Book Review: Lorenz J. Halbeisen: "Combinatorial Set Theory."

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Combinatorics is that area of mathematics where we abstract that much from any structural properties of the objects under study that the Pigeonhole Principle is of prominent use. This principle says that if 5 pigeons sit in 3 holes, then at least one hole is occupied by more that 1 pigeon, or more generally: there can be no injection $f\colon A \to B$ if the cardinality of the set A is strictly bigger than the cardinality of the set B.

The Pigeonhole Principle yields Ramsey's Theorem [5]. It's infinite version tells us that for all n and $k \in \mathbb{N}$, if every subset of $\mathbb{N}$ with n elements is colored by one of the colors 1, 2, ..., k, then there is an infinite $X \subset \mathbb{N}$ and some color $c \in \{1, 2, \ldots, k\}$ such that every subset of X with n elements is colored by c. Ramsey Theory studies (finite and infinite) variants and generalizations of Ramsey's Theorem and their applications.

An old result which exploits Ramsey theoretic methods, even though it actually predates F. Ramsey's formulation of his theorem, is a theorem of I. Schur [6] according to which for every $n > 1$ and for every sufficiently large prime p, the equation

$$x^n + y^n \equiv z^n (\text{mod } p)$$

has a solution x, y, z which is nontrivial, i.e., $p \nmid xyz$. A proof of this may be found in the classic [2] on Ramsey Theory.

By a theorem of Paris and Harrington [4] there is a *finite* variant of Ramsey's Theorem which is true but cannot be proven in Peano Arithmetic. Analogously, also many statements of *infinite* Ramsey Theory are independent in the following fashion.

For an arbitrary set B, $\mathcal{P}(B)$ denotes the power set of B, i.e., the set of all subsets of B; let us write $\mathcal{P}_\infty(B) \subset \mathcal{P}(B)$ for the set of all infinite subsets of B. A set $A \subset \mathcal{P}_\infty(\mathbb{N})$ is called *Ramsey* iff there is some infinite $X \subset \mathbb{N}$ such that

$$\mathcal{P}_\infty(X) \subset A \text{ or } \mathcal{P}_\infty(X) \cap A = \emptyset.$$

It is not difficult to use AC, the Axiom of Choice, to show that there is some $A \subset \mathcal{P}_\infty(\mathbb{N})$ which is not Ramsey. With the help of concepts from Descriptive Set Theory, we may however make sense of the following question, which then naturally arises: In the presence of AC, how "definable" can a non–Ramsey $A \subset \mathcal{P}_\infty(\mathbb{N})$ be? In order to answer this question, we have to specify the axiomatic background in which we work. This is because, as it turns out, the answer is independent from ZFC, the standard Zermelo–Fraenkel axiomatization of set theory with AC. The same applies if we drop assuming AC. A theorem of A.R.D. Mathias shows that in the absence of AC, every $A \subset \mathcal{P}_\infty(\mathbb{N})$ can be Ramsey [3].

The book under review provides a thorough and nicely written account of cominatorial set theory and infinite Ramsey theory together with a treatment of the underlying set theoretical axioms as well as of sophisticated methods which are involved in proving independence results.

Part I of the book introduces variants of Ramsey's Theorem and ZFC (with and without atoms) and discusses AC, the Banach–Tarski paradox, and ultrafilters on $\mathbb{N}$. It also deals with the topic of cardinal arithmetic in the absence of AC, where not every cardinality needs to be an $\aleph$. An amuzing open problem is: Without AC, if there is a surjection from $A \times A$ onto $\mathcal{P}(A)$, must it be the case that A has at most 4 elements?

Part I also introduces several cardinal characteristics related to combinatorial questions, such as as $\mathfrak{p}$, $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{s}$, $\mathfrak{r}$, $\mathfrak{a}$, and $\mathfrak{i}$. E.g., the concept of $A \subset \mathcal{P}_\infty(\mathbb{N})$ being Ramsey is related to the shattering number $\mathfrak{h}$. To give an example, let us present the almost disjoint number $\mathfrak{a}$ which is a bit easier to define than $\mathfrak{h}$. If $A, B \in \mathcal{P}_\infty(\mathbb{N})$, then A and B are called almost disjoint if $A \cap B$ is finite. A *mad* (maximal almost disjoint) family is a collection $\mathcal{A} \subset \mathcal{P}_\infty(\mathbb{N})$ such that any two distinct elements of $\mathcal{A}$ are almost disjoint and for every $A \in \mathcal{P}_\infty(\mathbb{N})$ there is some $B \in \mathcal{A}$ such that $A \cap B$ is infinite. By a diagonal argument, no mad family can be countable. Also, there is always a mad family of cardinality $2^{\aleph_0}$, the size of the continuum. The cardinal characteristic $\mathfrak{a}$ is defined to be the smallest cardinality of a mad family, so that $\aleph_0 < \mathfrak{a} \leq 2^{\aleph_0}$. An exciting area of set theory studies the possible values of cardinal characteristics and their relations to each other, cf. e.g. [1].

Part II of the book introduces the key method of this area: the technique of *forcing*, which originally was developped by P. Cohen to prove the independence of the Continuum Hypothesis from ZFC. Especially relevant for the separation of various cardinal characteristics from each other is *iterated* forcing. To give just one example (Proposition 18.5 in the book), it may be shown by adding a lot of Cohen reals to a set theoretic parent universe that $\mathfrak{a}$ can consistently be strictly smaller than $2^{\aleph_0}$. This is done by showing that in this special situation, a mad family from the parent universe is still mad in the forcing extension after adding Cohen reals.

In order to produce deeper results a more detailled analysis of different types of reals which may be added by forcing is called for. This is the topic of Part III of the book, which revisits Cohen reals but also introduces Laver, Silver, Miller, and Mathias reals. The forcings which add such reals may be iterated by *proper* forcing, a concept isolated by S. Shelah. Part III of the book brings us to the frontier of present–day research. For instance, it is shown how to build models of $\mathfrak{a} < \mathfrak{d} = \mathfrak{r}$ (Proposition 21.10), $\mathfrak{s} = \mathfrak{b} < \mathfrak{d}$ (Proposition 21.13), $\mathfrak{d} < \mathfrak{r}$ (Proposition 22.4), $\mathfrak{r} < \mathfrak{d}$ (Proposition 23.7), and $\mathfrak{p} < \mathfrak{h}$ (Proposition 2.12), respectively. All these statements may be translated into natural combinatorial statements obout infinite sets of natural numbers.

Each chapter of the book comes with historical information, suggestions for further reading, and it lists open problems.

Lorenz Halbeisen wrote a marvellous book. I can recommend this book to all graduate students, PostDocs, and researchers who are interested in set theoretical combinatorics, set theory in the absence of AC, (iterated) forcing, and cardinal invariants. However, also mathematicians from other areas who are interested in the foundational aspects of their subject will enjoy this book.

References

[1] J. Brendle, *Cardinal invariants of the continuum – a survey*, Surikaisekiken-kyusho Kokyuroku **1202** (2001), pp. 7–32.

[2] R.L. Graham, B.L. Rothschild, J.H. Spencer, *Ramsey Theory*, Wiley, New York 1980.

[3] A.R.D. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), pp. 59–111.

[4] J.B. Paris, L. Harrington, *A mathematical incompleteness in Peano Arithmetic*, in: Handbook of Mathematical Logic, J. Barwise (ed.), Amsterdam 1977, pp. 1133–1142.

[5] F. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. Ser. II **30** (1930), pp. 264–286.

[6] I. Schur, *Über die Kongruenz* $x^m + y^m \equiv z^m (mod\ p)$, Jahresberichte der Deutschen Math.–Vereinigung **25** (1916), pp. 114–117.