be τ–strong up to δ, then there is some $\alpha < \mu_{\kappa+1}$ such that

$$p_{\kappa+1} \Vdash \kappa \text{ is not } \tau\text{–strong up to } \alpha.$$

There is no problem with this construction.

Now set $A = \bigcup_{\kappa<\delta} A_\kappa$, so that $A \cap V_{\mu_\kappa} = A_\kappa$ for all κ. As δ is Woodin, by (a) we may pick some $\kappa = \mu_\kappa$ which is A–strong up to δ. Set $q = (a_\lambda\colon \lambda < \kappa + 1)$. By (b), (c), (d) we have that

$$q \Vdash \kappa \in \sigma \wedge \tau \cap \kappa = a_\kappa.$$

If q does not force κ to be τ–strong up to δ, then by (c), (e), and the definition of A, there is some $\alpha < \mu_{\kappa+1}$ with

$$p_{\kappa+1} \Vdash \kappa \text{ is not } A\text{–strong up to } \alpha,$$

which is nonsense.

q is thus as desired. □

Theorem 0.4 (Shelah) *Let δ be a Woodin cardinal. There is some semi–proper $\mathbb{P} \subset V_\delta$ with the δ–c.c. such that if G is $\mathbb{P}$–generic over V, then $V[G] \models$ "NS_{ω_1} is saturated."*

Proof. Let us assume that δ is Woodin with $\diamondsuit$. We perform an RCS iteration (cf. [1]) of length $\delta + 1$ of semi–proper forcings each of size $< \delta$, where in each successor step of the iteration, we either force with the poset $\mathbb{S}(\vec{S})$ to seal a given maximal antichain $\vec{S} \subset (\mathsf{NS}_{\omega_1})^+/\mathsf{NS}_{\omega_1}$, provided that $\mathbb{S}(\vec{S})$ is semi–proper, or else we force with $\mathrm{Col}(\omega_1, 2^{\aleph_2})$ (which is ω–closed, hence [semi–]proper). The choice of the maximal antichain $\vec{S}$ is according to the $\diamondsuit$–Woodinness of δ and will be left to the reader's discretion.

If $\vec{S}$ is a (not necessarily maximal) antichain, then the sealing forcing $\mathbb{S}(\vec{S})$ consists of all pairs (c, p) such that for some $\beta < \omega_1$ we have that $c\colon \beta + 1 \to \omega_1$, $p\colon \beta + 1 \to \vec{S}$, $\mathrm{ran}(c)$ is a closed subset of ω_1, and for all $\xi \leq \beta$, $c(\xi) \in \bigcup_{i<\xi} p(i)$.

$\mathbb{S}(\vec{S})$ is ordered by end–extension. The forcing $\mathbb{S}(\vec{S})$ is ω–distributive and preserves all the stationary subsets of all $S \in \vec{S}$, so that $\mathbb{S}(\vec{S})$ is stationary set preserving if $\vec{S}$ is maximal.

Let us write $\mathbb{P}$ for the entire iteration. Let us pick some G which is $\mathbb{P}$–generic over V. We aim to prove that in $V[G]$, every antichain in $(\mathsf{NS}_{\omega_1})^+/\mathsf{NS}_{\omega_1}$ has size $\leq \aleph_1$.

Suppose not, and let $\vec{S} = (S_i \colon i < \delta) \in V[G]$ be a maximal antichain. Let $\vec{S} = \tau^G$, where $\tau \in V^{\mathbb{P}} \cap V_{\delta+1}$. We may find some $\kappa < \delta$ such that

(i) κ is $\mathbb{P} \oplus \tau$–strong up to δ in V,

(ii) $\kappa = \omega_2^{V[G \restriction \kappa]}$, and

(iii) $\vec{S} \restriction \kappa = (S_i \colon i < \kappa) = (\tau \cap V_\kappa)^{G \restriction \kappa}$ is the maximal antichain in $V[G \restriction \kappa]$ which is picked at stage κ.

The forcing $\mathbb{S}(\vec{S} \restriction \kappa)$ for sealing $\vec{S} \restriction \kappa$, as defined in $V[G \restriction \kappa]$, cannot be semi–proper in $V[G \restriction \kappa]$, so that there is some $(c,p) \in \mathbb{S}(\vec{S} \restriction \kappa)$ such that the set

$$\tilde{T} = \{X \prec (H_{\kappa^+})^{V[G \restriction \kappa]} \colon \mathrm{Card}(X) = \aleph_0 \wedge (c,p) \in X \wedge \neg\, \exists Y \supset X (Y \prec (H_{\kappa^+})^{V[G \restriction \kappa]} \wedge$$

$$\mathrm{Card}(Y) = \aleph_0 \wedge Y \cap \omega_1 = X \cap \omega_1 \wedge \exists (d,q) \leq (c,p) \quad (d,q) \text{ is } Y\text{–generic })\}$$

is stationary in $V[G \restriction \kappa]$, and the κ^{th} forcing in the iteration $\mathbb{P}$ is $\mathrm{Col}(\omega_1, 2^{\aleph_2})$. In $V[G \restriction \kappa+1]$ there is a surjective $f \colon \omega_1 \to (H_{\kappa^+})^{V[G \restriction \kappa]}$. Because $\mathrm{Col}(\omega_1, 2^{\aleph_2})$ is proper, $\tilde{T}$ is still stationary in $V[G \restriction \kappa+1]$, and hence the set

$$T = \{\alpha < \omega_1 \colon f"\alpha \in \tilde{T} \wedge \alpha = f"\alpha \cap \omega_1\}$$

is stationary in $V[G \restriction \kappa+1]$. As the tail $\mathbb{P}_{[\kappa+2,\delta]}$ of the iteration $\mathbb{P}$ over $V[G \restriction \kappa+1]$ is semi–proper, T will remain stationary in $V[G]$, and as $\vec{S}$ is a maximal antichain there is some $i_0 < \delta$ such that

$$T \cap S_{i_0} \text{ is stationary in } V[G]. \tag{2}$$

Let $\lambda < \delta$, $\lambda > \max(i_0, \kappa+1)$ be such that $(\tau \cap V_\lambda)^{G \restriction \lambda} = \vec{S} \restriction \lambda$, so that $S_{i_0} = (\tau \cap V_\lambda)^{G \restriction \lambda}(i_0)$, the $(i_0)^{\text{th}}$ element of $(\tau \cap V_\lambda)^{G \restriction \lambda}$. Pick an elementary embedding

$$j \colon V \to M$$

such that $\mathrm{crit}(j) = \kappa$, M is transitive, ${}^\kappa M \subset M$, $V_{\lambda+\omega} \subset M$, $j(\mathbb{P}) \cap V_\lambda = \mathbb{P} \cap V_\lambda$, and $j(\tau) \cap V_\lambda = \tau \cap V_\lambda$.

Let H be generic for the segment $(\mathbb{P}_{[\lambda+1, j(\kappa)]})^{M[G \restriction \lambda]}$ of $j(\mathbb{P})$ over $M[G \restriction \lambda]$. We may lift $j \colon V \to M$ to an elementary embedding

$$j^* \colon V[G \restriction \kappa] \to M[G \restriction \lambda, H].$$

Notice that $(V_{\lambda+\omega})^{M[G \restriction \lambda]} = (V_{\lambda+\omega})^{V[G \restriction \lambda]}$.

Let $(X_i \colon i < \omega_1) \in V[G \restriction \kappa+1]$ be an increasing continuous chain of countable substructures of $(H_{j((2^\kappa)^+)})^{M[G \restriction \kappa+1]}$ with $\{\tau \cap V_\lambda, i_0\} \subset X_0$ and such that for all $i < \omega_1$,

(a) $i \in X_{i+1}$,

(b) $f"(X_i \cap \omega_1) \subset X_i$, and

(c) $j"(X_i \cap (2^\kappa)^{V[G \upharpoonright \kappa]}) \subset X_i$.

Write $\bar{G} = G \upharpoonright [\kappa+2, \lambda]$. We have that

$$\{X_i[\bar{G}] \cap \omega_1 \colon i < \omega_1\} \in V[G \upharpoonright \lambda]$$

is club in ω_1, so that by (2) we may find some $i < \omega_1$ with $X_i[\bar{G}] \cap \omega_1 = X_i \cap \omega_1 \in T \cap S_{i_0}$.

Write $X = X_i$ and $\alpha = X \cap \omega_1$. As $\mathrm{Col}(\omega_1, 2^{\aleph_2})$ is ω–closed, $X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]} \in V[G \upharpoonright \kappa]$. As $\alpha \in T$, $f"\alpha \in \tilde{T}$ and $\alpha = f"\alpha \cap \omega_1$, and hence by (b)

$$f"\alpha \subset X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]} \in V[G \upharpoonright \kappa].$$

This implies that $X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]} \in \tilde{T}$, and therefore

$$j^*(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]}) \in j^*(\tilde{T}). \tag{3}$$

As the segment $(\mathbb{P}_{[\lambda+1, j(\kappa)]})^{M[G \upharpoonright \lambda]}$ of $j(\mathbb{P})$ over $M[G \upharpoonright \lambda]$ is semi–proper, we have that $X[\bar{G}, H] \cap \omega_1 = X[\bar{G}] \cap \omega_1 = \alpha \in S_{i_0} = (\tau \cap V_\lambda)^{G \upharpoonright \lambda}(i_0) \in X[\bar{G}, H] \prec (H_{j((2^\kappa)^+)})^{M[G \upharpoonright \lambda, H]}$.

But now by (c),

$$j^*(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]}) = j^*"(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]}) \subset X[\bar{G}, H].$$

Therefore, $X[\bar{G}, H]$ witnesses that $j^*(X \cap (H_{\kappa^+})^{V[G \upharpoonright \kappa]})$ is *not* in $j^*(\tilde{T})$, as the condition $j((c,p)) = (c,p) \in \mathbb{S}(\vec{S} \upharpoonright \kappa) \subset j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$ from the definition of $\tilde{T}$ may be extended in $j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$ to some $X[\bar{G}, H]$–generic condition $(c^*, p^*) \in j(\mathbb{S}(\vec{S} \upharpoonright \kappa))$ with $\mathrm{dom}(c^*) = \mathrm{dom}(p^*) = \alpha + 1$, $c^*(\alpha) = \alpha$, and $p^*(i) = S_{i_0}$ for some $i < \alpha$.

This contradicts (3). □ (Theorem 0.4)

Theorem 0.5 (Woodin) *Suppose that NS_{ω_1} is saturated and $(\mathcal{P}(\omega_1))^\#$ exists. Then $\boldsymbol{\delta}^1_2 = \omega_2$.*

Proof sketch. (Cf. [4].) If $N \cong X \prec \mathcal{M} = ((\mathcal{P}(\omega_1))^\#; \in, \mathsf{NS}_{\omega_1})$, where N is countable and transitive, then N is generically (ω_1+1)–iterable via the preimage of NS_{ω_1} and its images. By the Boundedness Lemma, the ordinal height of every $(\omega_1)^{\text{th}}$ iterate of N is $< (\omega_1^V)^{+L[z]}$, where $z \in \mathbb{R}$ codes N. On the other hand, if $N_i \cong X_i = \mathrm{Hull}^{\mathcal{M}}(X \cup \{X_j \cap \omega_1 \colon j < i\}) \prec \mathcal{M}$ for $i \leq \omega_1$, then $(N_i \colon i \leq \omega_1)$, together with the obvious maps, is a generic iteration of N. Hence if $\beta \in X$, where $\beta < \omega_2$, $\beta < (\omega_1^V)^{+L[z]} < \boldsymbol{\delta}^1_2$. □ (Theorem 0.5)

[4] shows that if $\mathbb{P}$ is the poset of Theorem 0.4, as defined over M_1, and if G is $\mathbb{P}$–generic over M_1, then $\boldsymbol{\delta}^1_2 < \omega_2$ in $M_1[G]$. The following Theorem gives a bit more information.

Theorem 0.6 *Let $\mathbb{P}$ be the poset of Theorem 0.4, as defined over M_1, and let G be $\mathbb{P}$–generic over M_1. Then $(\boldsymbol{\delta}^1_2)^{M_1[G]} = (\boldsymbol{\delta}^1_2)^{M_1} < \omega_2^{M_1} < \omega_2^{M_1[G]}$.*

PROOF. Deny. Let $x \in \mathbb{R} \cap M_1[G]$ witness that $(\boldsymbol{\delta}^1_2)^{M_1[G]} > (\boldsymbol{\delta}^1_2)^{M_1}$. So if

$$(N_i, \pi_{ij} \colon i \leq j \leq \omega_1)$$

is the iteration of $x^\dagger = N_0$ of length $\omega_1 + 1$ which is obtained by hitting the bottom (total) measure of $x^\dagger$ and its images ω_1 times, then $(\omega_1^V)^{+N_{\omega_1}} > (\boldsymbol{\delta}^1_2)^{M_1}$.

As $x^\dagger \models$ "There is no inner model with a Woodin cardinal," we may let K denote the core model of $x^\dagger$ of height Ω, where Ω is the top measurable cardinal of $x^\dagger$. By [3], there is a normal iteration tree $\mathcal{T} \in x^\dagger$ on K with $[0,\infty)_\mathcal{T} = \emptyset$ and last model K^{N_1} such that $\pi_{01} = \pi^\mathcal{T}_{0\infty}$. Letting $\mathcal{T}^*$ be the concatenation of all $\pi_{0i}(\mathcal{T})$, $0 \leq i < \omega_1$, $\mathcal{T}^*$ is then a (non–normal) iteration tree on K with $[0,\infty)_{\mathcal{T}^*} = \emptyset$ and last model $K^{N_{\omega_1}}$ such that $\pi_{0\omega_1} \restriction K = \pi^{\mathcal{T}^*}_{0\infty}$. By absoluteness, K is in fact iterable in $M_1[G]$, and $\mathcal{T}^*$ is according to the (unique) relevant iteration strategy.

We claim that K iterates past $M_1|\omega_1$.

Otherwise suppose that $\alpha < \omega_1$ is such that $M_1|\alpha$ absorbs K. There is then, in $M_1[G]$, an iteration tree $\mathcal{U}$ on $M_1|\alpha$ of length $\omega_1 + 1$ such that $\mathcal{M}^\mathcal{U}_{\omega_1} \cap \mathrm{OR} \geq N_{\omega_1} \cap \mathrm{OR} > (\boldsymbol{\delta}^1_2)^{M_1}$. (Cf. [2] for a writeup of this argument.) On the other hand, by the Boundedness Lemma, if $z \in \mathbb{R} \cap M_1$ codes $M_1|\alpha$ and if γ denotes the supremum of all the ordinal heights of all $(\omega_1)^{\mathrm{th}}$ iterates of $M_1|\alpha$, then

$$\gamma < (\omega_1)^{+L[z]}.$$

In particular, $(\boldsymbol{\delta}^1_2)^{M_1} > (\omega_1)^{+L[z]} > \gamma > \mathcal{M}^\mathcal{U}_{\omega_1} \cap \mathrm{OR} > (\boldsymbol{\delta}^1_2)^{M_1}$.

This contradiction indeed shows that K iterates past $M_1|\omega_1$. But then ω_1 has to be an inaccessible cardinal of M_1, which is nonsense. $\square$ (Theorem 0.6)

Question. Is $M_1[G] \models \neg\mathsf{CH}$? Is $\mathbb{R} \cap M_1[G] \subset M_1$?

References

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