Modulo well-known results, the arguments of that paper may be simplified; Theorem 1 may also be slightly strengthened, as follows.

**Theorem 1** (BMM + $\exists\alpha$ a Woodin cardinal) There is no AC$\omega_1$ ad no $\Sigma_1$ formula $\psi$ in the language $\mathcal{L}_\in$ of set theory s.t. for all $S \subseteq \mathcal{P}(\omega_1)$, $S$ is stationary iff $\psi(S, \mathcal{A})$.

**Proof:** Let $S_0 = \{ X < H_{\omega_2} : \exists Y \in {\mathcal{N}}_1, X \text{ is transitive} \}$, a stationary set. Let $g$ be $\text{IP}_{<\delta}$-generic over $V$, $\delta$ being a Woodin cardinal, $\text{IP}_{<\delta}$ being the associated full stationary tower; let $S \in g$. Then if

$$ j : V \rightarrow M \subset V[g] $$

is the generic elementary embedding given by $g$, $M$ transitive, $j'' H_{\omega_2} \in j(S_0)$, so that
crit(j) = \omega_2^V.

We have \( P(\omega_1) \cap V \in M \): By BMM, \( 2^{\omega_1} = \aleph_2 \), so let \( f: \omega_2 \to P(\omega_1) \cap V \) be bijective, \( f \in V \). Then \( j(f) \upharpoonright \omega_2 = f \), and \( j(f) \in M \), so \( P(\omega_1) \cap V = \text{ran}(f) \in M \).

By MA\( \omega_1 \), NS\( \omega_1 \) is not \( \omega_1 \)-dense. This was shown by A.D. Taylor, see [Tay79], see also e.g. Lemma 6.51 of [Woo99]. As \( P(\omega_1) \cap V \) has size \( \omega_1^V = \aleph_1 \) in \( M \), there is then some stationary set \( S \subset \omega_1 \) in \( M \) s.t. \( T \setminus S \) is stationary for all stationary \( T \in P(\omega_1) \cap V \).

By \( <^{\omega_1} M \cap V[e] \subset M \), it is still true in \( V[e] \) that \( S \) is stationary and \( T \setminus S \) is stationary for all stationary \( T \in P(\omega_1) \cap V \).

*) This is a theorem of Todorcevic.
We may then continue as in the proof of Theorem 1 of "\( NS_1 \) is not \( \Pi_1 \) definable," see pp. 4 bottom - 6.

The same simplification may be applied in the proof of Theorem 2 of "\( NS_{\omega_1} \) is not \( \Pi_1 \) definable."

If \( p, q \in P_{\text{max}}, q \leq_{P_{\text{max}}} p \) as being witnessed by \( i : p \rightarrow p^* \), then \( p^* = X_i \) in \( q \), hence by \( q \models MA_{\omega_1} \), there is a \textit{stationary} positive set \( S \) in \( q \) such that \( T \setminus S \) is positive for all positive \( T \in p^* \) (here, "positive" refers to the distinguished ideal \( I^*_q \) of \( q \); \cite{Tay79} shows there is no \( \omega_1 \)-dense normal ideal under \( MA_{\omega_1} \)). We may then let \( L = \{ X \in P(\omega_1)^q : \exists Y \in I^*_q \setminus S \cap Y \} \), and proceed from there as on pp. 9-10 of "\( NS_1 \) is not \( \Pi_1 \) definable."


\cite{Woo99} W.H. Woodin, "The axiom of determinacy, forcing axioms, and the nonstationary ideal," de Gruyter 1999.