PROBLEM LIST. BERKELEY CONFERECE ON INNER MODEL THEORY 2019

1. (Schindler) Let a core model, K, be a fully iterable pure extender model satisfying (1) weak covering in that $\{\alpha < \mathrm{ORD} | \alpha^{+K} = \alpha^{+}\}$ is stationary and (2) for any fully iterable pure extender model N satisfying this form of weak covering there is an elementary embedding $j: K \to N$. (Aside: Can there be two of them? Must it (they) be rigid?) How many Woodin cardinals can K have? (I.e., what is consistent with ZFC in this regard?)

E.g. (a) 0 Woodins is possible, (b) 1 Woodin is possible: $K^{M_{\mathrm{sw}}} = (\mathcal{M}_\infty)^{M_{\mathrm{sw}}}$, (c) exactly 2 is impossible: if $K \models$ "there are 2 Woodins," then $K \models$ "there is a strong cardinal which is a limit of Woodins," cf. Sargsyan-Schindler, "The number of Woodin cardinals in a core model,"
`https://ivv5hpp.uni-muenster.de/u/rds/number_of_woodins.pdf`
Moreover, there is a universe in which there is a K which has a strong cardinal that is a limit of Woodins, cf. Stefan Miedzianowski's PhD thesis.

What can a fully iterable pure extender model having the above form of weak covering look like?

2. (Wilson) Let δ be Woodin. Can the δ-c.c. property of the extender algebra be proved directly from $\langle V_\delta, V_{\delta+1}, \in\rangle \models \mathsf{SWVP}$? ($\mathsf{SWVP}$ being the semi-weak Vopenka principle.) "Directly" means without using elementary embeddings of models of set theory.

3. (Steel) (a) Does δ inaccessible + $\mathbb{B}_\delta$ (the ω-generator version of the extender algebra) is δ-c.c. imply that δ is Woodin?

(Ketchersid-Zoble: Con($\mathbb{B}_\delta$ is δ-c.c. + $V_\delta^{\#}$ exists) $\Longrightarrow$ Con(ZFC + $\exists$ a Woodin).)

(b) Do you need $V_\delta^{\#}$ exists here?

(c) What about the δ-generator version of the extender algebra?

4. (Trang) Assume there are arbitrarily large Woodins. Are the following equivalent?

(i) LSA-over-UB

(ii) Sealing, i.e., for all generic g, $L(\mathsf{Hom}_\infty^{V[g]}) \cap \mathcal{P}(\mathbb{R}^{V[g]}) = \mathsf{Hom}_\infty^{V[g]}$, and $\forall g\, \forall h = g * l$ there is an elementary embedding $L(\mathsf{Hom}_\infty^{V[g]}) \to L(\mathsf{Hom}_\infty^{V[h]})$

(iii) Sealing (-1), i.e., for all generic g, $L(\mathsf{Hom}_\infty^{V[g]}) \cap \mathcal{P}(\mathbb{R}^{V[g]}) = \mathsf{Hom}_\infty^{V[g]}$

(iv) Sealing (-2), i.e., for all generic g, $L(\mathsf{Hom}_\infty^{V[g]}) \models$ "there is no ω_1-sequence of pairwise distinct reals"

Reference: Sargsyan-Trang, "The exact consistency strength of generic absoluteness for the universally Baire sets."

5. (Ben-Neria) Let U be an ultrafilter on a cardinal κ. Definition: $\mathcal{F} \subset U$ *generates* U iff $\mathcal{F} \subset U$ and for every $X \in U$ there is some $Y \in \mathcal{F}$ such that $Y \subset X$. What is the consistency strength of: κ is measurable, $2^\kappa \geq \kappa^{++}$, and $\exists U$ (U is a κ-complete ultrafilter on κ and $\exists \mathcal{F}$ ($|\mathcal{F}| = \kappa^+$ and $\mathcal{F}$ generates U))?

Known: 1) Lower bound is $o(\kappa) \geq \kappa^{++}$ (because Con(κ measurable + $2^\kappa = \kappa^{++}$) $\implies$ Con($o(\kappa) \geq \kappa^{++}$) by Gitik). Upper bound: $o(\kappa) = \lambda$, where $\lambda > \kappa$ and λ is weakly compact (reference: Ben-Neria-Garti,"Consistency results about cardinal characteristics above the continuum,"
`https://arxiv.org/pdf/1905.06067.pdf`

Rmk. For $\kappa = \omega$, there are theorems of Shelah.

Rmk. For $\kappa = \omega_1$, this is an old question of Kunen: Con($\exists$ a uniform ultrafilter on ω_1 generated by fewer than $2^{\aleph_1}$ sets)?

6. (Schlutzenberg) Let $M_1^\# \leq_T x$. Let κ be a cardinal of $L[x]$. What is the κ-mantle of $L[x]$? Definition: The κ-mantle is $\mathbb{M} = \bigcap\{W|\exists\mathbb{P} \in W(W \models |\mathbb{P}| < \kappa \wedge \exists g\, \mathbb{P}\text{-generic over } W \text{ such that } V = W[g])\}$.

In particular, for $\kappa = \aleph_\omega^{L[x]}$, let $M = \bigcap\{N|N$ is an iterate of M_1 via a normal tree $\mathcal{T}$ in $L_\kappa[x]\}$, let $M_\infty =$ the direct limit of these N, and let $*$ be the usual $*$-map. Then $M_\infty[*] \subset M$. But $M_\infty[*] \neq M$, since Prikry generics over M_∞ exist in M. Question: $M \models \mathsf{ZFC}$?

Rmk. κ strong limit $\implies$ κ-mantle $\models \mathsf{ZF}$ (Usuba). What about other κ's?

Rmk. $x, y \geq_T M_1^\# \implies \mathrm{Th}(L[x]) \equiv \mathrm{Th}(L[y])$.

7. (Glazer) What is the consistency strength of ZFC + $\exists\kappa$ there is no OD-surjection of ${}^{\mathrm{cf}(\kappa)}\kappa$ onto κ^+?

(a) κ must be singular.

(b) Lower bound: one Woodin (K can't exist).

(c) Unknown whether it is consistent at all.

(d) Can it be that κ is singular and there is no OD-injection from κ^+ into ${}^{\mathrm{cf}(\kappa)}\kappa$?

Remark. Woodin's **HOD Conjecture** implies that this does not happen at any $\kappa >$ an extendible cardinal.

Known: From a supercompact + an inaccessible above get a model of ZFC + κ is a strong limit cardinal + $\forall A \subset \kappa\, (\kappa^+)^{\mathsf{HOD}_A} < \kappa^+$. Reference: Gitik + Merimovich, "Some applications of supercompact extender based forcings to HOD,"
`http://www.math.tau.ac.il/~gitik/somepapers.html`
Maybe related to ¬ (c), (d)

8. (Ben-Neria) Suppose there is a singular λ such that λ^+ is ω-strongly measurable (means: $\exists\eta < \lambda^+((2^\eta)^{\mathsf{HOD}} < \lambda^+ \wedge \neg\exists$ a partition $\langle S_\alpha|\alpha < \eta\rangle$ of λ^+ into statonary sets such that $\langle S_\alpha|\alpha < \eta\rangle \in \mathsf{HOD}$)). Is there an inner model with a Woodin? (Schindler: yes.) What is the consistency strength?

What is the consistency strength of $\exists\kappa$ (κ and κ^+ are singular)?

9. (Ben-Neria, Zeman) Suppose every $\langle S_n|n < \omega\rangle$ such that $S_n \subset \omega_{n+2} \cap \mathrm{cof}(\omega_1)$ and S_n's stationary is mutually stationary. Must $\mathsf{AD}^{L(\mathbb{R})}$?

Rmk. Consistent from ω-many supercompacts (Ben-Neria)

Rmk. Known lower bound is PD. (Ben-Neria, Zeman, Adolf, Schindler, Steel)

10. (Adolf) Asume $(\omega_3, \omega_2) \implies (\omega_2, \omega_1)$. Does mouse reflection hold?: Given a nice mouse operator on H_{ω_2} ($X \mapsto M_X$, $X \geq Y_0$, $X \in H_{\omega_2}$); does it extend to H_{ω_3}?