\[ P \neq NP \] for infinite time Turing machines

Ralf Schindler\(^a\)*

\(^a\)Institut für formale Logik, Universität Wien, 1090 Wien, Austria

rds@logic.univie.ac.at
http://www.logic.univie.ac.at/~rds/

Abstract. We state different versions of the \( P =?NP \) problem for infinite time Turing machines. It is observed that \( P \neq NP \) collapses to the fact that there are analytic sets which are not Borel.

In this note we study versions of the \( P =?NP \) problem for infinite time Turing machines. The analytic sets of reals may be construed as an infinite analog to the class \( NP \) (cf. for example [3, §3.9]). The \( P =?NP \) problems for infinite time Turing machines can therefore naturally be translated as questions about analytic sets. These questions have classical answers.

1 Analytic sets.

We shall have to consider Polish spaces, i.e., complete separable metric spaces. In particular we shall be interested in the Cantor space \( \omega^2 \) and in the Baire space \( \omega^\omega \). We refer the reader to [2] for background information. However, the descriptive set theory which we shall need is pretty elementary indeed. In order to make the paper self-contained modulo [1] this section develops all the necessary descriptive set theoretic tools.

Let \( \mathcal{X} \) be a Polish space and let \( (\mathcal{O}_n; n < \omega) \) be a recursive enumeration of basic open sets. Let \( \mathcal{O}_\mathcal{X} \subset \mathcal{X} \times \omega^2 \) be defined by

\[(x, y) \in \mathcal{O}_\mathcal{X} \iff x \in \bigcup_{y(n)=1} \mathcal{O}_n.\]

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Then $O^X$ is a universal open set, i.e., $O^X$ is open in $X \times \omega_2$ and if $O$ is an open subset of $X$ then there is some $y \in \omega_2$ with

$$O = \{ x \mid (x, y) \in O^X \}.$$ 

A set $A \subseteq X$ is analytic if and only if there is a closed $C \subseteq X \times \omega$ such that

$$x \in A \iff \exists y \in \omega (x, y) \in C.$$ 

Let $U^X \subseteq X \times \omega_2$ be defined by

$$(x, y) \in U^X \iff \exists z \in \omega_\omega ((x, z), y) \notin O^{X \times \omega}.$$ 

Then $U^X$ is a universal analytic set, i.e., $U^X$ is analytic in $X \times \omega_2$ and if $A$ is an analytic subset of $X$ then there is some $y \in \omega_2$ with

$$A = \{ x \mid (x, y) \in U^X \}.$$ 

Now let $X = \omega_2$, and let us write $U$ for $U^X = U^\omega_2$. The set $\Delta = \{ x \mid (x, x) \in U \}$ is analytic. If $\Delta$ were coanalytic, i.e., if $\omega_2 \setminus \Delta$ were analytic, then there would be some $y \in \omega_2$ with

$$\Delta = \{ x \mid (x, y) \notin U \};$$

but then $y \in \Delta$ iff $(y, y) \notin U$ iff $y \notin \Delta$. Hence $\Delta$ is not coanalytic, and therefore not Borel.

Let $G \subseteq \omega_2$ be the set of all $x \in \omega_2$ which are not eventually constant (equivalently, such that there are arbitrary large $m$ and $m'$ with $x(m) = 0$ and $x(m') = 1$). $G$ is a $G_\delta$ subset of $\omega_2$. We may define a bijection $\varphi: G \rightarrow \omega_\omega$ by $\varphi(x) = y$ if and only if the $n$th block of 1's in $x$ contains exactly $y(n)$ 1's. If $A \subseteq \omega_2 \times \omega$, say, then we may define $A^\varphi \subseteq \omega_2 \times \omega_2$ by

$$(x, y) \in A^\varphi \iff y \in G \land (x, \varphi(y)) \in A.$$ 

Now let $A \subseteq \omega_2$ be lightface analytic which means that there is a recursive $R(-,-)$ such that, if $C$ denotes the closed set

$$[R] = \{ (x, y) \in \omega_2 \times \omega \mid \forall n < \omega R(x \downharpoonright n, y \downharpoonright n) \},$$

then we have that

$$x \in A \iff \exists y \in \omega \omega (x, y) \in C.$$ 

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We then also have that

\[ x \in A \iff \exists y \in \omega^2 \ (x, y) \in C^\varphi. \]

It is straightforward to verify that \( C^\varphi \) is no longer closed. Rather, \( C^\varphi \) is a lightface \( G_\delta \) subset of \( \omega^2 \).

Now the set \( \Delta \subset \omega^2 \) defined above is a lightface analytic set. In fact, the witnessing recursive \( R(-, -) \) is easily given by the complement of \( O^{\omega^2 \times \omega} \); we here need that \( (O_n : n < \omega) \) be recursive. We then have that

\[ x \in \Delta \iff \exists y \in \omega^2 \ (x, y) \in \mathcal{G}, \]

where \( \mathcal{G} \) is a lightface \( G_\delta \) subset of \( \omega^2 \). It is this latter representation of \( \Delta \) which we shall need later on.

## 2 \( P \neq NP \) for infinite time Turing machines.

Infinite time Turing machines were introduced in [1]. They have exactly the same hardware as traditional Turing machines. The difference is that one allows transfinite running times. We refer the reader to [1] for exact definitions. An acquaintance with §§1 and 2 (i.e., pp. 569-575) of [1] will basically suffice for our purposes. In what follows, by “Turing machine” we shall always mean an infinite time Turing machine.

It will be convenient to think of a Turing machine to come with two halting states, the accept state, and the reject state.

**Definition 2.1** Let \( A \subset \omega^2 \). We say that \( A \) is decidable in polynomial time, or \( A \in P \), if there are a Turing machine \( T \) and some \( m < \omega \) such that

(a) \( T \) decides \( A \) (i.e., \( x \in A \) iff \( T \) accepts \( x \)), and

(b) \( T \) halts on all inputs after \( < \omega^m \) many steps.

With infinite time Turing machines, all inputs (i.e., elements of the Cantor space \( \omega^2 \)) may be counted as having the same length, namely \( \omega \). So it appears reasonable to have a polynomial time Turing machine being one which always halts after \( < \omega^m \) many steps, for some fixed \( m < \omega \).

The following just generalizes Definition 2.1.
**Definition 2.2** Let $A \subset \omega^2$, and let $\alpha \leq \omega_1 + 1$. We say that $A$ is in $P_\alpha$ if there are a Turing machine $T$ and some $\beta < \alpha$ such that
(a) $T$ decides $A$ (i.e., $x \in A$ iff $T$ accepts $x$), and
(b) $T$ halts on all inputs after $< \beta$ many steps.

Of course, $P = P_{\omega^2}$. Moreover, $P_{\omega_1 + 1}$ is just the class of all $A \subset \omega^2$ which are decided by some Turing machine.

**Lemma 2.3** ([1, Theorem 2.6]) Let $A \subset \omega^2$. Then $A \in P_{\omega^2}$ if and only if $A$ is an arithmetic set.

**Lemma 2.4** Let $A \subset \omega^2$. Then $A \in P_{\omega^2}$ if and only if $A$ is a lightface $G_\delta$ set.

**Proof.** This is straightforward. The less trivial direction is given by the proof of [1, Theorem 2.6]. $\Box$

**Lemma 2.5** Let $A \subset \omega^2$. Then $A \in P_{\omega_1}^e$ if and only if $A$ is a hyperarithmetic set. If $A \in P_\omega$, then $A$ is a Borel set.

**Proof.** The first part is [1, Theorem 2.7]. The second part is an immediate consequence of the proof thereof. $\Box$

It is on the other hand not true that every Borel set is in $P_\omega$. Lemma 2.7 will characterize $P_{\omega_1}$.

**Definition 2.6** Let $A \subset \omega^2$. If $\alpha < \omega_1$, then we say that $A \in \Delta^1_1(\alpha)$ if $A \in \Delta^1_1(x)$ uniformly for every real $x$ coding $\alpha$. We say that $A$ is $\Delta^1_1$ in a countable ordinal if there is some $\alpha < \omega_1$ such that $A \in \Delta^1_1(\alpha)$.

**Lemma 2.7** \footnote{This was observed independently by J.D. Hamkins.} $A \in P_{\omega_1}$ if and only if $A$ is $\Delta^1_1$ in a countable ordinal. In fact, if $\alpha$ is admissible then $A \in P_\alpha$ if and only if $A \in \Delta^1_1(\beta)$ for some $\beta < \alpha$.

**Proof.** This follows from revisiting the proof of [1, Theorem 2.7]. “$\Rightarrow$” is immediate. As to “$\Leftarrow$,” note that we may pick a real $x$ coding $\beta$ such that $\alpha \geq \omega_1^x$ (= the least $x$-admissible $> \omega$). The Borel code for $A$ is the a tree with rank $< \omega_1^x \leq \alpha$. $\Box$

We now turn to the class $NP$. 

1 This was observed independently by J.D. Hamkins.
Definition 2.8 Let $A \subseteq \omega^2$. We say that $A$ is verifiable in polynomial time, or $A \in NP$, if there are a Turing machine $T$ and some $m < \omega$ such that
(a) $x \in A$ if and only if $(\exists y \ T$ accepts $x \oplus y)$, and
(b) $T$ halts on all inputs after $< \omega^m$ many steps.

Definition 2.9 Let $A \subseteq \omega^2$, and let $\alpha \leq \omega_1 + 1$. We say that $A$ is in $NP_\alpha$, if there are a Turing machine $T$ and some $\beta < \alpha$ such that
(a) $x \in A$ if and only if $(\exists y \ T$ accepts $x \oplus y)$, and
(b) $T$ halts on all inputs after $< \beta$ many steps.

Again, $NP = NP_{\omega^2}$. $NP_\alpha$ is the class of all projections of sets in $P_\alpha$. It is now immediate that $P \neq NP$.

Theorem 2.10 $NP_{\omega+1} \setminus P_{\omega_2} \neq \emptyset$.

Proof. Let $\Delta$ and $\mathcal{G}$ be as in section 1. In particular, $\Delta$ is a lightface analytic subset of $\omega^2$ which is not Borel, $\mathcal{G}$ is a lightface $G_\delta$ set, and

$$x \in \Delta \iff \exists y \in \omega^2 (x, y) \in \mathcal{G}.$$ 

By Lemma 2.4, $\mathcal{G} \in P_{\omega^2}$. Hence $\Delta \in NP_{\omega+2}$. However, by Lemma 2.5, $\Delta$ cannot be in $P_{\omega_2}$, as it is not Borel. \(\square\)

Another version of the $P = NP$ problem counts an input $x \in \omega^2$ as having length $\omega^x_1$ (= the least $x$-admissible $> \omega$). Note that no admissible ordinal is clockable (cf. [1, Theorem 8.8]. This leads to:

Definition 2.11 Let $A \subseteq \omega^2$. We say that $A \in P^+$ if there is a Turing machine $T$ such that
(a) $x \in A$ if and only if $T$ accepts $x$, and
(b) $T$ halts on all inputs $x$ after $< \omega^x_1$ many steps.

Definition 2.12 Let $A \subseteq \omega^2$. We say that $A \in NP^+$ if there is a Turing machine $T$ such that
(a) $x \in A$ if and only if $(\exists y \ T$ accepts $x \oplus y)$, and
(b) $T$ halts on all inputs $x \oplus y$ after $< \omega^x_1$ many steps.

Again we'll have that $P \neq NP$. 


Theorem 2.13 \( P^+ = P_{\omega^1_{CK}} = \Delta^1_1 \).

Proof. Let \( A \in P^+ \). It is straightforward that there is then a \( \Sigma_1 \) formula \( \Psi \) (saying that there is a certain sequence of snapshots) such that

\[
x \in A \iff L_{\omega^1_{CK}}[x] = \Psi(x).
\]

This implies that \( A \) is coanalytic (i.e., \( \Pi^1_1 \)). Of course, we also have that \( \omega^2 \setminus A \in P^+ \), so that by the same argument \( \omega^2 \setminus A \in \Pi^1_1 \). Therefore, \( P^+ \subset \Delta^1_1 \).

On the other hand, we have \( \Delta^1_1 = P_{\omega^1_{CK}} \subset P^+ \). \( \square \)

Corollary 2.14 \( NP^+ \setminus P^+ \neq \emptyset \).

3 Some open problems.

We may allow a Turing machine to take even more time to reach its decision. Recall that if \( \lambda + n \) is clockable for \( n < \omega \) then so is \( \lambda \). We arrive at:\(^2\)

Definition 3.1 Let \( A \subset \omega^2 \). We say that \( A \in P^{++} \) if there is a Turing machine \( T \) such that
(a) \( x \in A \) if and only if \( T \) accepts \( x \), and
(b) \( T \) halts on all inputs \( x \) after \( \leq \omega_1^\omega + \omega \) many steps.

Definition 3.2 Let \( A \subset \omega^2 \). We say that \( A \in NP^{++} \) if there is a Turing machine \( T \) such that
(a) \( x \in A \) if and only if \( (\exists y \ T \) accepts \( x \oplus y \)), and
(b) \( T \) halts on all inputs \( x \oplus y \) after \( \leq \omega_1^\omega + \omega \) many steps.

\( P^{++} \) is a larger class than \( P^+ \):

Theorem 3.3 Every lightface analytic set is in \( P^{++} \).

Proof. Let \( A \) be a lightface analytic set. There is a recursive \( R(\_, \_, \_) \) such that

\[
x \in A \iff \exists y \in \omega \ \forall n < \omega \ R(x \upharpoonright n, y \upharpoonright n).
\]

\(^2\)In spirit this has been suggested by P. Welch.
For $x \in \omega^2$ consider the tree

$$T_x = \{ s \mid R(x \upharpoonright lh(s), s) \}.$$  

Then

$$x \in A \iff T_x \text{ is illfounded}.$$  

We can design a Turing machine $T$ which, on input $x$, first produces $T_x$ and then crosses out the wellfounded part of $T_x$. This wellfounded part has rank $\leq \omega_x^x$ (as every wellfounded tree which is recursive in $x$ has rank $< \omega_x^x$).

The machine $T$ is finally supposed to check if there is something left after crossing out the wellfounded part of $T_x$. This will take another $\omega$ many steps of computation. On input $x$, $T$ has therefore a running time $\leq \omega_x^x + \omega$. $\square$

Of course, $P^{++}$ is also closed under complements.

**Question.** $P^{++} \neq NP^{++}$?

**Definition 3.4** Let $f : D \to \omega_1$. Let $A \subseteq \omega^2$. We say that $A \in P^f$ if there is a Turing machine $T$ such that

(a) $x \in A$ if and only if $T$ accepts $x$, and

(b) $T$ halts on all inputs $x$ after $< f(x)$ many steps.

**Definition 3.5** Let $f : D \to \omega_1$. Let $A \subseteq \omega^2$. We say that $A \in NP^f$ if there is a Turing machine $T$ such that

(a) $x \in A$ if and only if $\exists y T$ accepts $x \oplus y$, and

(b) $T$ halts on all inputs $x \oplus y$ after $< f(x)$ many steps.

**Question.** For which $f : D \to \omega_1$ is $P^f \neq NP^f$?

**References**


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