Woodin's axiom $(*)$, or Martin's Maximum, or both?

Ralf Schindler

This paper is dedicated to W. Hugh Woodin on the occasion of his 60th birthday.

Abstract. We present and discuss a new axiom, Martin's Maximum$^{*,++}$, cf. Definition 2.10, which amalgamates Woodin's $\mathbb{P}_{\max}$ axiom $(*)$ and Martin's Maximum^{++}.

If a mathematical object can be imagined in a reasonable way, then it exists!

Menachem Magidor

1. Introduction

Building upon earlier work of J. Steel and R. Van Wesep, cf. [**StVW82**], W. Hugh Woodin introduced in [**Woo83**] an axiom which he called $*$, cf. [**Woo83**, p. 189]. He shows that ZFC plus $*$ is consistent relative to[1] ZF plus AD and that $*$ implies that $\boldsymbol{\delta}^1_2 = \aleph_2$ and NS_{ω_1} is saturated, cf. [**Woo83**, Theorems 3.2 and 3.3]. The axiom $*$ of [**Woo83**] is a precursor to the axiom $(*)$ which W. Hugh Woodin introduced in [**Woo99**, Definition 5.1].

Recall that the axiom $(*)$ of [**Woo99**] says that

(1) AD holds in $L(\mathbb{R})$, and
(2) there is some G which is $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$ such that $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[G]$.

As $*$, the $\mathbb{P}_{\max}$ axiom $(*)$ also implies that $\boldsymbol{\delta}^1_2 = \aleph_2$ and NS_{ω_1} is saturated – the latter under the additional hypothesis that $V = L(\mathbb{R})[G]$, where G witnesses $(*)$, cf. [**Woo99**, Theorems 4.50 and 4.53]. Besides, "$\delta^1_2 = \aleph_2$," $(*)$ yields many other interesting statements whose complexity is Π_2 over H_{ω_2}, e.g. ϕ_{AC}, [**Woo99**, Corollary 5.7], and ψ_{AC}, [**Woo99**, Lemma 5.18], cf. also [**Woo99**, Theorems 5.74 and 5.76].

The axiom $(*)$ may in fact be construed as a maximality principle with respect to truths which are Π_2 over H_{ω_2}. E.g., every sentence of that complexity which

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[1] Here, AD is the axiom of determinacy.

holds true in V already holds true in every $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, cf. [**Woo99**, Theorem 4.64], and $(*)$ implies that every sentence which is Π_2 over H_{ω_2} and which is Ω–consistent holds true in V, cf. [**Woo99**, Theorem 10.149].

Another way of spelling out the Π_2 maximality feature of $(*)$ is given by [**AspSch**, Theorem 2.7] which states that $(*)$ is in fact *equivalent* to a generalized version of Bounded Martin's Maximum^{++}. The main theorem of the present paper, Theorem 4.2, will be an expansion of [**AspSch**, Theorem 2.7].

There is a discussion in [**Woo99**] of the relationship of $(*)$ with forcing axioms, but to this date it still remains a mystery.

Martin's Maximum, MM (cf.[**FoMaSh88**]), expresses the idea that V is maximal in the sense that if certain Σ_1 truths may be forced to hold in stationary set preserving forcing extensions of V, then these truths already hold in V. Cf. e.g. [**ClaSch**, Theorem 1.3] for a precise formulation. Many consequences of $(*)$ which are Π_2 over H_{ω_2} have been verified to follow also from MM, cf. [**Woo99**, Theorems 3.17, 5.9, and 5.14], [**ClaSch09**], and [**DoeSch09**].

Recall that Martin's Maximum^{++}, MM^{++} for short, is the statement that for every stationary set preserving poset $\mathbb{P}$, for every family $\{D_i \colon i < \omega_1\}$ of dense subsets of $\mathbb{P}$, and for every collection $\{\tau_i \colon i < \omega_1\}$ of names for stationary subsets of ω_1 there is a filter G such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$ and $\tau_i^G = \{\xi < \omega_1 \colon \exists p \in G\, p \Vdash \check{\xi} \in \tau_i\}$ is stationary in ω_1 for every $i < \omega_1$.

It is fair to say that ZFC plus $(*)$ and ZFC plus MM^{++} are the two most prominent axiomatizations of set theory which both negatively decide the continuum problem. However, the following questions are still wide open, cf. [**Woo99**, pp. 769ff. and p.924 Question (18) a)] and [**Lar08**, Question 7.2].

- (Q1) Assuming ZFC plus the existence of large cardinals, must there be a (semi–proper) forcing $\mathbb{P}$ such that if G is $\mathbb{P}$–generic over V, then $V[G] \models (*)$?
- (Q2) Is Martin's Maximum^{++} consistent with $(*)$?
- (Q3) Assume Martin's Maximum^{++}. Must $(*)$ hold true?

The reader may consult [**Woo99**, Theorem 10.14 and 10.70], [**Lar00**], [**Lar08**], and [**SchWoo**∞] to find out what is known concerning these questions.

Inspired by our work on Jensen's $\mathcal{L}$–forcing which led to the papers [**ClaSch09**], [**DoeSch09**], and [**AspSch**, Definition 2.6], the present paper proposes a new axiom which I shall call *Martin's Maximum*$^{*,++}$, $\mathsf{MM}^{*,++}$, and which amalgamates $(*)$ and MM^{++}. Cf. Definition 2.10 below.[2] $\mathsf{MM}^{*,++}$ may be thought of as resulting from MM^{++} in the formulation of [**ClaSch**, Theorem 1.3] by replacing "may be forced to hold in stationary set preserving forcing extensions of V" by "is honestly consistent" (cf. Definition 2.8 below), where honest consistency in turn states a form of Ω–consistency which also guarantees that the structure NS_{ω_1} be respected.

We would like to mention that in recent work, J. Steel takes the alternate route by propagating a determinacy hypothesis, AD_2, cf. [**Lar**∞], from which $\mathsf{MM}^{++}(c^+)$ may (probably) be deduced to hold in $\mathbb{P}_{\max}$ extensions of a determinacy model V, compare [**Woo99**, Theorem 9.44]. The ultimate hope might be to design a global determinacy hypothesis which gives MM^{++} in $\mathbb{P}_{\max}$ extensions of a determinacy model V. We do not know how AD_2 relates to $\mathsf{MM}^{*,++}$.

We construe this paper as proposing a framework for discussing the above questions (Q1) – (Q3).

[2] I first presented $\mathsf{MM}^{*,++}$ in my talk "An axiom" at the Fields Institute, Toronto, on November 12, 2012.

Section 2 is elementary and introduces $\mathsf{MM}^{*,++}$. In Section 3, we shall use inner model theory to obtain $\mathbb{P}_{\max}$ conditions which are A–iterable for $A \subset \mathbb{R}$. This will be used in Section 4 to formulate and prove an equivalence of a strong form of ($*$) with a bounded version of Martin's Maximum$^{*,++}$. In the Appendix, Section 5, we will include a proof that the Unique Branch Hypothesis gives universally Baire iterations strategies for collapses of countable substructures of V.

The new results of this paper are Theorems 3.14 and 4.2. No new techniques had to be developed to prove these results that were not already made available on the market by D.A. Martin, J.R. Steel, W.H. Woodin, and others, and to make the paper more self–contained we allowed ourselves to include the presentation of some tools which are relevant to our questions and which to a large extent play also a crucial role in the *core model induction*, a method first explored by W.H. Woodin, cf. [**SchSt**∞].

2. From MM^{++} and ($*$) to $\mathsf{MM}^{*,++}$

We shall write $\mathbb{R}$ for ${}^{\omega}\omega$ and refer to it as the set of real numbers. We say that $x \in \mathbb{R}$ *codes a transitive set* iff

$$E_x = \{(n,m) \colon x(\langle n,m\rangle) = 0\}$$

is extensional and well–founded.[3] If x codes a transitive set, then we shall write $\text{decode}(x)$ for $\pi_x(0)$, where

$$\pi_x \colon (\omega; E_x) \cong (M_x; \in)$$

is the transitive collapse of $(\omega; E_x)$. That way, every $z \in \text{HC}$ is coded by a real in the sense that there is some $x \in \mathbb{R}$ coding a transitive set such that $z = \text{decode}(x)$. If x, $x' \in \mathbb{R}$, then we shall write

$$x \cong x'$$

to express that fact that both x and x' code transitive sets and $\text{decode}(x) = \text{decode}(x')$.

Let us write $\mathbb{C}$ for the set of all reals coding a transitive set. Then $\mathbb{C}$ is a Π^1_1 set, and $\cong$ is a Σ^1_1 equivalence relation on $\mathbb{C}$.

If $f \colon \mathbb{R} \to \mathbb{R}$ is a function, then we say that f is *universally Baire* iff the graph of f is universally Baire as a subset of $\mathbb{R} \times \mathbb{R}$, cf. [**FeMaWo92**], i.e., if there are trees T and U, both on $\omega \times \omega \times \text{OR}$, such that

(a) $f = p[T]$, and
(b) for all posets $\mathbb{P}$, $V^{\mathbb{P}} \models p[U] = \mathbb{R}^2 \setminus p[T]$.

By absoluteness, for all posets $\mathbb{P}$ and for all $x \in \mathbb{R} \cap V^{\mathbb{P}}$, there is at most one $y \in \mathbb{R} \cap V^{\mathbb{P}}$ with $(x,y) \in p[T]$, i.e., $p[T] \cap V^{\mathbb{P}}$ is a (possibly partial) function in $V^{\mathbb{P}}$.

Definition 2.1. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a universally Baire function. We then say that f is *total and code invariant in all set generic extensions* iff the following holds true. Let the trees T and U witness that f is universally Baire, with $f = p[T]$. Then for all posets $\mathbb{P}$,

(a) $V^{\mathbb{P}} \models \forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (x,y) \in p[T]$, and
(b) $V^{\mathbb{P}} \models \forall \{x, x', y, y'\} \subset \mathbb{R}\, ((x,y) \in p[T] \wedge (x',y') \in p[T] \wedge x \cong x' \longrightarrow y \cong y')$.

[3] Here, n, $m \mapsto \langle n,m\rangle$ denotes the Gödel pairing function from $\omega \times \omega$ onto ω.

If f is universally Baire as being witnessed by T, U as well as by T', U', with $f = p[T] = p[T']$, then for every poset $\mathbb{P}$, $V^{\mathbb{P}} \models p[T] = p[T']$, cf. [**FeMaWo92**]. Hence the truth value of (a) and (b) in Definition 2.1 is not sensitive to the choice of T, U, so that being total and code invariant in all set generic extensions is really a property of the function f.

For the record, let us note the following easy criterion.

Lemma 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that there is a tree T on $\omega \times \omega \times$ OR such that*

$$f = p[T], \text{ and}$$
$$\text{for all posets } \mathbb{P},\ V^{\mathbb{P}} \models \forall x \in \mathbb{R}\, \exists y \in \mathbb{R}\, (x, y) \in p[T].$$

f is then universally Baire.

Proof. Let the tree U on $\omega \times \omega \times (\omega \times \mathrm{OR})$ search for $(x, z, (y, \vec{\alpha}))$ such that $(x, y, \vec{\alpha}) \in T$ and $z \neq y$. T, U then witness that f is universally Baire. □

Let f be a universally Baire function which is total and code invariant in all set generic extensions. Let $\mathbb{P}$ be a poset, and let g be $\mathbb{P}$–generic over V. We may then define inside $V[g]$ a natural (total) map

$$f^{\mathbb{P},g} : V[g] \to V[g] \tag{2.1}$$

as follows. Let the trees T and U witness that f is universally Baire, with $f = p[T]$. Let $X \in V[g]$, let $\theta \geq \mathrm{Card}(\mathrm{TC}(\{X\}))$,[4] let H be $\mathrm{Col}(\omega, \theta)$–generic over $V[g]$, and let $x \in \mathbb{R} \cap V[g][H]$ code a transitive set such that $X = \mathrm{decode}(x)$. By (a) of Definition 2.1 applied to $V[g][H]$, there is then some $y \in \mathbb{R} \cap V[g][H]$ with $(x, y) \in p[T]$, and by (b) of Definition 2.1 applied with $x' = x$ and $y' = y$, y codes a transitive set. Set

$$f^{\mathbb{P},g}(X) = \mathrm{decode}(y).$$

If H' is $\mathrm{Col}(\omega, \theta)$–generic over $V[g][H]$, if $x' \in \mathbb{R} \cap V[g][H']$ codes a transitive set such that $X = \mathrm{decode}(x)$ and if $y' \in \mathbb{R} \cap V[g][H']$ is such that $(x', y') \in p[T]$, then by (b) of Definition 2.1 applied to $V[g][H][H']$, $y \cong y'$, hence $\mathrm{decode}(y') = \mathrm{decode}(y)$. Therefore $f^{\mathbb{P},g}(X)$ is sensitive neither to the choice of H nor of x, so that $f^{\mathbb{P},g}(X) \in V[g]$ and the function $f^{\mathbb{P},g}$ is in fact well–defined inside $V[g]$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a total function such that the graph of f is an analytic subset of $\mathbb{R} \times \mathbb{R}$ and in V,

$$\forall \{x, x', y, y'\} \subset \mathbb{R}\, (y = f(x) \wedge y' = f(x') \wedge x \cong x' \longrightarrow y \cong y').$$

Then f is universally Baire, and if T, U witness this with $f = p[T]$, then both (a) and (b) of Definition 2.1 hold true for any poset $\mathbb{P}$, as those statements may be rephrased in a $\mathbf{\Pi}^1_2$ fashion.

More complex examples may be given in the presence of large cardinals, cf. Theorem 2.5, Corollary 2.6, and Section 3.

Definition 2.3. Let $F : \mathrm{HC} \to \mathrm{HC}$ be a function. We say that F is *universally Baire in the codes* iff there is a universally Baire function $f : \mathbb{R} \to \mathbb{R}$ such that if $z \in \mathrm{HC}$ and $x \in \mathbb{R}$ codes a transitive set with $z = \mathrm{decode}(x)$, then $f(x)$ codes a transitive set with $F(z) = \mathrm{decode}(f(x))$.

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[4] $\mathrm{TC}(\{X\})$ is the transitive closure of $\{X\}$.

We also say that F is *strongly universally Baire in the codes* iff F is universally Baire in the codes as being witnessed by a universally Baire function $f\colon \mathbb{R} \to \mathbb{R}$ which is total and code invariant in all generic extensions.

Let $F\colon \mathrm{HC} \to \mathrm{HC}$ be strongly universally Baire in the codes as being witnessed by f. Let $\mathbb{P}$ be a poset, and let g be $\mathbb{P}$–generic over V. We may then define inside $V[g]$ a natural (total) map

$$F^{\mathbb{P},g}\colon V[g] \to V[g] \tag{2.2}$$

by setting $F^{\mathbb{P},g}(X) = f^{\mathbb{P},g}(X)$. We claim that $F^{\mathbb{P},g}$ is in fact well–defined in that the definition given is not sensitive to the choice of f. To see this, let $f\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \to \mathbb{R}$ be universally Baire functions which are total and code invariant in all geeric extensions and which both witness that $F\colon \mathrm{HC} \to \mathrm{HC}$ is universally Baire in the codes. Let T, U witness that f is universally Baire, with $f = p[T]$, and let T', U' witness that h is universally Baire, with $h = p[T']$. Suppose that $\mathbb{P}$ is a poset, g is $\mathbb{P}$–generic over V, and $f^{\mathbb{P},g} \neq h^{\mathbb{P},g}$, say $f^{\mathbb{P},g}(X) \neq h^{\mathbb{P},g}(X)$, where $X \in V[g]$. Let $\theta \geq \mathrm{Card}(\mathrm{TC}(\{X\}))$, let H be $\mathrm{Col}(\omega,\theta)$–generic over $V[g]$, and let $x \in \mathbb{R} \cap V[g][H]$ code a transitive set such that $X = \mathrm{decode}(x)$. There are then reals y, y' such that $(x,y) \in p[T]$, $(x,y') \in p[T']$, and

$$\mathrm{decode}(y) = f^{\mathbb{P},g}(X) \neq h^{\mathbb{P},g}(X) = \mathrm{decode}(y'),$$

so that $y \not\cong y'$. As $\mathbb{C}$ and $\cong$ are both universally Baire, cf. p. 3, we may in V construct a tree S searching for reals $\bar{x}$, $\bar{y}$, $\bar{y}' \in \mathbb{C}$ such that $(\bar{x},\bar{y}) \in p[T]$, $(\bar{x},\bar{y}') \in p[T']$, and $\bar{y} \not\cong \bar{y}'$. As S is ill–founded in $V[g][H]$, S has to be ill–founded in V by absoluteness, and if $\bar{x}$, $\bar{y}$, $\bar{y}' \in \mathbb{C} \cap V$ are given by an infinite branch through S, then $f(\bar{x}) = \bar{y} \not\cong \bar{y}' = h(\bar{x})$, so that if $\bar{X} = \mathrm{decode}(\bar{x})$, $\bar{Y} = \mathrm{decode}(\bar{y})$, and $\bar{Y}' = \mathrm{decode}(\bar{y}')$, then $F(X) = Y \neq Y' = F(X)$ by the choice of f and h. Contradiction!

It is worth pointing out that of course

$$F^{\mathbb{P},g} \supset F. \tag{2.3}$$

If $A \subset \mathbb{R}$ is universally Baire, if $\mathbb{P}$ is a poset, and if g is $\mathbb{P}$–generic over V, then we will follow [**FeMaWo92**] and denote by A_g the set $p[T] \cap V[g]$, where T is such that there is some U such that T, U witnesses that A is universally Baire, with $A = p[T]$. A_g is the "new version" of A in $V[g]$ and is not sensitive to the choice of T, U.

The following is a crude corollary to seminal theorems by D.A. Martin, J.R. Steel, and W.H. Woodin.

Theorem 2.4. *Assume that there is a proper class of Woodin cardinals. Let $A \subset \mathbb{R}$ be universally Baire, let $\varphi(A)$ be any statement which is projective in A, let $\mathbb{P}$ be any poset, and let g be $\mathbb{P}$–generic over V. Then*

$$V \models \varphi(A) \iff V[g] \models \varphi(A_g).$$

Proof. Let us assume throughout this proof that there is a proper class of Woodin cardinals.

The reader may find definitions of the concepts of "homogeneously Souslin" and "weakly homogeneously Souslin" e.g. in [**Sch14**, p. 322f.].[5] By a theorem of W.H.

[5] In what follows, by "homogeneously Souslin" and "weakly homogeneously Souslin" we mean ∞–homogeneously Souslin and ∞–weakly homogeneously Souslin, respectively.

Woodin, if $A \subset \mathbb{R}$ is universally Baire, then A is weakly homogeneously Souslin. For a proof, cf. e.g. [**St**, Theorem 1.2]. By a theorem of D.A. Martin and J.R. Steel, if $A \subset \mathbb{R}^2$ is weakly homogeneously Souslin, then $\mathbb{R} \setminus pA$ is homogeneously Souslin, where $pA = \{x \in \mathbb{R} \colon \exists y \in \mathbb{R}\,(x,y) \in A\}$ is the projection of A. For a proof, cf. [**MaSt89**, Theorem 5.11]. If $A \subset \mathbb{R}$ is weakly homogeneously Souslin, then A is universally Baire, cf. e.g. [**Sch14**, Problem 13.4, p. 323].

These results imply that $A \subset \mathbb{R}$ is universally Baire iff A is (weakly) homogeneously Souslin, and an inspection of the proof of [**Sch14**, Problem 13.4, p. 323], say, then gives the conclusion of the theorem. $\square$

Corollary 2.5. *Assume that there is a proper class of Woodin cardinals. Let $f \colon \mathbb{R} \to \mathbb{R}$ be universally Baire such that in V,*

$$\forall \{x, x', y, y'\} \subset \mathbb{R}\,(y = f(x) \wedge y' = f(x') \wedge x \cong x' \longrightarrow y \cong y').$$

Then f is total and code invariant in all generic extensions.

Proof. If f is as in the statement of this corollary, then by Theorem 2.4 both (a) and (b) of Definition 2.1 hold true for any poset $\mathbb{P}$, as those statements may be rephrased in a $\Pi^1_2(f)$ fashion. $\square$

Corollary 2.6. *Assume that there is a proper class of Woodin cardinals. If $F \colon \mathrm{HC} \to \mathrm{HC}$ is universally Baire in the codes, then F is strongly universally Baire in the codes. Hence if $\mathbb{P}$ is a poset and if g is $\mathbb{P}$–generic over V, then the function $F^{\mathbb{P},g} \colon V[g] \to V[g]$ as in (2.2) is well–defined.*

We will discuss examples of functions $F \colon \mathrm{HC} \to \mathrm{HC}$ which are universally Baire in the codes and which arise from inner model theory in Section 3.

Definition 2.7. Let $F \colon \mathrm{HC} \to \mathrm{HC}$ be strongly universally Baire. Let θ be arbitrary, let g be $\mathrm{Col}(\omega, \theta)$–generic over V, and let $\mathfrak{A} \in V[g]$ be a transitive set. We say that $\mathfrak{A}$ is *closed under F*, or *F–closed*, iff

(a) $(F^{\mathrm{Col}(\omega,\theta),g})\,``\,\mathfrak{A} \subset \mathfrak{A}$ and
(b) $(\mathfrak{A}, F^{\mathrm{Col}(\omega,\theta),g} \restriction \mathfrak{A})$ is amenable, i.e., $F^{\mathrm{Col}(\omega,\theta)} \restriction M \in \mathfrak{A}$ for every $M \in \mathfrak{A}$.

If $F \colon \mathrm{HC} \to \mathrm{HC}$ is $\Sigma^1_1(z)$ in the codes, $z \in \mathbb{R}$, then every transitive model $\mathfrak{A}$ of ZFC^- with $z \in \mathfrak{A}$ is closed under F by downward Π^1_2 absoluteness.[6]

There is an obvious correspondence between the notion of closure in the sense of Definition 2.7 and the concept of A–closure in the sense of [**Woo99**, Definition 10.139], cf. also [**BaCaLa**]. To discuss this correspondence, let us assume that there is a proper class of Woodin cardinals to have the arguments from the proof of Theorem 2.4 at our disposal.

Let $F \colon \mathrm{HC} \to \mathrm{HC}$ be universally Baire in the codes, and let the universally Baire function $f \colon \mathbb{R} \to \mathbb{R}$ be a witness to this fact. Let

$$A = \{(x, m, n) \colon \exists y (f(x) = y \wedge y(m) = n)\}. \tag{2.4}$$

Then A is universally Baire, and if the transitive model $\mathfrak{A} \in V$ is A–closed in the sense of [**Woo99**, Definition 10.139], then by [**Woo99**, Lemma 10.143], if g is generic over $\mathfrak{A}$ for some poset $\mathbb{P} \in \mathfrak{A}$, then $f``\mathfrak{A}[g] \subset \mathfrak{A}[g]$ and in fact $f \restriction \mathbb{R} \cap \mathfrak{A}[g] \in \mathfrak{A}[g]$ and $f \restriction \mathbb{R} \cap \mathfrak{A}[g]$ is definable in $\mathfrak{A}[g]$ from parameters in $\mathfrak{A}$. This is easily seen to imply that $\mathfrak{A}$ is F–closed in the sense of Definition 2.7.

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[6] Here and in what follows, ZFC^- denotes ZFC without the power set axiom.

On the other hand, let $A \subset \mathbb{R}$ be universally Baire, say as being witnessed by T, U, with $A = p[T]$. For a countable poset $\mathbb{P}$, for $p \in \mathbb{P}$, and for a "nice" name $\tau \in V^{\mathbb{P}}$ for a real,

(a) $p \Vdash_V^{\mathbb{P}} \tau \in p[T]$ iff there is $j \colon N \to V_{\theta+1}$, N countable and transitive, $\{\mathbb{P}, p, \tau, T \restriction \theta\} \subset \operatorname{ran}(j)$, and $p \Vdash_N^{\mathbb{P}} \tau \in p[j^{-1}(T)]$, and

(b) $p \not\Vdash_V^{\mathbb{P}} \tau \in p[T]$ iff there is $q \leq_{\mathbb{P}} p$, $j \colon N \to V_{\theta+1}$, N countable and transitive, $\{\mathbb{P}, q, \tau, U \restriction \theta\} \subset \operatorname{ran}(j)$, and $q \Vdash_N^{\mathbb{P}} \tau \in p[j^{-1}(U)]$,

which is easily seen to imply that the function $F \colon \mathrm{HC} \to \mathrm{HC}$ with

$$\mathbb{P}, p, \tau \overset{F}{\mapsto} \begin{cases} 0 & \text{if } \mathbb{P} \text{ is not a poset, } p \notin \mathbb{P}, \text{ or } \tau \text{ is not a "nice" name for a real,} \\ & \text{or otherwise:} \\ 1 & \text{if } p \Vdash_V^{\mathbb{P}} \tau \in p[T], \text{ and} \\ 2 & \text{if } p \not\Vdash_V^{\mathbb{P}} \tau \in p[T] \end{cases}$$

is universally Baire in the codes. But then if $\mathfrak{A}$ is F–closed in the sense of Definition 2.7, then $\mathfrak{A}$ is A–closed in the sense of [**Woo99**, Definition 10.139].

We therefore arrive at an equivalent formulation of the concept of Ω–consistency if in [**Woo99**, Definition 10.144] we replace the requirement that the certifying model be A–closed in the sense of [**Woo99**, Definition 10.139] by the requirement that it be F–closed in the sense of Definition 2.7 for some $F \colon \mathrm{HC} \to \mathrm{HC}$ which is universally Baire in the codes. We shall be interested in a version of Ω–consistency which allows parameters in H_{ω_2} or beyond.

In what follows we shall frequently consider the languages

$$\mathcal{L}_{\dot\in, \dot I_{\mathsf{NS}_{\omega_1}}} \text{ and } \mathcal{L}_{\dot\in, \dot I_{\mathsf{NS}_{\omega_1}}, \dot A}$$

which arise from the usual first order language of set theory with the binary relation symbol $\dot\in$ for membership by adding a unary relation symbol $\dot I_{\mathsf{NS}_{\omega_1}}$ and also, in the latter case, a unary constant symbol $\dot A$. In transitive models $\mathfrak{A}$ of ZFC^- + "ω_1 exists," $\dot\in$ is always to be interpreted by $\in \restriction \mathfrak{A}$, $\dot I_{\mathsf{NS}_{\omega_1}}$ is to be interpreted by what $\mathfrak{A}$ thinks is the collection of all nonstationary subsets of ω_1, and $\dot A$ will be interpreted by a given universally Baire set of reals.

The following defines a strong form of consistency for a statement in $\mathcal{L}_{\dot\in, \dot I_{\mathsf{NS}_{\omega_1}}}$.

Definition 2.8. Let $\Psi(v_0)$ be a formula in the language $\mathcal{L}_{\dot\in, \dot I_{\mathsf{NS}_{\omega_1}}}$, and let $M \in V$. Let $\theta = \aleph_1 + \operatorname{Card}(\mathrm{TC}(\{M\}))$. We say that $\Psi(M)$ is *honestly consistent* iff for every $F \colon \mathrm{HC} \to \mathrm{HC}$ which is strongly universally Baire in the codes, if g is $\operatorname{Col}(\omega, 2^\theta)$–generic over V, then in $V[g]$ there is some transitive model $\mathfrak{A}$ such that

(a) $\mathfrak{A}$ is F–closed,
(b) $\mathfrak{A} \models \mathsf{ZFC}^-$,
(c) $(H_{\theta^+})^V \subset \mathfrak{A}$,
(d) if $S \in V$, $V \models$ "S is a stationary subset of ω_1," then $\mathfrak{A} \models$ "S is a stationary subset of ω_1," and
(e) $\mathfrak{A} \models \Psi(M)$.

Notice that $\mathfrak{A}$ will typically be countable in $V[g]$ and by a straightforward absoluteness argument, if there is an $\mathfrak{A}$ with (a) through (d) in *some* outer model[7] $W \supset V$ of ZFC, then there is some such $\mathfrak{A}$ in $V[g]$.

[7]We assume to have made sense of "F–closure" for outer models which are not generic extensions.

Item (d) in Definition 2.8 may of course also be written as

$$(\dot{I}_{\mathsf{NS}_{\omega_1}})^{\mathfrak{A}} \cap V = (\mathsf{NS}_{\omega_1})^V. \tag{2.5}$$

Whereas every true statement is obviously honestly consistent, not every honestly consistent statement can be true: e.g., let $M = (\omega_2)^V$ and $\Psi(v_0) \equiv$ "v_0 has size $\aleph_1$."

Definition 2.9. Let $\mathcal{M} = (M; \in, \vec{R})$ be a model, where M is transitive and $\vec{R} = (R_i \colon i < \omega_1)$ is a list of $\aleph_1$ relations on M. Let $\varphi(v_0)$ be a Σ_1 formula in the language $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}_{\omega_1}}}$. We then let $\Psi(\mathcal{M}, \varphi)$ be the follwing assertion.

> "There is $\pi \colon \bar{\mathcal{M}} = (\bar{M}; \in, (\bar{R}_i \colon i < \omega_1)) \to \mathcal{M}$ such that π is (fully) elementary (in the language associated with $\mathcal{M}$), and $\varphi(\bar{\mathcal{M}})$ holds true."

Notice that $\Psi(\mathcal{M}, \varphi)$ is again a Σ_1 sentence in the language $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}_{\omega_1}}}$ with parameter $\mathcal{M}$. If $\mathcal{M}$ is as in Definition 2.9, then $\varphi(\mathcal{M})$ trivially implies $\Psi(\mathcal{M}, \varphi)$ in every transitive ZFC^- model $\mathfrak{A}$ which contains $\mathcal{M}$. If $\mathrm{Card}(M) = \aleph_1$ and e.g. for every $x \in M$ there is some $i < \omega_1$ with $R_i(y) \iff y = x$, then $\Psi(\mathcal{M}, \varphi)$ is simply equivalent with $\varphi(\mathcal{M})$ in every transitive ZFC^- model $\mathfrak{A}$ which contains $\mathcal{M}$.

Definition 2.10. By *Martin's Maximum*$^{*,++}$, abbreviated by $\mathsf{MM}^{*,++}$, we mean the statement that whenever $\mathcal{M} = (M; \in, \vec{R})$ is a model, where M is transitive and $\vec{R}$ is a list of $\aleph_1$ relations on M and whenever $\varphi(v_0)$ is a Σ_1 formula in the language $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}_{\omega_1}}}$ such that $\varphi(\mathcal{M})$ is honestly consistent, then $\Psi(\mathcal{M}, \varphi)$ is true in V.

By remarks before Definition 2.10, $\mathsf{MM}^{*,++}$ could also be phrased as the the more cumbersome statement that for $\mathcal{M}$ and $\varphi(v_0)$ as in Definition 2.10, if $\Psi(\mathcal{M}, \varphi)$ is honestly consistent, then $\Psi(\mathcal{M}, \varphi)$ is true in V.

Theorem 2.11. $\mathsf{MM}^{*,++}$ *implies* MM^{++}.

Proof. Trivially, if $F \colon \mathrm{HC} \to \mathrm{HC}$ is strongly universally Baire in the codes, then for every poset $\mathbb{P}$, every $\mathbb{P}$–generic extension of V is F–closed.

Now let $\mathbb{P}$ be a stationary set preserving poset $\mathbb{P}$, let $\vec{D} = \{D_i \colon i < \omega_1\}$ be a family of dense subsets of $\mathbb{P}$, and let $\vec{\tau} = \{\tau_i \colon i < \omega_1\}$ be a collection of names for stationary subsets of ω_1. For $i < \omega$, let $T_i = \{(p, \xi) \in \mathbb{P} \times \omega_1 \colon p \Vdash^{\mathbb{P}} \check{\xi} \in \tau_i\}$. Let $\kappa = \mathrm{Card}(\mathrm{TC}(\{\mathbb{P}\})) + \aleph_1$. Let

$$\mathcal{M} = (H_{\kappa^+}; \in, \mathbb{P}, \vec{D}, (T_i \colon i < \omega_1)).$$

Let g be $\mathbb{P}$–generic over V. Set $\mathfrak{A} = (H_{(2^\kappa)^+})^{V[g]}$. We may pick some g' which is $\mathrm{Col}(\omega, 2^{2^\kappa})$–generic over V such that $g \in V[g']$, so that $\mathfrak{A} \in V[g']$.

Let $F \colon \mathrm{HC} \to \mathrm{HC}$ be strongly universally Baire in the codes. Then $\mathfrak{A}$ is F–closed, $\mathfrak{A} \models \mathsf{ZFC}^-$, and $(H_{(2^\kappa)^+})^V \subset \mathfrak{A}$. As $\mathbb{P}$ is stationary set preserving, (2.5) also holds true. The following assertion, call it φ, may be written as a Σ_1 statament in $\mathcal{L}_{\dot{\in}, \dot{I}_{\mathsf{NS}_{\omega_1}}}$ and in the parameter $\mathcal{M}$.

> There is some filter $h \subset \mathbb{P}$ such that $h \cap D_i \neq \emptyset$ for all $i < \omega$ and such that $\{\xi < \omega_1 \colon \exists p \in h \, (p, \xi) \in T_i\}$ is stationary for every $i < \omega_1$.

This statement is true with $h = g$. This gives that $\varphi(\mathcal{M})$ is true in $\mathfrak{A}$.

We have verified that $\varphi(\mathcal{M})$ is honestly consistent. By $\mathsf{MM}^{*,++}$, $\Psi(\mathcal{M},\varphi)$ is therefore true. But it is straightforward to verify that if

$$\pi\colon \bar{\mathcal{M}} = (\bar{M}; \in, \bar{\mathbb{P}}, (\bar{D}_i\colon i<\omega_1), (\bar{T}_i\colon i<\omega)) \to (H_{\kappa^+}; \in, \vec{D}, (T_i\colon i<\omega_1))$$

witesses $\Psi(\mathcal{M},\varphi)$ and h is as in $\varphi(\mathcal{M})$, then π"h generates a filter G such that $G \cap D_i \neq \emptyset$ for every $i<\omega_1$ and τ_i^G is stationary for every $i<\omega_1$. □

The proof of [**ClaSch**, Theorem 1.3] actually gives more information than Theorem 2.11, and [**ClaSch**, Theorem 1.3] as well as [**AspSch**, Theorem 2.7] was the inspiration for formulating $\mathsf{MM}^{*,++}$.

By [**ClaSch**, Theorem 1.3], which is part of the folklore, MM^{++} is equivalent to the statement that for all $\mathcal{M}$ and φ as in Definition 2.9, if there is some stationary set preserving poset $\mathbb{P}$ and some g which is $\mathbb{P}$–generic over V such that $V[g] \models \Psi(\mathcal{M},\varphi)$, then $\Psi(\mathcal{M},\varphi)$ holds true in V. $\mathsf{MM}^{*,++}$ results from this statement by replacing "may be forced by some stationary set preserving forcing" by the apparently more liberal "is honestly consistent."

This leads to an obvious question which we formulate as a conjecture, as it is an obvious variant of Woodin's Ω–conjecture, cf. e.g. [**Woo99**, Question (23)].

CONJECTURE 2.12. *Assume that there is a proper class of Woodin cardinals. Let $\Psi(v_0)$ be a Σ_1 formula in the language $\mathcal{L}_{\dot\in,\dot I_{\mathsf{NS}_{\omega_1}}}$, and let $M \in V$. If $\Psi(M)$ is honestly consistent, then there is a stationary set preserving poset $\mathbb{P}$ such that if g is $\mathbb{P}$–generic over V, then $\Psi(M)$ holds true in $V[g]$.*

If Conjecture 2.12 holds true, then in the presence of a proper class of Woodin cardinals, MM^{++} is equivalent with $\mathsf{MM}^{*,++}$. It is conceivable that Conjecture 2.12 is provable relative to some natural extra hypothesis, cf. [**SchWoo∞**].

The new axiom $\mathsf{MM}^{*,++}$ is an attempt to amalgamate MM^{++} and $(*)$. The following is a version of [**AspSch**, Theorem 2.7].

THEOREM 2.13. $\mathsf{MM}^{*,++}$ *implies* $(*)$.

[**AspSch**, Theorem 2.7] in fact proves an equivalence of $(*)$ with a bounded version of $\mathsf{MM}^{*,++}$. In Section 4 we shall prove a generalized form of [**AspSch**, Theorem 2.7]. Let us introduce the relevant bounded version of $\mathsf{MM}^{*,++}$.

From now on we shall write Γ^∞ for the set of all sets of reals which are universally Baire.

DEFINITION 2.14. Let $A \in \Gamma^\infty$. Let $\Psi(v_0)$ be a formula in the language $\mathcal{L}_{\dot\in,\dot I_{\mathsf{NS}_{\omega_1}},\dot A}$, and let $M \in V$. Let $\theta = \aleph_1 + \mathrm{Card}(\mathrm{TC}(\{M\}))$. We say that $\Psi(M)$ is *honestly consistent at A* iff for every $F\colon \mathrm{HC} \to \mathrm{HC}$ which is strongly universally Baire in the codes, if g is $\mathrm{Col}(\omega, 2^\theta)$–generic over V, then in $V[g]$ there is some transitive model $\mathfrak{A}$ such that

(a) $\mathfrak{A}$ is F–closed,
(b) $\mathfrak{A} \models \mathsf{ZFC}^-$,
(c) $(H_{\theta^+})^V \subset \mathfrak{A}$,
(d) if $S \in V$, $V \models$ "S is a stationary subset of ω_1," then $\mathfrak{A} \models$ "S is a stationary subset of ω_1," and
(e) $\mathfrak{A} \models \Psi(M)$ with the understanding that in $\mathfrak{A}$, $\dot A$ is interpreted by A_g, i.e.,

$$\dot A^{\mathfrak{A}} = A_g \cap \mathfrak{A}.$$

The following definition results from [**AspSch**, Definition 2.6] by crossing out the hypothesis that NS_{ω_1} be precipitous.

Definition 2.15. Let $A \in \Gamma^\infty$. By *A–Bounded Martin's Maximum*$^{*,++}$, abbreviated by A–$\mathsf{BMM}^{*,++}$, we mean the statement that if $M \in H_{\omega_2}$ and whenever $\Psi(v_0)$ is a Σ_1 formula in the language $\mathcal{L}_{\dot\in, \dot{I}_{\mathsf{NS}_{\omega_1}}, \dot{A}}$ such that $\Psi(M)$ is honestly consistent at A, then $\Psi(M)$ holds true in V with the understanding that in V, $\dot{A}$ is interpreted by A.

If $\Gamma \subset \Gamma^\infty$, then by *$\Gamma$–Bounded Martin's Maximum*$^{*,++}$, abbreviated by Γ–$\mathsf{BMM}^{*,++}$, we mean the statement that *A–Bounded Martin's Maximum*$^{*,++}$ is true for every $A \in \Gamma$.

We shall prove below, cf. Corollary 4.6, that $\mathsf{MM}^{*,++}$ implies Γ^∞–$\mathsf{BMM}^{*,++}$. We don't know an elementary proof of this fact, though.

3. A–iterable mice and A–iterable $\mathbb{P}_{\max}$ conditions

The new result of this section is Theorem 3.14 which produces the existence of A–iterable $\mathbb{P}_{\max}$ conditions in a way that it may be used to prove (a) $\Longrightarrow$ (b) of Theorem 4.2. (Here, $A \subset \mathbb{R}$.) Along the way, we shall discuss how the hypotheses of Theorems 3.14 and 4.2 may be realized, cf. Theorem 3.13.

We start out with the following concept which is due to W.H. Woodin. It encapsulates, in terms of inner model theory, a form of saying that a given $A \subset \mathbb{R}$ is universally Baire, cf. Theorem 3.3.

Definition 3.1. Let $A \subset \mathbb{R}$. Let N be a countable premouse,[8] let $\delta \in N$, and assume that $N \models$ "ZFC^- plus δ is a Woodin cardinal." Let Σ be an iteration strategy for N witnessing that N be $\omega_1 + 1$ iterable.[9] Let $\tau \in N^{\mathrm{Col}(\omega,\delta)}$.

We then say that $(N, \delta, \tau, \Sigma)$ *captures* A provided that the following hold true.

(a) Σ satisfies hull condensation[10] and branch condensation[11] and is positional.[12]

(b) If $\mathcal{T}$ is an iteration tree on N of successor length $\theta + 1 < \omega_1$ which is built according to Σ such that there is no drop on the main branch $[0, \theta]_\mathcal{T}$, if

$$\pi^\mathcal{T}_{0\theta} \colon N \to \mathcal{M}^\mathcal{T}_\theta$$

is the associated iteration map, and if $g \in V$ is $\mathrm{Col}(\omega, i(\delta))$–generic over $\mathcal{M}^\mathcal{T}_\theta$, then

$$\pi^\mathcal{T}_{0\theta}(\tau)^g = A \cap \mathcal{M}^\mathcal{T}_\theta[g].$$

We also say that (N, Σ) *captures* A iff there is $\{\delta, \tau\} \subset N$ such that $(N, \delta, \tau, \Sigma)$ captures A.

[8]Here and in what follows, in particular in Conjecture 3.4, we actually allow N to be a hod premouse, cf. [**Sa15**, Definition 1.34].

[9]The relevant iteration trees here and in what follows are finite stacks of normal trees.

[10]A hull of a tree according to Σ is again according to Σ. Cf. [**Sa15**, Definition 1.31] for a precise statement.

[11]If a non–dropping branch model of a tree according to Σ embeds into a non–dropping limit model on a tree according to Σ in a commuting way, then the branch is according to Σ. Cf. [**Sa15**, Definition 2.14] for a precise statement.

[12]The iteration strategy for a Σ–iterate N^* of N which is induced by Σ does not depend on how to get from N to N^*. Cf. [**Sa15**, Definition 2.35 (4)]; the last "positional" in [**Sa15**, Definition 2.35 (4)] should read "weakly positional," though.

Capturing gives rise to "Suslin capturing," cf. [**SchSt**∞, Section 1.4]. Let us present the relevant constructions.

Suppose that $(N, \delta, \tau, \Sigma)$ captures $A \subset \mathbb{R}$. For every $x \in \mathbb{R}$ there are $\mathcal{T}$, θ, and g as in (b) of Definition 3.1 such that $x \in \mathcal{M}_\theta^\mathcal{T}[g]$; in fact, we may choose $\mathcal{T}$ here in such a way that $\mathcal{T}$ does not have any drops. Therefore,

$$A = \bigcup \{\pi_{0\theta}^\mathcal{T}(\tau)^g : \mathcal{T},\ \theta, \text{ and } g \text{ as in (b) of Definition 3.1}\}. \tag{3.1}$$

Let $I(N, \Sigma)$ be the collection of all $\mathcal{M}_\theta^\mathcal{T}$, where $\mathcal{T}$ and θ are as in (b) of Definition 3.1. Let $\mathcal{M}_\theta^\mathcal{T}$ and $\mathcal{M}_{\theta'}^{\mathcal{T}'}$ both be in $I(N, \Sigma)$. As Σ is positional by (a) of Definition 3.1, every $\mathcal{M}_\theta^\mathcal{T} \in I(N, \Sigma)$ has an iteration strategy $\Sigma_{\mathcal{M}_\theta^\mathcal{T}}$ which is induced by Σ and which only depends on the model $\mathcal{M}_\theta^\mathcal{T}$ and not the particular tree $\mathcal{T}$. Moreover, as Σ also satisfies hull condensation by (a) of Definition 3.1, Σ is commuting,[13] which means that every $\Sigma_{\mathcal{M}_\theta^\mathcal{T}}$ for $\mathcal{M}_\theta^\mathcal{T} \in I(N, \Sigma)$ has the Dodd–Jensen property in the sense of [**Sa15**, Definition 2.35].

We may therefore let

$$N_\infty^{<\omega_1},\ (\pi_{P,\infty} : P \in I(N, \Sigma)) \tag{3.2}$$

be the direct limit of the directed system consisting of $I(N, \Sigma)$, together with the (unique) respective iteration maps between any two points in $I(N, \Sigma)$.

As Σ satisfies branch condensation by (a) of Definition 3.1, the system giving rise to (3.2) induces tree representations for A and its complement as follows.

We let $x \in p[T]$ iff there is some iteration tree $\mathcal{T}$ on N of length $\theta + 1 < \omega_1$ such that

(i) $\mathcal{T}$ has no drops at all,
(ii) there is a system $(\psi_i : i \leq \theta)$ of elementary embeddings such that $\psi_0 = \pi_{N,\infty}$ and for all $i \leq_\mathcal{T} j \leq \theta$, $\psi_i : \mathcal{M}_i^\mathcal{T} \to N_\infty^{<\omega_1}$ and $\psi_j \circ \pi_{ij}^\mathcal{T} = \psi_i$, and
(iii) there is some g which is $\mathrm{Col}(\omega, \pi_{0\theta}^\mathcal{T}(\delta))$–generic over $\mathcal{M}_\theta^\mathcal{T}$ such that $x \in \mathcal{M}_\theta^\mathcal{T}[g]$, and in fact $x \in \pi_{0\theta}^\mathcal{T}(\tau)^g$.

We let $x \in p[U]$ be defined in exacly the same way except for that in clause (c), we replace "$x \in \pi_{0\theta}^\mathcal{T}(\tau)^g$" by "$x \notin \pi_{0\theta}^\mathcal{T}(\tau)^g$." It is then easy to see that

$$A = p[T] \text{ and } \mathbb{R} \setminus A = p[U]. \tag{3.3}$$

Still suppose that (N, Σ) captures A. Let $\kappa \geq \aleph_1$ be a cardinal. It is straightforward to verify that there is *at most* one $\tilde{\Sigma} \supset \Sigma$ such that $\tilde{\Sigma}$ is an iteration strategy for N witnessing that N be $\kappa^+ + 1$ iterable and

(c) $\tilde{\Sigma}$ satisfies hull condensation,

and any such $\tilde{\Sigma}$ will be positional and satisfy branch condensation.

Let us assume that there is such a $\tilde{\Sigma}$. We may then define a direct limit

$$N_\infty^{<\kappa^+},\ (\pi_{P,\infty} : P \in I(N, \tilde{\Sigma}, \kappa^+)) \tag{3.4}$$

in much the same way as we defined (3.2), where $I(N, \tilde{\Sigma}, \kappa^+)$ is the collection of all $\mathcal{M}_\theta^\mathcal{T}$, where $\mathcal{T}$ is an iteration tree on N of successor length $\theta + 1 < \kappa^+$ which is built according to $\tilde{\Sigma}$ such that there is no drop on the main branch $[0, \theta]_\mathcal{T}$. We may then define trees T^κ and U^κ exactly as we defined T and U above, except for

[13] Cf. [**Sa15**, Definition 2.35 (9)] and [**Sa15**, Proposition 2.36].

that "$\psi_i \colon \mathcal{M}_i^{\mathcal{T}} \to N_\infty^{<\omega_1}$" gets replaced by "$\psi_i \colon \mathcal{M}_i^{\mathcal{T}} \to N_\infty^{<\kappa^+}$." Of course, we will again have that

$$A = p[T^\kappa] \text{ and } \mathbb{R} \setminus A = p[U^\kappa]. \tag{3.5}$$

We claim that T^κ and U^κ witness that A is κ–universally Baire. By (3.5), we just need to see that in $V[g]$, $p[T^\kappa] \cup p[U^\kappa] = \mathbb{R}$.

Fix $x \in V[g]$. Let $x = \rho^g$, where $\rho \in V^{\mathrm{Col}(\omega,\kappa)}$. Inside V, let us construct an iteration tree $\mathcal{T}$ on N of length $\leq \kappa^+ + 1$ such that $\mathcal{T}$ has no drops at all and is according to Σ, as follows.[14] Say $\mathcal{T} \restriction i+1$ has been defined, where $i < \kappa$. If there is some $E_\nu^{\mathcal{M}_i^{\mathcal{T}}} \neq \emptyset$ which is total on $\mathcal{M}_i^{\mathcal{T}}$ and such that for some $p \in \mathrm{Col}(\omega,\kappa)$,[15]

$$p \Vdash^{\mathrm{Col}(\omega,\kappa)} \rho \text{ violates the axiom of } \mathbb{B}^{\mathcal{M}_i^{\mathcal{T}}}_{\pi_{0i}^{\mathcal{T}}(\delta)} \text{ associated with } E_\nu^{\mathcal{M}_i^{\mathcal{T}}}, \tag{3.6}$$

then we let $E_\alpha^{\mathcal{T}} = E_\nu^{\mathcal{M}_i^{\mathcal{T}}}$ for the least such ν. If there is no such ν, then we stop the construction and set $\mathrm{lh}(\mathcal{T}) = i+1$. This defines $\mathcal{T}$.

We claim that $\mathrm{lh}(\mathcal{T}) < \kappa^+$. Otherwise there is some $p \in \mathrm{Col}(\omega,\kappa)$ and a stationary set $S \subset [0,\kappa^+)_{\mathcal{T}}$ such that (3.6) holds true for all i for which there is some $j \leq i$, $j \in S$, such that $i+1 = \min([0,\kappa^+)_{\mathcal{T}} \setminus (j+1))$. We may then pick some g' with $p \in g'$ such that g' is $\mathrm{Col}(\omega,\kappa)$–generic over V. Then S is still stationary in $V[g']$, and the usual hull argument yields some $j \in S$ such that setting $i+1 = \min([0,\kappa^+)_{\mathcal{T}} \setminus (j+1))$, $\rho^{g'}$ satisfies the axiom of $\mathbb{B}^{\mathcal{M}_i^{\mathcal{T}}}_{\pi_{0i}^{\mathcal{T}}(\delta)}$ associated with $E_i^{\mathcal{T}}$. This is a contradiction.

Setting $\theta = \mathrm{lh}(\mathcal{T}) < \kappa^+$, we now have that x is $\mathbb{B}^{\mathcal{M}_\theta^{\mathcal{T}}}_{\pi_{0\theta}^{\mathcal{T}}(\delta)}$–generic over $\mathcal{M}_\theta^{\mathcal{T}}$. Setting $\psi_i = \pi_{\mathcal{M}_i^{\mathcal{T}},\infty}$ for $i \leq \theta$, it is now easy to see that $\mathcal{T}$ witnesses that $x \in p[T^\kappa] \cup p[U^\kappa]$, as desired.

Definition 3.2. Let $A \subset \mathbb{R}$. We say that (N,Σ) *strongly capures* A provided that (N,Σ) captures A and for every cardinal $\kappa \geq \aleph_1$ there is some iteration strategy $\tilde{\Sigma} \supset \Sigma$ witnessing that N is $\kappa^+ + 1$ iterable such that (c) above holds true.

We have shown:

Theorem 3.3. *Let $A \subset \mathbb{R}$. If (N,Σ) strongly captures A, then A is universally Baire.*

Theorem 3.3 verifies (C) $\Longrightarrow$ (A) of the following bold conjecture. (A) $\Longrightarrow$ (B) is true by the above–mentioned results of Martin, Steel, and Woodin, cf. the proof of Theorem 2.4.

Conjecture 3.4. *Assume* MM *plus the existence of a proper class of Woodin cardinals. Let $A \subset \mathbb{R}$. The following are equivalent.*

(A) *A is universally Baire.*
(B) *A is determined.*
(C) *There is some (N,Σ) such that (N,Σ) strongly captures A.*

Now assume that N is a premouse and Σ is a iteration strategy for N. If X is any self–wellordered transitive set and if $N \in L_1(X)$,[16] then $\mathcal{M}$ is a Σ–*premouse*

[14] Cf. [**ScTr**∞, Definition 4.13].

[15] $\mathbb{B}^{\mathcal{M}_i^{\mathcal{T}}}_{\pi_{0i}^{\mathcal{T}}(\delta)}$ denotes the extender algebra at $\pi_{0i}^{\mathcal{T}}(\delta)$ in the sense of $\mathcal{M}_i^{\mathcal{T}}$.

[16] I.e., N is simply definable from X.

over X iff $\mathcal{M}$ is a J–model of the form $J_\alpha[\vec{E}, \vec{S}, X]$ where $\vec{E}$ codes a sequence of (partial and total) extenders satisfying the usual axioms for X–premice with the necessary adjustments due to the following aditional feature. $\vec{S}$ codes a partial iteration strategy for N, organized as follows.[17]

> Let $\gamma < \gamma + \delta \leq \alpha$ be such that $J_\gamma[\vec{E}, \vec{S}, X] \models \mathsf{ZFC}^-$ and γ is the largest cardinal of $J_{\gamma+\delta}[\vec{E}, \vec{S}, X]$. Suppose that $\mathcal{T} \in J_\gamma[\vec{E}, \vec{S}, X]$ is $J_\gamma[\vec{E}, \vec{S}, X]$–least such that $\mathcal{T}$ is an iteration tree on N of limit length, $\mathcal{T}$ is according to $\vec{S} \restriction \gamma$, but $(\vec{S} \restriction \gamma)(\mathcal{T})$ is undefined. Suppose also that $\delta = \mathrm{lh}(\mathcal{T})$, and δ does not have measurable cofinality in $J_{\gamma+\delta}[\vec{E}, \vec{S}, X]$. Then $\Sigma(\mathcal{T})$ is defined, and $\vec{S}(\gamma + \delta)$ is an amenable code for $(\mathcal{T}, \Sigma(\mathcal{T}))$.[18]

It is easy to see that if $\kappa < \lambda < \alpha$ and both κ and λ are regular cardinals of $\mathcal{M} = J_\alpha[\vec{E}, \vec{S}]$, then $\Sigma \restriction J_\kappa[\vec{E}, \vec{S}] \in J_{\kappa+\omega}[\vec{E}, \vec{S}]$.

If $\mathcal{M}$ is a Σ–premouse over X and if Γ is an iteration strategy for $\mathcal{M}$, then we say that Γ *moves* Σ *correctly* iff every iterate $\mathcal{M}^*$ of $\mathcal{M}$ which is obtained via Γ is again a Σ–premouse over X. We call $\mathcal{M}$ a Σ*–mouse over* X iff for every sufficiently elementary $\sigma \colon \bar{\mathcal{M}} \to \mathcal{M}$ with $\bar{\mathcal{M}}$ being countable and transitive there is some iteration strategy Γ for $\bar{\mathcal{M}}$ which witnesses that $\mathcal{M}$ is $\omega_1 + 1$ iterable and which moves Σ correctly.

Theorem 3.14 will make use of the following concept.

Definition 3.5. Let N be a countable premouse, and let Σ be an iteration strategy for N. Let X be a self–wellordered transitive set such that $N \in L_1(X)$, and let $n < \omega$. Then we denote by

$$M_n^{\#,\Sigma}(X)$$

the unique Σ–mouse $\mathcal{M}$ over X, if it exists, such that $\mathcal{M}$ is sound above X, $\mathcal{M}$ is not n–small above X, but every proper initial segment of $\mathcal{M}$ is n–small above X.

As $M_n^{\#,\Sigma}(X)$ is sound above X, it also projects to X, and it is its own least initial segment which satisfies "There is a measure above n Woodin cardinals."

Lemma 3.6. *Suppose that there is a proper class of Woodin cardinals. Let N be a countable premouse, and let Σ be an iteration strategy for N witnessing that N is $<$ OR iterable. Assume that Σ satisfies hull condensation.*

Then for every self–wellordered transitive set X with $N \in L_1(X)$ and for every $n < \omega$, $M_n^{\#,\Sigma}(X)$ exists, and there is an iteration strategy Γ witnessing that $M_n^{\#,\Sigma}(X)$ be $<$ OR–iterable which moves Σ correctly.

Proof. (Sketch.) Let Y be any self–wellordered transitive set such that $N \in L_1(Y)$. We perform an $(n+1)$–small $L[E, \Sigma](Y)$ construction, an obvious variant of the $L[E](Y)$ construction, where we also keep feeding in Σ. By hull condensation, all models from the $L[E, \Sigma](Y)$ construction will be Σ–premice over Y. Moreover, if M is the transitive collapse of a countable substructure of an ω–small model (with no definable Woodin cardinal) from the $L[E, \Sigma](Y)$ construction, then the realization strategy will witness that M be countably iterable and it will move Σ correctly. In particular, the $(n+1)$–small $L[E, \Sigma](X)$ construction reaches a model $\mathcal{M}$ which is not n–small, but the $L[E, \Sigma](Y)$ constructions for Y with $X \in L_1(Y)$

[17] Cf. [**ScTr**∞, Section 3] and the discussion in [**ScTr**∞, Appendix B].

[18] Hence if $\mathcal{T}$ on N is according to $\vec{S}$, then $\mathcal{T}$ is according to Σ.

produce the relevant $\mathcal{Q}$–structures to show that $\mathcal{M}$ is actually $<$ OR iterable via an iteration stratagy which moves Σ correctly. $\square$

Let N be a countable premouse, and let Σ be an iteration strategy for N witnessing that N is $<$ OR iterable. Assume that Σ satisfies hull condensation. Suppose also that $\Sigma \restriction \mathrm{HC}$ is strongly universally Baire in the codes and that there is a proper class of Woodin cardinals. It is straightforward to verify that then for every poset $\mathbb{P}$ and for every g which is $\mathbb{P}$–generic over V, $(\Sigma \restriction \mathrm{HC})^{\mathbb{P},g}$ is an iteration strategy for N with hull condensation which witnesses that N is $<$ OR iterable in $V[g]$, and hence $\Sigma = (\Sigma \restriction \mathrm{HC})^{\mathbb{P},g} \restriction V$.[19]

Lemma 3.7. *Suppose that there is a proper class of Woodin cardinals. Let N be a countable premouse, and let Σ be an iteration strategy for N witnessing that N is $<$ OR iterable. Assume that Σ satisfies hull condensation and that $\Sigma \restriction \mathrm{HC}$ is strongly universally Baire in the codes.*

Then the function

$$X \mapsto M_n^{\#,\Sigma}(X), \text{ where } X \in \mathrm{HC} \text{ is self–wellordered and } N \in L_1(X)$$

is strongly universally Baire in the codes.

Proof. (Sketch.) We use Lemma 2.2. Let the trees T^*, U^* witness that $\Sigma \restriction \mathrm{HC}$ is strongly universally Baire in the codes. Let κ be an uncountable cardinal, and let $\theta >> \kappa$ be a cardinal such that there are at least $n+1$ Woodin cardinals between κ and θ. Say that $T^* \restriction 2^\theta$ and $U^* \restriction 2^\theta$ witness that $p[T^*] \cap V$ is θ–universally Baire.

It is straightforward to design a tree T searching for x, y, M, $\bar{\Sigma}$, P, g, σ such that

(i) $x \in \mathbb{R}$ codes some $X \in \mathrm{HC}$ with $N \in L_1(X)$,
(ii) $y \in \mathbb{R}$ codes M, a $\bar{\Sigma}$–premouse over X which is sound above X, is not n–small above X, but every of its proper initial segments is n–small above X,[20]
(iii) $\sigma \colon P \to H_{(2^\theta)^+}$ is a fully elementary embedding such that P is countable and transitive and $\{N, H_{\kappa^+}, \theta, T^* \restriction 2^\theta, U^* \restriction 2^\theta\} \subset \mathrm{ran}(\sigma)$,
(iv) g is $\mathrm{Col}(\omega, \sigma^{-1}(\kappa))$–generic over P,
(v) $X \in \sigma^{-1}(H_{\kappa^+})[g]$,
(vi) $p[\sigma^{-1}(T^* \restriction 2^\theta)]$ is consulted in $P[g]$ to yield $\mathcal{T} \mapsto \bar{\Sigma}(\mathcal{T})$ à la (2.2), and
(vii) M results from a $L[E,\bar{\Sigma}](X)$–construction performed inside $P[g]$ and using extenders with critical point above $\sigma^{-1}(\kappa)$.

Notice that if h is $\mathrm{Col}(\omega, \sigma^{-1}(\theta)$–generic over $P[g]$, then

$$p[\sigma^{-1}(T^* \restriction 2^\theta)] \cap P[g][h] = p[T^*] \cap P[g][h],$$

so that an M as above will actually be a Σ–premouse over X.

We claim that in fact every M as above is a Σ–mouse over X. To see this, let the strategy Γ for countable trees on M be defined as follows. Suppose that $\mathcal{T}$ on M is of countable limit length and according to Γ. Then $\mathcal{T}$ induces a (non–dropping) tree $\mathcal{U}(\mathcal{T})$ on the background universe $P[g]$ as in [**MiSt94**, §12]. We may construe $\mathcal{U}(\mathcal{T})$ as an iteration tree on P. Construed that way, we may let b be a cofinal

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[19] We here use the notation from (2.2).

[20] In the light of what is to follow, we my and shall in fact arrange that y depends on x, i.e., $x \mapsto y$ will be a function.

branch through $\mathcal{U}$ such that the direct limit model $\mathcal{M}_b^{\mathcal{U}(\mathcal{T})}$ may be emdedded back into $H_{2^{\theta+}}$ in a commuting way. Notice that M is $(n+1)$–small, so that $\mathcal{U}(\mathcal{T})$ is "simple" enough so that there is such a branch.[21] We may then let $\Gamma(\mathcal{T}) = b$, construed as a branch through $\mathcal{T}$.

Now let $\mathcal{T}$ have countable successor length $\alpha+1$ and be according to Γ. Let $\mathcal{U}(\mathcal{T})$ be the induced tree on $P[g]$. Then $\mathcal{M}_\alpha^{\mathcal{T}}$ embeds into a model $\mathcal{N}_\xi$ of the $L[E,\tilde{\Sigma}](X)$–construction of $\mathcal{M}_\alpha^{\mathcal{U}(\mathcal{T})}$, where the predicate $\tilde{\Sigma}$ is given by consulting

$$p[\pi_{0\alpha}^{\mathcal{U}(\mathcal{T})}(\sigma^{-1}(T^* \restriction 2^\theta))]$$

inside $\mathcal{M}_\alpha^{\mathcal{U}(\mathcal{T})}$. However, $\{\sigma^{-1}(T^* \restriction 2^\theta), \sigma^{-1}(U^* \restriction 2^\theta)\} \subset P$, and $\mathcal{M}_\alpha^{\mathcal{U}(\mathcal{T})}$, construed as an iterate of P, may be reembedded into $H_{2^{\theta+}}$, say via $k\colon \mathcal{M}_\alpha^{\mathcal{U}(\mathcal{T})} \to H_{2^{\theta+}}$, in a way such that

$$k \circ \pi_{0\alpha}^{\mathcal{U}(\mathcal{T})} = \sigma.$$

But then $p[\pi_{0\alpha}^{\mathcal{U}(\mathcal{T})}(\sigma^{-1}(T^* \restriction 2^\theta))]$ and $p[T^*]$, restricted to the relevant generic extensions of $\mathcal{M}_\alpha^{\mathcal{U}(\mathcal{T})}$, are equal with one another. This implies that $\tilde{\Sigma}$ conforms with Σ, i.e., $\mathcal{M}_\alpha^{\mathcal{T}}$ is a Σ–premouse over X. We have shown that Γ moves Σ correctly. □

Definition 3.8. Let $A \subset \mathbb{A}$. We say that (N,Σ) *u.B.–strongly captures* A[22] provided that (N,Σ) strongly captures A and $\Sigma \restriction \mathrm{HC}$ is universally Baire in the codes.

Let $A \subset \mathbb{R}$. How can we construct a pair (N,Σ) which u.B.–strongly captures A? One key idea, capturing by self–justifying systems, cf. Definition 3.9, is due to W.H. Woodin. The proof of Lemma 3.10 is as in [**St**∞], cf. [**St**∞, Lemma 3.7], cf. also [**SchSt**∞, Section 3.7].

Definition 3.9. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ be a countable collection of sets of reals. We say that $\mathcal{A}$ is a *self justifying system* iff the following holds true. Every $A \in \mathcal{A}$ admits a scale $(\leq_n\colon n<\omega)$ such that each individual $\leq_n$ belongs to $\mathcal{A}$, too, and such that if $A \in \mathcal{A}$, then $\mathbb{R} \setminus A \in \mathcal{A}$.

We refer the reader to [**Wi15**] for a thorough discussion of how self justifying systems may be constructed.

Lemma 3.10. (Woodin) (Term Condensation) *Let $\mathcal{A}$ be a self–justifying system. Let M be a transitive model of* ZFC^-*, and let $\delta \in M$. Let $\mathcal{C} \subset {}^\omega\delta$ be a comeager set of $\mathrm{Col}(\omega,\delta)$–generics over M and suppose that for each $A \in \mathcal{A}$ there is a term $\tau_A \in M^{\mathrm{Col}(\omega,\delta)}$ such that whenever $G \in \mathcal{C}$, then $\tau_A^G = A \cap M[G]$. Let $\pi\colon \bar{M} \to M$ be elementary with $\{\delta\} \cup \{\tau_A\colon A \in \mathcal{A}\} \subset \mathrm{ran}(\pi)$. Let $\pi(\bar{\delta}) = \delta$ and $\pi(\bar{\tau}_A) = \tau_A$ for $A \in \mathcal{A}$.*

Then whenever g is $\mathrm{Col}(\omega,\bar{\delta})$–generic over $\bar{M}$, then $\bar{\tau}_A^g = A \cap \bar{M}[g]$ for all $A \in \mathcal{A}$.

Proof. Fix any $A \in \mathcal{A}$ for a while, and let $(\psi_n : n<\omega)$ be a scale on A such that for every $n<\omega$, if $\leq_n$ is the prewellorder on $\mathbb{R}$ given by ψ_n then $\leq_n \in \mathcal{A}$. Let $\tau_n \in M$ be such that $\tau_n^G = \leq_n \cap M[G]$ for all $G \in \mathcal{C}$. Let $\phi_n \in M^{\mathrm{Col}(\omega,\delta)}$ be such that for every G which is $\mathrm{Col}(\omega,\delta)$–generic over M, ϕ_n^G is the norm on $A \cap M[G]$

[21] Cf. [**MaSt94**] on the existence of such a realizable branch.

[22] Of course, "u.B." here stands for "universally Baire."

given by τ_n^G. Let $U_n \in M^{\mathrm{Col}(\omega,\delta)}$ be a term for the n^{th} level of the tree associated to these norms, i.e., for all G being $\mathrm{Col}(\omega,\delta)$–generic over M,

$$\dot{U}_n^G = \{(x \restriction n, (\phi_0^G(x), \ldots, \phi_{n-1}^G(x))) : x \in A \cap M[G]\}.$$

Now let G_0 be $\mathrm{Col}(\omega,\delta)$–generic over M, and let G_1 be $\mathrm{Col}(\omega,\delta)$–generic over $M[G_0]$. Then for any appropriate s, $\vec{\alpha}$, and $h \in \{0,1\}$ we have that $(s,\vec{\alpha}) \in \dot{U}_n^{G_h}$ iff there is some $p_h \in G_h$ such that

$$p_h \Vdash_M^{\mathrm{Col}(\omega,\delta)} (\check{s}, \check{\vec{\alpha}}) \in \dot{U}_n.$$

As $\mathcal{C}$ is comeager, we may build $G_0^* \in \mathcal{C}$ and $G_1^* \in \mathcal{C}$ such that for some real y, $(p_0 {}^\frown y) \restriction n \in G_0^*$ and $(p_1 {}^\frown y) \restriction n \in G_1^*$ for all $n < \omega$. In particular, we have $M[G_0^*] = M[G_1^*]$, which implies $\tau_n^{G_0^*} = \leq_n \cap M[G_0^*] = \leq_n \cap M[G_1^*] = \tau_n^{G_1^*}$, and so $\dot{U}_n^{G_0^*} = \dot{U}_n^{G_1^*}$. Hence $(s,\vec{\alpha}) \in \dot{U}_n^{G_0}$ iff $(s,\vec{\alpha}) \in \dot{U}_n^{G_0^*}$ iff $(s,\vec{\alpha}) \in \dot{U}_n^{G_1^*}$ iff $(s,\vec{\alpha}) \in \dot{U}_n^{G_1}$.

This means that $\dot{U}_n^G$ is independent from the particular choice of the $\mathrm{Col}(\omega,\delta)$–generic G, and therefore there is $U_n \in M$ such that $U_n = \dot{U}_n^G$ for all G which are $\mathrm{Col}(\omega,\delta)$–generic over M. In fact, $U_n \in \mathrm{ran}(\pi)$ for every $n < \omega$. Let U be the tree whose n^{th} level is U_n. (Possibly $U \notin M$.)

Claim 3.11. Whenever G is $\mathrm{Col}(\omega,\delta)$–generic over M, $A \cap M[G] \subset p[U] \subset A$.

To verify the claim, notice that $A \cap M[G] \subset p[U]$ is obvious from the definition of U. Let $(x,f) \in [U]$. Let G be $\mathrm{Col}(\omega,\delta)$–generic over M. Let $n < \omega$. Then the n^{th} level of U is U_n^G, and so we can find a real $x_n \in A$ with $x_n \restriction n = x \restriction n$ and $\forall i < n (\phi_i^G(x_n) = f(i))$. So for any $i < \omega$, $\phi_i^G(x_n)$ is eventually constant as $n \to \omega$. Hence $\psi_i(x_n)$ is eventually constant as $n \to \omega$. But $(\psi_i : i < \omega)$ is a scale on A, thus $x \in A$. This shows $p[U] \subset A$. We have shown Claim 3.11.

For any $n < \omega$, we now have that

$$\Vdash_M^{\mathrm{Col}(\omega,\delta)} \forall x\,(x \in \tau_A \rightarrow (x \restriction n, (\phi_0(x), \ldots, \phi_{n-1}(x))) \in U_n).$$

The elementarity of π gives that

$$\Vdash^{\mathrm{Col}(\omega,\bar{\delta})} \forall x\,(x \in \bar{\tau}_A \rightarrow (x \restriction n, (\pi^{-1}(\phi_0)(x), \ldots, \pi^{-1}(\phi_{n-1})(x))) \in \bar{U}_n],$$

where $\bar{U}_n = \pi^{-1}(U_n)$. Let $\bar{U}$ be the tree whose n^{th} level is $\bar{U}_n$. Of course, $p[\bar{U}] \subset p[U]$. But now if $x \in \bar{\tau}_A^g$, where g is $\mathrm{Col}(\omega,\bar{\delta})$–generic over $\bar{M}$, then $x \in p[\bar{U}] \subset p[U] \subset A$, by Claim 3.11. So $\bar{\tau}_A^g \subset A$.

However, the same reasoning with $\mathbb{R}\backslash A \in \mathcal{A}$ and $\tau_{\mathbb{R}\backslash A}$ instead of A and τ_A shows that $\bar{\tau}_{\mathbb{R}\backslash A}^g \subset \mathbb{R}\backslash A$, and thus in fact $\bar{\tau}_A^g = A \cap \bar{M}[g]$, as $\bar{\tau}_{\mathbb{R}\backslash A}^g = (\mathbb{R} \cap \bar{M}[g]) \setminus \bar{\tau}_A^g$. $\square$

The following hypothesis was introduced by D.A. Martin and J.R. Steel, cf. [**MaSt94**, p. 47].

Definition 3.12. Let $\mathcal{E}$ be a class of (total) V–extenders. We say that the *Unique Branch Hypothesis*, UBH for short, holds true for $\mathcal{E}$ iff whenever $\mathcal{T}$ is an iteration tree on V of limit length which only uses extenders from $\mathcal{E}$ and its images, $\mathcal{T}$ has at most one cofinal well–founded branch.

Let us fix $A \subset \mathbb{R}$, $A \in \Gamma^\infty$. Suppose that $\mathcal{E}$ is a class of (total) V–extenders such that UBH holds true for $\mathcal{E}$.

Let N be a premouse, and let Σ_0 be an iteration strategy for N witnessing that N is $<$ OR iterable. Assume that Σ_0 satisfies hull condensation.[23]

For some self–wellordered transitive set X, let us perform an $L[E, \Sigma_0](X)$ construction in much the same way as in the proof of Lemma 3.6, with the following changes, though.

(i) We don't impose any smallness restrictions on the premice occuring in the $L[E, \Sigma_0](X)$–construction, but on the other hand
(ii) we only use extenders from $\mathcal{E}$ as background certificates.

By UBH, the construction cannot break down as all the models appearing in that construction will be fully iterable.

Let us pretend $X = \emptyset$, let us write $L[E, \Sigma_0]$ for the resulting model, and let us suppose that $\mathcal{E}$ witnesses that δ is a Woodin cardinal. Let H be $\mathrm{Col}(\omega, 2^\delta)$–generic over V. Suppose that in $V[H]$,

(†) there is a self–justifying system $\mathcal{A} = \{A_n : n < \omega\}$ with $A_H \in \mathcal{A}$, there are $\tau_n \in L[E, \Sigma_0]^{\mathrm{Col}(\omega,\delta)}$, $n < \omega$, and there is some comeager set $\mathcal{C}$ of $\mathrm{Col}(\omega, \delta)$–generics over $L[E, \Sigma_0]$ such that if $G \in \mathcal{C}$, then
$$\tau_n^G = A_n \cap L[E, \Sigma_0][G]$$
for every $n < \omega$.

Write $\lambda = \delta^{++L[E,\Sigma_0]}$. We may assume that $\tau_n \in L_\lambda[E, \Sigma_0]$ for every $n < \omega$. In V, let us pick an elementary embedding
$$\sigma \colon P \to L_\lambda[E, \Sigma_0],$$
where P is countable and transitive and $\{\delta\} \cup \{\tau_n : n < \omega\} \subset \mathrm{ran}(\sigma)$. For $n < \omega$, write $\bar{\tau}_n = \sigma^{-1}(\tau_n)$.

UBH for $\mathcal{E}$ yields a canonical iteration strategy Σ for P. Namely, if $\mathcal{T} \in V$ is on P, of limit length, and according to Σ, then we may first use σ to copy $\mathcal{T}$ onto $L[E, \Sigma]$, getting $\sigma\mathcal{T}$ on $L[E, \Sigma_0]$, and we may then lift $\sigma\mathcal{T}$ to a (non–dropping) tree $\mathcal{U}(\sigma\mathcal{T})$ on V as in [**MiSt94**, §12], cf. the proof of Lemma 3.7. The unique cofinal well–founded branch through $\mathcal{U}(\sigma\mathcal{T})$ will then give rise to $\Gamma(\mathcal{T})$: formally, in fact, $\Gamma(\mathcal{T})$ *is* the unique cofinal well–founded branch through $\mathcal{U}(\sigma\mathcal{T})$.

For future reference, cf. Theorem 3.13, we shall refer to the strategy Σ thus defined as the *$\mathcal{E}$–induced pullback strategy for P*. We will actually have that $\Sigma \restriction \mathrm{HC}$ is universally Baire in the codes, cf. Theorem 5.2. We defer a proof of this result to the appendix, Section 5, cf. Lemma 5.2.

Let us assume that $\mathcal{T} \in V \cap \mathrm{HC}$ on P is according to Σ. For $i < \mathrm{lh}(\mathcal{T})$, we may write
$$\sigma^i \colon \mathcal{M}_i^{\mathcal{T}} \to \mathcal{M}_i^{\sigma\mathcal{T}}$$
for the canonical copying map, and we may write
$$\pi^i \colon \mathcal{M}_i^{\sigma\mathcal{T}} \to (\mathcal{N}_{\xi^i})^{\mathcal{M}_i^{\mathcal{U}(\sigma\mathcal{T})}}, \text{ some } \xi,$$
for the map obtained by lifting $\sigma\mathcal{T}$ to $\mathcal{U}(\sigma\mathcal{T})$; here, $(\mathcal{N}_{\xi^i})^{\mathcal{M}_i^{\mathcal{U}(\sigma\mathcal{T})}}$ is a model from the $L[E, \pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}(\Sigma_0)]$–construction of $\mathcal{M}_i^{\mathcal{U}(\sigma\mathcal{T})}$.[24] If $[0, i)_{\mathcal{T}}$ does not drop, then, writing

[23] Σ_0 is going to play a pasive role in what follows, we shall need that its Wadge rank is large enough so as to allow (†) to hold true.

[24] We have $\pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}(\Sigma_0) = \Sigma_0 \restriction \mathcal{M}_i^{\mathcal{U}(\sigma\mathcal{T})}$, as Σ_0 has hull condensation, but we don't need that here.

$\lambda^* = \pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}(\lambda)$,

$$\pi^i \circ \sigma^i \colon \mathcal{M}_i^{\sigma\mathcal{T}} \to L_{\lambda^*}[E, \pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}(\Sigma_0)]^{\mathcal{M}_i^{\mathcal{U}(\sigma\mathcal{T})}}$$

is elementary with $\pi^i \circ \sigma^i \circ \pi_{0i}^{\mathcal{T}} = \pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})} \circ \sigma = \pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}(\sigma)$, and hence by absoluteness, as P and $\mathcal{T}$ are countable in $\mathcal{M}_i^{\mathcal{U}(\sigma\mathcal{T})}$, inside $\mathcal{M}_i^{\mathcal{U}(\sigma\mathcal{T})}$ there is some elementary map

$$\varphi^i \colon \mathcal{M}_i^{\mathcal{T}} \to L_{\lambda^*}[E, \pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}(\Sigma_0)]^{\mathcal{M}_i^{\sigma\mathcal{T}}}$$

with $\varphi^i \circ \pi_{0i}^{\mathcal{T}} = \pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}(\sigma)$. By elementarity of $\pi_{0i}^{\mathcal{U}(\sigma\mathcal{T})}$, there is hence an elementary map

$$\psi^i \colon \mathcal{M}_i^{\mathcal{T}} \to L_\lambda[E, \Sigma_0]$$

such that $\psi^i \circ \pi_{0i}^{\mathcal{T}} = \sigma$. In particular, if $n < \omega$, then $\psi^i(\pi_{0i}^{\mathcal{T}}(\bar{\tau}_n)) = \tau_n$, so that by the Term Condensation Lemma 3.10, if $g \in V$ is $\mathrm{Col}(\omega, \pi_{0i}^{\mathcal{T}}(\sigma^{-1}(\delta)))$–generic over $\mathcal{M}_i^{\mathcal{T}}$, then

$$(\pi_{0i}^{\mathcal{T}}(\bar{\tau}_0))^g = A_H \cap \mathcal{M}_i^{\mathcal{T}}[g] = A \cap \mathcal{M}_i^{\mathcal{T}}[g].$$

We have shown that (b) of Definition 3.1 holds true for $(P, \sigma^{-1}(\delta), \bar{\tau}_0, \Sigma)$.

This comes close to having $(P, \sigma^{-1}(\delta), \bar{\tau}_0, \Sigma)$ capture A, but UBH for $\mathcal{E}$ does not seem to abstractly yield (a) of Definition 3.1, whereas in practice it will, cf. the discussion after the statement of Theorem 3.13.

In order to verify (†), let us assume the following. Inside $V[H]$,

(††) there is a self–justifying system $\mathcal{A}$ with $A_H \in \mathcal{A}$ such that for every $n < \omega$,

$$C_{\Sigma_1^1(A_n)}(L_\kappa[E, \Sigma_0]) \subset L[E, \Sigma_0],$$

where $\kappa = \delta^{+L[E,\Sigma_0]}$.

Here, $\Sigma_1^1(D)$ is the set of all sets of reals which are Σ_1^1 in D, and for $\Delta \subset \mathcal{P}(\mathbb{R})$ and x, $y \in \mathbb{R}$, $y \in C_\Delta(x)$ iff y is $\Delta(x)$ in a countable ordinal; if M is a countable transitive set and $Y \subset M$, then $Y \in C_\Delta(M)$ iff for comeager many $\mathrm{Col}(\omega, M)$–generics G, $\{n < \omega \colon (\cup G)(n) \in M\} \in C_\Delta(\cup G)$. (Cf. [**St**$\infty$].)

Assuming (††), let us fix $n < \omega$. We aim to produce $\tau = \tau_n$ as in (†) by an argument as in [**SchSt**∞, Section 3.7]. Let us write $D = A_n$. We let $(p, \sigma) \in \tau$ iff

> $p \in \mathrm{Col}(\omega, \delta)$, $\sigma \in L[E, \Sigma_0]^{\mathrm{Col}(\omega,\delta)}$ is a standard term for a real, and for comeager many g being $\mathrm{Col}(\omega, \delta)$–generic over $L[E, \Sigma_0]$, if $p \in g$, then $\sigma^g \in D$.

Trivially, $\tau \subset L[E, \Sigma_0]^{\mathrm{Col}(\omega,\delta)}$, in fact, writing $\kappa = \delta^{+L[E,\Sigma_0]}$, we have that $\tau \subset L_\kappa[E, \Sigma_0]$.

We claim that

$$\tau \in L[E, \Sigma_0]. \tag{3.7}$$

To this end, let $x \in \mathbb{R}$ be $\mathrm{Col}(\omega, L_\kappa[E, \Sigma_0])$–generic over $L[E, \Sigma_0]$, i.e.,

$$(\omega; E_x) \overset{\varphi}{\cong} (L_\kappa[E, \Sigma_0]; \in).$$

Let us write $\tau_x = \{n < \omega \colon \varphi(n) \in \tau\}$ for the "real" coding τ relative to x. If $(p, \sigma) \in \tau$, then the comeager set of g witnessing $(p, \sigma) \in \tau$ may be taken of the form $\bigcap_{n<\omega} \mathcal{O}_n$, where each $\mathcal{O}_n$ is open dense. It is then clear that

$$\{\tau_x\} \in \Sigma_1^1(D)(\{x\}). \tag{3.8}$$

However, $C_{\Sigma^1_1(D)}(L_\kappa[E,\Sigma_0]) \subset L[E,\Sigma_0]$ and (3.8) imply that $\tau_x \in L[E,\Sigma_0][x]$ and hence also $\tau \in L[E,\Sigma_0][x]$. If x and x' are mutually $\mathrm{Col}(\omega, L_\kappa[E,\Sigma_0])$–generic over $L[E,\Sigma_0]$, then $\tau \in L[E,\Sigma_0][x] \cap L[E,\Sigma_0][x']$. But $\tau \subset L[E,\Sigma_0]$, and therefore (3.7).

We now claim that in $V[H]$, there is a comeager set $\mathcal{C}$ of $\mathrm{Col}(\omega,\delta)$–generics over $L[E,\Sigma_0]$ such that if $G \in \mathcal{C}$, then $\tau^G = D \cap L[E,\Sigma_0][G]$. For $p \in Col(\omega,\delta)$ and $\sigma \in L[E,\Sigma_0]^{\mathrm{Col}(\omega,\delta)}$ a standard term for a real, let $C_{p,\sigma} = \{G \colon p \in G \wedge \sigma^G \in D\}$ and $C'_{p,\sigma} = \{G \colon p \in G \wedge \sigma^G \notin D\}$, with the understanding that G is always $\mathrm{Col}(\omega,\delta)$–generic over $L[E,\Sigma_0]$. We have $\tau = \{(p,\sigma) \colon C_{p,\sigma}$ is comeager in $U_p\}$, where $U_p = \{G \colon p \in G\}$.

We claim that for all σ, $\{p \in \mathrm{Col}(\omega,\delta) \colon C_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager in $U_p\}$ is dense in $\mathrm{Col}(\omega,\delta)$. Fix σ. Let $q \in Col(\omega,\kappa)$. Suppose that $C_{q,\sigma}$ is not comeager. As $C_{q,\sigma}$ has the property of Baire, there is an open set $\mathcal{O}$ such that $(\mathcal{O} \setminus C_{q,\sigma}) \cup (C_{q,\sigma} \setminus \mathcal{O})$ is meager. If $\mathcal{O} = \emptyset$, then $C'_{q,\sigma}$ is comeager. Let us assume that $\mathcal{O} \neq \emptyset$. Then there is some p such that $U_p \setminus C_{q,\sigma}$ is meager, where $U_p = \{G \colon p \in G\}$. We may assume that $\mathrm{lh}(p) \geq \mathrm{lh}(q)$. We must have that $p \leq q$, as otherwise $U_p \setminus C_{q,\sigma} = U_p$, which is not meager. But now $C_{p,\sigma}$ is comeager in U_p, as $U_p \setminus C_{p,\sigma} \subset U_q \setminus C_{p,\sigma}$.

If $C_{p,\sigma}$ or $C'_{p,\sigma}$ is comeager, then let $C^*_{p,\sigma}$ denote the comeager one of them. There are only countably many such p's and σ so that

$$\mathcal{C} = \bigcap_{p,\sigma} C^*_{p,\sigma}$$

is a comeager set.

Now let $G \in \mathcal{C}$. Then $\sigma^G \in \tau^G$ implies that there is some $p \in G$ with $(p,\sigma) \in \tau$, so that there is some $p \in G$ with $C_{p,\sigma}$ being comeager in U_p, hence $\sigma^G \in A$. As $C_{p,\sigma}$ is not comeager in U_p iff $C'_{p,\sigma}$ is comeager in U_p, the same reasoning yields that $\sigma^G \notin \tau^G$ implies that $\sigma^G \notin A$. We have verified (†).

Recall that by a theorem of J. Steel, cf. [**Lar04**, Theorem 3.3.19], hypothesis (1) of Theorem 3.13 implies that the pointclass Γ^∞ admits the scale property. We have thus shown the following Theorem, via (††) and (†) and also Theorem 5.2.

Theorem 3.13. *Assume that $\mathcal{E}$ is a class of V–extenders such that*

(1) *$\mathcal{E}$ witnesses that there is a proper class of Woodin cardinals, and*
(2) UBH *holds true for $\mathcal{E}$.*

Let us also assume the following.

(3) *Let $\mathcal{A} \subset \Gamma^\infty$ be countable. There is then some premouse N and some iteration strategy Σ_0 for N witnessing that N is $<\mathrm{OR}$–iterable such that Σ_0 has hull condensation, and if δ is a Woodin cardinal and if H is $\mathrm{Col}(\omega, 2^\delta)$–generic over V, then in $V[H]$,*

$$C_{\Sigma^1_1(D)}(L_\kappa[E,\Sigma_0]) \subset L[E,\Sigma_0]$$

for every $D \in \mathcal{A}$.

(4) *$\mathcal{E}$–induced pullback strategies for collapses of countable substructures of sufficiently small initial segments of $L[E,\Sigma_0]$ satisfy hull condensation and branch condensation and are positional.*

For every $A \in \Gamma^\infty$ there is then some (P,Σ) u.B.–strongly capturing A.

It is conceivable that (3) follows from (1) and (2) (or even just from (1)), and one may attempt to prove this by an induction on the Wadge rank of the set $A \in \Gamma^\infty$ in question. This leads to the core model induction, cf. [**SchSt∞**] and [**Sa15**], and specifically to *Strong Mouse Set Capturing*, cf. [**Sa15**, p. 8]. (4) should also follow from (1) and (2), cf. e.g. [**Sa15**, Theorem 2.42] on "positional."

The new theorem of this section is now a generalized version of [**AspSch**, Lemma 2.12].

Woodin is able to produce iterable "coarse" mice which capture a given set of reals in a determinacy model by using the HOD of a slightly stronger determinacy model, cf. [**KoeWoo10**, Theorem 5.40] and [**St**∞, Lemma 3.12]. Those coarse mice do not seem to be useful for our purposes, though, as we don't seem to be able to make sense of a directed system generated by a coarse mouse N giving rise to a version of $N_\infty^{<\kappa^+}$ as in (3.4), and also the coarse mice don't seem to produce a substitute for the total code invariant universally Baire function derived from $X \mapsto M_2^{\#,\Sigma}(N,X)$ and obtained via Theorem 3.14 which plays a crucial role in the proof of (a) $\Longrightarrow$ (b) of Theorem 4.2, cf. (4.13).

$\mathbb{P}_{\max}$ conditions are defined in [**Woo99**, Definition 4.33]. A transitive model $(M;\in,I)$ is called a $\mathbb{P}_{\max}$ *precondition* iff there is some a such that $(M;\in,I,a)$ is a $\mathbb{P}_{\max}$ condition, cf. [**Lar10**, Definition 2.1]. Cf. [**Woo99**, Definition 4.3] for A–iterability, where $A\subset\mathbb{R}$.

Theorem 3.14. *Let $A\in\Gamma^\infty$, and suppose that (N,δ,τ,Σ) captures A. Let $X\in\mathrm{HC}$, and suppose that*

$$M = M_2^{\#,\Sigma}(N,X)$$

exists. Let δ_0 be the bottom Woodin cardinal of M, let $g_0\in V$ be $(\mathrm{Col}(\omega_1,<\delta_0))^M$–generic over M, and let $g_1\in V$ be $\mathbb{Q}$–generic over $M[g_0]$, where $\mathbb{Q}\in M[g_0]$ is the standard c.c.c. forcing for Martin's Axiom.

Then

$$p = (M[g_0,g_1];\in,(\mathsf{NS}_{\omega_1})^{M[g_0,g_1]}) \tag{3.9}$$

is an A–iterable $\mathbb{P}_{\max}$ precondition.

Proof. As $M[g_0]\models$ "NS_{ω_1} is presaturated," cf. e.g. [**Woo99**, Theorem 2.61], well–known arguments show that p is an iterable $\mathbb{P}_{\max}$ precondition, cf. [**Woo99**, Lemma 3.10 and Remark 3.11]. We thus need to see that

$$A\cap M[g_0,g_1]\in M[g_0,g_1] \tag{3.10}$$

and if

$$i\colon p\to p^* = (M^*;\in,I^*) \tag{3.11}$$

arises from a countable generic iteration of p, then

$$i(A\cap M[g_0,g_1]) = A\cap M^*. \tag{3.12}$$

To this end, let us first show that $A\cap M$ is δ_1–universally Baire inside M, where δ_1 is the top Woodin cardinal of M.

Let

$$T = (T^{\delta_1})^M \text{ and } U = (U^{\delta_1})^M$$

be the trees as defined on p. 11 for $\kappa=\delta_1$ and running the definition inside M. This is possible, as $\Sigma\restriction M$ is amenable to M.

Let us fix some $G\in V$ which is $\mathrm{Col}(\omega,\delta_1)$–generic over M. We claim that

$$p[T]\cap M[G] = A\cap M[G]. \tag{3.13}$$

Let us first assume that $x \in p[T] \cap M[G]$, as being witessed by $\mathcal{T}$, N^*, and h. As Σ has branch condensation in V, item (b) gives that $\mathcal{T}$ is according to Σ. Therefore, $x \in A$, as $(N, \delta, \tau, \Sigma)$ captures A.

Now let $x \in A \cap M[G]$. The argument on pp. 12f. then shows that $x \in p[T]$.

The very same argument shows that

$$p[U] \cap M[G] = ({}^\omega\omega \setminus A) \cap M[G]. \tag{3.14}$$

Now by (3.13), $A \cap M[g_0, g_1] = p[T] \cap M[g_0, g_1] = A \cap M[g_0, g_1]$, so that (3.10) holds true.

We now aim to verify that (3.12) holds true for all i as in (3.11). Suppose otherwise, and let $i \colon p \to p^* = (M^*; \in, I^*)$ be as in (3.11) such that (3.12) is false, i.e., $i(A \cap M[g_0, g_1]) \neq A \cap M^*$. Exploiting the fact that δ_1 is a Woodin cardinal in $M[g_0, g_1]$ as being witnessed by lifts of extenders from the M–sequence, we may iterate $M[g_0, g_1]$ using such lifts to get some countable iterate $M^+[g_0, g_1]$ of $M[g_0, g_1]$ together with some iteration map

$$j \colon M[g_0, g_1] \to M^+[g, g_1]$$

such that for some G^+ which is $\mathrm{Col}(\omega, j(\delta_1))$–generic over $M^+[g_0, g_1]$,

$$i \restriction (M[g_0, g_1] | (\delta_0^{+3})^M) \in M^+[g_0, g_1][G^+].$$

(Cf. [**St10**].) As $M = M_2^{\#,\Sigma}(N, z)$ is a Σ–mouse, the proof showing (3.13) and (3.14) also yields that

$$p[j(T)] \cap M^+[g_0, g_1][G^+] = A \cap M^+[g_0, g_1][G^+] \tag{3.15}$$

and

$$p[j(U)] \cap M^+[g_0, g_1][G^+] = ({}^\omega\omega \setminus A) \cap M^+[g_0, g_1][G^+]. \tag{3.16}$$

This means that $M^+[g_0, g_1][G^+]$ knows that $i \restriction (M[g_0, g_1] | (\delta_0^{+3})^M)$ is a generic iteration such that $i(A \cap M[g_0, g_1]) \neq p[j(T)] \cap M^* | i(\delta_0^{+2})^{M^*}$. This assertion is forced to be true over $M^+[g_0, g_1]$, so that by the elementarity of j we may conclude that there is some G^* which is $\mathrm{Col}(\omega, \delta_1)$–generic over $M[g_0, g_1]$ such that there is some generic iteration

$$i' \colon (M[g_0, g_1] | (\delta_0^{+3})^M) \to M'$$

with $i'(A \cap M[g_0, g_1]) \neq p[T] \cap M'$. However, i' may be lifted to a generic iteration $i'' \supset i$ with

$$i'' \colon M[g_0, g_1] \to M''.$$

By (3.13) and (3.14), T and U witness that $A \cap M[g_0, g_1]$ is δ_1–universally Baire in $M[g_0, g_1]$. Assume that $i'(A \cap M[g_0, g_1]) \setminus p[T] \cap M' \neq \emptyset$, and let $x \in i'(A \cap M[g_0, g_1]) \setminus p[T] \cap M'$. As $i' \restriction (M[g_0, g_1] | (\delta_0^{+3})^M) \in M[g_0, g_1][G^*]$, $x \in M[g_0, g_1][G^*]$, so that $x \notin p[T]$ implies that $x \in p[U] \subset p[i''(U)]$. By elementarity $i'(A \cap M[g_0, g_1]) = i''(A \cap M[g_0, g_1]) = p[i''(T)] \cap M''$, so that $x \in p[i''(T)] \cap p[i''(U)]$. By absoluteness, $M'' \models p[i''(T)] \cap p[i''(U)] \neq \emptyset$, and hence by elementarity $M[g_0, g_1] \models p[T] \cap p[U] \neq \emptyset$. This contradicts (3.13) and (3.14). A symmetric argument shows that $(p[T] \cap M') \setminus i'(A \cap M[g_0, g_1]) \neq \emptyset$ is impossible.

We reached a contradiction! □

4. $(*)_{\Gamma^\infty}$ is equivalent to Γ^∞–$\mathsf{BMM}^{++,*}$

We formulate a natural strengthening of the axiom $(*)$ of [**Woo99**] and prove that it is equivalent with Γ^∞–$\mathsf{BMM}^{*,++}$. For $\Gamma \subset \mathcal{P}(\mathbb{R})$, a filter $G \subset \mathbb{P}_{\max}$ is Γ–generic iff $G \cap D \neq \emptyset$ for every open $D \subset \mathbb{P}_{\max}$ for which there is some $D^* \in \Gamma$ with

$$D = \{p \in \mathbb{P}_{\max} \colon \exists x \in D^* (x \text{ codes a transitive set and } p = \text{decode}(x))\}, \tag{4.1}$$

i.e., such that D may be coded by a set of reals in Γ.

Definition 4.1. Let $\Gamma \subset \mathcal{P}(\mathbb{R})$. By $(*)_\Gamma$ we mean the statement that
(1) every $A \in \Gamma$ is determined, and
(2) there is some Γ–generic filter $G \subset \mathbb{P}_{\max}$ such that $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[G]$.

Therefore, $(*)$ is equal to $(*)_{\mathcal{P}(\mathbb{R})\cap L(\mathbb{R})}$.

Theorem 4.2. *Assume that there is a proper class of Woodin cardinals. Assume further that for every $A \in \Gamma^\infty$ there is some (N, Σ) which u.B.–strongly captures A.*
The following are equivalent.
(a) $(*)_{\Gamma^\infty}$.
(b) Γ^∞–$\mathsf{BMM}^{*,++}$.

Proof. We first aim to verify (b) $\Longrightarrow$ (a). To this end, let us fix some $A_0 \subset \omega_1$ such that $\omega_1^{L[A_0]} = \omega_1$. Let G be the set of all $p = (M_0; \in, J_0, a_0) \in \mathbb{P}_{\max}$ such that there is some generic iteration

$$((\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1), (G_i \colon i < \omega_1))$$

of $\mathcal{M}_0 = p$ such that $\pi_{0,\omega_1}(a_0) = A_0$ and, writing $\mathcal{M}_{\omega_1} = (M_{\omega_1}; \in, J_{\omega_1}, A_0)$, every set in

$$J^+_{\omega_1} = (\mathcal{P}(\omega_1) \cap M_{\omega_1}) \setminus J_{\omega_1}$$

is stationary in V.

We claim that G is a Γ^∞–generic filter and that

$$L(\mathbb{R})[G] = L(\mathbb{R})[A_0] = L(\mathcal{P}(\omega_1)) \tag{4.2}$$

holds true for G. In order to verify this, we shall need to prove the following three Claims 4.3, 4.4, and 4.5.

Claim 4.3. G is a filter.

Claim 4.4. If D is an open dense subset of $\mathbb{P}_{\max}$ for which there is some $D^* \in \Gamma^\infty$ with (4.1), then $D \cap G \neq \emptyset$.

By a standard $\mathbb{P}_{\max}$–argument, if $p \in G$, then there is a *unique* generic iteration

$$((\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1), (G_i \colon i < \omega_1))$$

of $\mathcal{M}_0 = p$ such that $\pi_{0,\omega_1}(a_0) = A_0$. Assuming Claims 4.3 and 4.4 and following [**Woo99**], we shall then write $\mathcal{P}(\omega_1)_G$ for the set of all $X \subset \omega_1$ for which there is some $p \in G$ such that if

$$((\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1), (G_i \colon i < \omega_1))$$

is the generic iteration of $\mathcal{M}_0 = p$ with $\pi_{0\omega_1}(a_0) = A_0$, then $X \in \text{ran}(\pi_{i,\omega_1})$ for some $i < \omega_1$.

Claim 4.5. $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$.

If NS_{ω_1} were assumed to be saturated, then Claim 4.3 would be given by [**Woo99**, Theorem 4.74] and Claim 4.5 would follow from [**Woo99**, Lemma 3.12 and Corollary 3.13]. Assuming the existence of a measurable cardinal κ above a Woodin cardinal δ as well as Bounded Martin's Maximum^{++}, though, one can prove Claims 4.3 and 4.5 by an easy modification of the forcing developed in [**ClaSch09**], as follows. Let g be $\mathrm{Col}(\omega_1, < \delta)$–generic over V. Inside $V[g]$, $(\mathsf{NS}_{\omega_1})^{V[g]}$ is pre–saturated and hence precipitous, cf. [**Woo99**, Theorem 2.61]. From this, [**ClaSch09**] designs a stationary set preserving forcing which (for a given regular cardinal $\theta \geq (2^\kappa)^+$) adds a generic iteration

$$(\mathcal{M}_i, \pi_{i,j} \colon i \leq j \leq \omega_1)$$

of a countable model $\mathcal{M}_0 = (M_0; \in, I_0)$ such that $\mathcal{M}_{\omega_1} = ((H_\theta)^{V[g]}; \in, (\mathsf{NS})^{V[g]})$. This immediately gives Claim 4.5 by Bounded Martin's Maximum^{++}. Also, if p, $q \in G$, then we may assume without loss of generality that p, q, $A_0 \cap \omega_1^{M_0} \in M_0$, so that Bounded Martin's Maximum^{++} also yields Claim 4.3.

However, Claims 4.3 and 4.5 also follow by a simplified variant of the following argument which shows Claim 4.4 from our hypothesis (b) together with a proper class of Woodin cardinals.

Let us fix $D \subset \mathbb{P}_{\max}$, an open dense set in $\mathbb{P}_{\max}$, such that there is some $D^* \in \Gamma^\infty$ with (4.1). Let δ be a Woodin cardinal, and let $\kappa > \delta$ be measurable. Write $\rho = 2^{2^\kappa}$. As D^* is 2^ρ–universally Baire, we may pick trees T and U on $\omega \times 2^{2^\rho}$ such that

$$D^* = p[T] \text{ and } \Vdash_{\mathrm{Col}(\omega,\, 2^\rho)} \; p[\check{U}] = {}^\omega\omega \setminus p[\check{T}].$$

Let g be $\mathrm{Col}(\omega_1, < \delta)$–generic over V, and let h be $\mathrm{Col}(\omega, \rho)$–generic over $V[g]$. By [**Woo99**, Theorem 2.61] and [**Woo99**, Lemma 3.10 and Remark 3.11],

$$p_0 = ((H_{2^\kappa})^V[g]; \in, (\mathsf{NS}_{\omega_1})^{V[g]}, A_0)$$

is then a $\mathbb{P}_{\max}$ condition in $V[g,h]$. The statement

$$\forall p \in \mathbb{P}_{\max} \exists q \in \mathbb{P}_{\max} (q \leq_{\mathbb{P}_{\max}} p \wedge q \in D) \tag{4.3}$$

which expresses that D is dense in $\mathbb{P}_{\max}$ is Π^1_2 in any set of reals coding $\mathbb{P}_{\max} \oplus D$ in a natural way, so that by absoluteness, cf. Theorem 2.4, there is some $q = (N_0; \in, J_0, A'_0) \in V[g,h]$ belonging to the set of $\mathbb{P}_{\max}$–conditions coded by $(D^*)_{g,h}$ and such that $q <_{\mathbb{P}_{\max}} p_0$. Let

$$j_0 \colon p_0 \to (\bar{N}_0; \in, \bar{J}_0, A'_0) \tag{4.4}$$

be a generic iteration of p_0 such that p_0, $j_0 \in N_0$, i.e., (4.4) witnesses that $q < p_0$.

Inside $V[g,h]$, let

$$(S_\xi \colon \xi < (\rho^+)^V) \in V[g,h]$$

be a partition of $(\rho^+)^V = (\omega_1)^{V[g,h]}$ into stationary sets. Working inside $V[g,h]$, we may then build a generic iteration

$$(\mathcal{N}_i, \sigma_{i,j} \colon i \leq j \leq (\rho^+)^V),$$

of $\mathcal{N}_0 = (N_0; \in, J_0, A'_0) = q$ such that, writing $\mathcal{N}_{(\rho^+)^V} = (N; \in, J, A')$,

$$\forall S \in (\mathcal{P}((\rho^+)^V) \cap N) \setminus J \; \exists \xi < (\rho^+)^V \; \exists \beta < (\rho^+)^V \; S_\xi \setminus \beta \subset S.$$

(Cf. e.g. [**ClaSch09**, proof of Lemma 5].) In particular,

$$J = (\mathsf{NS}_{\omega_1})^{V[g,h]} \cap N. \tag{4.5}$$

Writing

$$j = \sigma_{0,(\rho^+)^V}(j_0)\colon p_0 \to \sigma_{0,(\rho^+)^V}((\bar{N}_0; \in, \bar{J}_0, A'_0)), \tag{4.6}$$

we then have that

$$\sigma_{0,(\rho^+)^V}((\bar{N}_0; \in, \bar{J}_0, A'_0)) = (\sigma_{0,(\rho^+)^V}(\bar{N}_0); \in, J \cap \sigma_{0,(\rho^+)^V}(\bar{N}_0), A') \tag{4.7}$$

and

$$J \cap \sigma_{0,(\rho^+)^V}(\bar{N}_0) = (\mathsf{NS}_{\omega_1})^{V[g,h]} \cap \sigma_{0,(\rho^+)^V}(\bar{N}_0). \tag{4.8}$$

We may lift the generic iteration j of (4.6) to a generic iteration

$$\hat{\jmath}\colon (V[g]; \in, (\mathsf{NS}_{\omega_1})^{V[g]}, A_0 \to (M; \in, (\mathsf{NS}_{\omega_1})^{V[g,h]} \cap M, A').$$

Let us write

$$\bar{M} = \bigcup \{\hat{\jmath}((V_\alpha)^V)\colon \alpha \in \mathrm{OR}\}$$

and

$$\hat{\jmath}_0\colon V \to \bar{M}.$$

Notice that by elementarity, if $S \in \mathcal{P}((\omega_1)^{\bar{M}}) \cap \bar{M}$ and $\bar{M} \models$ "S is stationary," then S is also stationary in M and hence S is stationary in $V[g,h]$ by (4.8).

Now there is a canonical Σ_1 statement $\varphi(A_0, \dot{D}^*, \dot{I}_{\mathsf{NS}_{\omega_1}})$ expressing the existence of a $\mathbb{P}_{\max}$–condition in G coded by a real in D^*:

> There is some $x \in \dot{D}^*$ coding a $\mathbb{P}_{\max}$–condition p and there is some generic iteration of p of length $\omega_1 + 1$ with iteration map $j\colon p \to (Q; \in, \tilde{J}, A_0)$ such that $\tilde{J} = \dot{I}_{\mathsf{NS}_{\omega_1}} \cap Q$.

By D^*–Bounded Martin's Maximum$^{*,++}$, in order to finish off the proof of Claim 4.4 it will suffice to verify that $\varphi(A_0, \dot{D}^*, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is honestly consistent.

Let $x \in p[T] \cap V[g,h]$ code q, so that also

$$x \in p[\hat{\jmath}_0(T)]. \tag{4.9}$$

Let us now fix a universally Baire function $F\colon \mathbb{R} \to \mathbb{R}$ in V, let $\theta >> \kappa$, say a fixed point under $\hat{\jmath}$, and let $\overline{T}$ and $\overline{U}$ a pair of trees on $\omega \times 2^\theta$ witnessing the θ–universal Baireness of F, with $F = p[\overline{T}]$. Let k be $\mathrm{Col}(\omega, \theta)$–generic over V such that g, $h \in V[k]$. Write $T^* = \hat{\jmath}(\overline{T})$ and $U^* = \hat{\jmath}(\overline{U})$, so that $p[\overline{T}] = p[T^*]$ and $p[\overline{U}] = p[U^*]$ in $V[k]$.

In $V[k]$, there is a $p[T^*]$–closed model $\mathfrak{A}$ such that $H^{\bar{M}}_{\omega_2} \subset \mathfrak{A}$, every set in $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^{\bar{M}}$ is stationary in $\mathfrak{A}$, and such that

$$\mathfrak{A} \models \mathsf{ZFC}^- \wedge \varphi(\hat{\jmath}_0(\mathsf{A}_0), \dot{\mathsf{D}}^*, \dot{\mathsf{I}}_{\mathsf{NS}_{\omega_1}}), \tag{4.10}$$

with the understanding that $\dot{D}^*$ be interpreted by $p[\hat{\jmath}_0(T)] \cap \mathfrak{A}$. To get such an $\mathfrak{A}$ in $V[k]$, just let $\mathfrak{A}$ be some appropriate rank–initial segment of $V[g,h]$, cf. (4.7), (4.8), and (4.9); also, $\hat{\jmath}(A_0) = j(A_0) = A'_0$. By absoluteness between $\bar{M}[k]$ and $V[k]$, there is then inside $\bar{M}[k]$ some $p[T^*]$–closed model $\mathfrak{A}$ such that $H^{\bar{M}}_{\omega_2} \subset \mathfrak{A}$, every set in $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^{\bar{M}}$ is stationary in $\mathfrak{A}$, and (4.10) holds true, where again $\dot{D}^*$ is interpreted by $p[\hat{\jmath}_0(T)] \cap \mathfrak{A}$. By the homogeneity of $\mathrm{Col}(\omega, \theta)$, the existence of such an $\mathfrak{A}$ is forced by the trivial condition $1_{\mathrm{Col}(\omega,\theta)}$ over $\bar{M}$, so that by the elementarity of $\hat{\jmath}_0$, there is then in $V[k]$ some $p[\overline{T}]$–closed model $\mathfrak{A}$ such that $H^V_{\omega_2} \subset \mathfrak{A}$, every set in $(\mathcal{P}(\omega_1) \setminus \mathsf{NS}_{\omega_1})^V$ is stationary in $\mathfrak{A}$, and

$$\mathfrak{A} \models \mathsf{ZFC}^- \wedge \varphi(\mathsf{A}_0, \dot{\mathsf{D}}^*, \dot{\mathsf{I}}_{\mathsf{NS}_{\omega_1}}), \tag{4.11}$$

with the understanding that $\dot{D}^*$ is interpreted by $p[\overline{T}] \cap \mathfrak{A}$.

As F was arbitrary, we verified that $\varphi(A_0, \dot{D}^*, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is honestly consistent.

We are now going to prove (a) $\Longrightarrow$ (b).

Let us fix $B \in \Gamma^\infty$ and $A \in H_{\omega_2}$. Let $\varphi(A, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ be a Σ_1 formula which is honestly consistent in the sense of Definition 2.8, with the understanding that $\dot{B}$ is to be interpreted by (the version of) B (in the generic extension). We aim to show that $\varphi(A, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ holds true in V, now with the understanding that $\dot{B}$ is to be interpreted by B.

Suppose not. We may assume without loss of generality that $A \subset \omega_1$ and in fact that A is $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$ (cf. [**Woo99**, Theorem 4.60]). Let $\dot{A}$ be the canonical name for A. Now say that

$$p = (M, \in, I, a) \Vdash \neg\varphi(\dot{A}, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}}), \tag{4.12}$$

where $p \in G_A = \{q = (N, \in, I', a') \in \mathbb{P}_{\max} \colon a' = A \cap \omega_1^N \wedge$ there exists some generic iterate $(N^*, \in, I^*, A)$ of q with $I^* = \mathsf{NS}_{\omega_1}\}$. We shall derive a contradiction by finding some $q <_{\mathbb{P}_{\max}} p$ with $q \Vdash \varphi(\dot{A}, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$.

By our hypothesis, we may pick some $(N, \delta, \tau, \Sigma)$ which u.B.–strongly captures B. The function

(4.13) $F \colon X \mapsto M_2^{\#,\Sigma}(X)$, where $X \in \mathrm{HC}$ is self–wellordered and $N \in L_1(X)$

is then well–defined, total, and strongly universally Baire in the codes, cf. Lemmata 3.6 and 3.7.

Let $\theta \geq 2^{\aleph_1}$, and let g be $\mathrm{Col}(\omega, \theta)$–generic over V. Let $\mathfrak{A} \in V[g]$ be an F–closed witness to the fact that $\varphi(A, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$ is honestly consistent. Let $X \in \mathfrak{A}$ be transitive and such that $(\mathcal{P}(\omega_1) \cap \mathfrak{A}) \cup \{(\mathsf{NS}_{\omega_1})^{\mathfrak{A}}\} \in X$. Write $M = M_2^{\#,\Sigma}(X)$, and let δ_0, g_0, g_1, and $\mathbb{Q}$ be as in the statement of Theorem 3.14. By the conclusion of Theorem 3.14, inside $V[g]$ we have that

$$q = (M[g_0, g_1]; \in, \mathsf{NS}^{M[g_0,g_1]}, A)$$

is a B_g–iterable $\mathbb{P}_{\max}$ condition with $q <_{\mathbb{P}_{\max}} p$, and

$$q \models \varphi(A, B_g \cap M[g_0, g_1], (\mathsf{NS}_{\omega_1})^{M[g_0,g_1]}),$$

so that $q \Vdash \varphi(\dot{A}, \dot{B}, \dot{I}_{\mathsf{NS}_{\omega_1}})$.

However, the assertion that there is such a q is absolute between V and $V[g]$, cf. Theorem 2.4. We obtained a contradiction! □

Corollary 4.6. *Assume that there is a proper class of Woodin cardinals. Assume also that for every $A \in \Gamma^\infty$ there is some (N, Σ) which u.B.–strongly captures A.*

Then $\mathsf{MM}^{*,++}$ *implies* Γ^∞–$\mathsf{BMM}^{*,++}$.

5. Appendix: UBH and universally Baire iteration strategies

Let $\mathcal{T}$ be a normal iteration tree on V. We say that $\mathcal{T}$ is $2^{\aleph_0}$–*closed* iff for every $i < \mathrm{lh}(\mathcal{T})$, $\mathcal{M}_i^{\mathcal{T}} \models$ "The support of $E_i^{\mathcal{T}}$ is $2^{\aleph_0}$–closed." Let b be a maximal branch through an iteration tree $\mathcal{T}$. Then $\mathcal{T}$ is said to be *continuously ill–founded off* b iff there is some sequence $(\alpha_i \colon i < \mathrm{lh}(\mathcal{T}) \setminus b)$ of ordinals such that for all $i, j \in \mathrm{lh}(\mathcal{T}) \setminus b$ with $i <_{\mathcal{T}} j$, $\alpha_j < \pi_{ij}^{\mathcal{T}}(\alpha_i)$.

The following is essentially proven in [**MaSt94**].

Lemma 5.1. *Let $\mathcal{T}$ be a normal $2^{\aleph_0}$–closed iteration tree on V such that $\lambda = \mathrm{lh}(\mathcal{T})$ is a countable limit ordinal. Suppose that $\mathcal{T}$ has exactly one cofinal well–founded branch, b. Then $\mathcal{T}$ is continuously ill–founded off b.*

Proof. If $\pi\colon N \to V_\theta$ is an elementary embedding, where N is transitive and $\mathcal{T} \in \mathrm{ran}(\pi)$, then $\bar{\mathcal{T}} = \pi^{-1}(\mathcal{T})$ is an iteration tree on N with the same length and tree order as $\mathcal{T}$. Let us write π_i for $\pi \restriction \mathcal{M}_i^{\bar{\mathcal{T}}}$, where λ.

Let us also assume that ${}^{2^{\aleph_0}}V_\theta \subset V_\theta$ and that $\mathrm{Card}(N) = 2^{\aleph_0}$. We claim that there is $((X_i^n\colon n<\omega)\colon i<\lambda)$ such that for each $i<\lambda$, $\mathcal{M}_i^{\bar{\mathcal{T}}} = \bigcup\{X_i^n\colon n<\omega\}$ and for each $n<\omega$, $\pi_n \restriction X_i^n \in \mathcal{M}_i^{\mathcal{T}}$.

Such $(X_i^n\colon n<\omega)$ is easily constructed by recursion on $i<\lambda$. It is trivial for $i=0$. If $j = \mathcal{T}\text{–pred}(i+1) < \lambda$, then $\pi_i \restriction X_i^n \in \mathcal{M}_i^{\mathcal{T}}$ implies $\pi_i \restriction (X_i^n \cap \mathrm{supp}(E_i^{\mathcal{T}})) \in \mathcal{M}_{i+1}^{\mathcal{T}}$, as $\mathcal{M}_i^{\mathcal{T}} \models$ "The support of $E_i^{\mathcal{T}}$ is $2^{\aleph_0}$–closed." But then if $f \in X_j^m$ and $a \in X_i^n \cap \mathrm{supp}(E_i^{\mathcal{T}})$,

$$\pi(\pi_{ji+1}^{\bar{\mathcal{T}}}(f)(a)) = (\pi_{ji+1}^{\mathcal{T}}(\pi_j \restriction X_j^m)(f))((\pi_i \restriction X_i^n)(a)),$$

so that we may let[25]

$$X_{i+1}^{\langle m,n\rangle} = \{\pi_{ji+1}^{\bar{\mathcal{T}}}(f)(a)\colon f \in X_j^m \wedge a \in X_i^n \cap \mathrm{supp}(E_i^{\mathcal{T}})\}.$$

If $i<\lambda$ is a limit ordinal and $(i_n\colon n<\omega)$ is cofinal in i, then we may simply set

$$X_{i+1}^{\langle m,n\rangle} = \pi_{i_n i}^{\bar{\mathcal{T}}}{}''X_{i_n}^m.$$

Let us now fix a strictly increasing sequence $(\lambda_n\colon n<\omega)$ which is cofinal in λ and $\lambda_n \in b$ for all $n<\omega$. We shall assume in what follows that for all $i \in (\lambda \setminus b)$ and for all n, $m<\omega$, $X_i^n \subset X_i^{n+1}$, $X_i^n \prec \mathcal{M}_i^{\bar{\mathcal{T}}}$, and if $j = \max([0,i]_{\mathcal{T}} \cap \lambda_m)$, then $\pi_{ji}^{\bar{\mathcal{T}}}{}''X_j^n \subset X_i^n$.

Let us finally also assume that ${}^\omega N \subset N$. There cannot be a cofinal branch $c \notin b$ through $\bar{\mathcal{T}}$ together with some elementary $\sigma^*\colon \mathcal{M}_c^{\bar{\mathcal{T}}} \to V_\theta$. This is because then $c \in N$ and $\mathcal{M}_c^{\bar{\mathcal{T}}}$ is well–founded, which by elementarity implies that $c \neq b$ is a well–founded branch through $\mathcal{T}$

We now let the tree U search for a cofinal branch $c \neq b$ through $\bar{\mathcal{T}}$ together with some elementary $\sigma^*\colon \mathcal{M}_c^{\bar{\mathcal{T}}} \to V_\theta$. Formally, we let $(i,\sigma) \in U$ iff $i \notin b$, $\sigma\colon X_i^{n(i)} \to V_\theta$ is elementary, and $\sigma \circ \pi_{0i}^{\bar{\mathcal{T}}}(x) = \pi(x)$ for all $x \in (\pi_{0i}^{\bar{\mathcal{T}}})^{-1}{}''X_i^{n(i)}$. For (j,σ'), $(i,\sigma) \in U$ we let $(j,\sigma') <_U (i,\sigma)$ iff $i <_{\mathcal{T}} j$, there is some $n<\omega$ with $i < \lambda_n < j$, and $\sigma' \circ \pi_{ij}^{\bar{\mathcal{T}}}(x) = \sigma(x)$ for all $x \in X_i^{n(i)} \cap (\pi_{ij}^{\bar{\mathcal{T}}})^{-1}{}''X_j^{n(j)}$.

For $i \notin b$, we now let

$$\alpha_i = ||(i, \pi_i \restriction X_i^{n(i)})||_{\pi_{0i}^{\mathcal{T}}(U)}.$$

This makes sense, as $\pi_i \restriction X_i^{n(i)} \in \mathcal{M}_i^{\mathcal{T}}$ and $\pi_i \circ \pi_{0i}\bar{\mathcal{T}} = \pi_{0i}^{\mathcal{T}} \circ \pi = \pi_{0i}^{\mathcal{T}}(\pi)$, so that $(i, \pi_i \restriction X_i^{n(i)}) \in \pi_{0i}^{\mathcal{T}}(U)$. Moreover, if $x \in \mathcal{M}_i^{\bar{\mathcal{T}}}$, $i \le_{\mathcal{T}} k = \max([0,j]_{\mathcal{T}} \cap \lambda_{n(i)})$, $\pi_{ik}^{\bar{\mathcal{T}}}(x) \in X_k^m$, and $m \le n(j)$, then $\pi_{ij}^{\bar{\mathcal{T}}}(x) \in X_j^{n(j)}$. This implies that every infinite branch through U would give rise to a cofinal branch c through $\bar{\mathcal{T}}$ together with some elementary $\sigma^*\colon \mathcal{M}_c^{\bar{\mathcal{T}}} \to V_\theta$. Therefore, $\alpha_i < \infty$.

[25] $\langle\cdot,\cdot\rangle$ denotes Gödel pairing

If $i <_{\mathcal{T}} j$ and $n < \omega$ is such that $i < \lambda_n < j$, then $\pi_j \circ \pi^{\bar{\mathcal{T}}}_{ij} \upharpoonright X^{n(i)}_i = \pi^{\mathcal{T}}_{ij} \circ \pi_i \upharpoonright X^{n(i)}_i = \pi^{\mathcal{T}}_{ij}(\pi_i \upharpoonright X^{n(i)}_i)$, so that $(j, \pi_j \upharpoonright X^{n(j)}_j) <_{\pi^{\mathcal{T}}_{ij}(U)} (i, \pi^{\mathcal{T}}_{ij}(\pi_i \upharpoonright X^{n(i)}_i)$, i.e., $\pi^{\mathcal{T}}_{ij}(\alpha_i) > \alpha_j$.

We have shown that $(\alpha_i \colon i \notin b)$ witesses that $\mathcal{T}$ is continuously ill–founded off b. □

Lemma 5.2. *Let κ be such that for all $A \subset \mathbb{R}$, A is universally Baire iff A is κ–universally Baire. Suppose that $\mathcal{E}$ is a class of V–extenders with critical point $> \kappa$ whose support is $2^{\aleph_0}$–closed such that $\mathcal{E}$ witnesses that there is a proper class of Woodin cardinals. Assume that* UBH *holds true for $\mathcal{E}$. Let $\Omega > \theta > \kappa$, and let*

$$\sigma \colon N \to H_\theta$$

be an elementary embedding such that N is countable and $\mathcal{E} \cap H_\theta \in \operatorname{ran}(\sigma)$.

There is then an iteration strategy Σ for countable iteration trees on N which use extenders from $\sigma^{-1}(\mathcal{E} \cap H_\theta)$ and its images such that Σ is universally Baire in the codes.

Proof. The strategy Σ is of course given by copying a given countable tree $\mathcal{T}$ on N of limit length which uses extenders from $\sigma^{-1}(\mathcal{E} \cap H_\theta)$ and its images onto V via σ and pulling back the unique maximal well–founded branch.

It is easy to design a tree T searching for x, y, $(\alpha_i \colon i < \rho)$ such that if $x \in \mathbb{R}$ codes a countable iteration tree $\mathcal{T}$ on N of limit length $\operatorname{lh}(\mathcal{T})$ which uses extenders from $\sigma^{-1}(\mathcal{E} \cap H_\theta)$ and its images, then

(i) $y \in \mathbb{R}$ codes a maximal branch b through $\mathcal{T}$,[26] and
(ii) writing $\lambda = \sup(b) \leq \operatorname{lh}(\mathcal{T})$, if $\sigma\mathcal{T} \upharpoonright \lambda$ is the tree on V obtained by copying $\mathcal{T}$ onto V via σ, then $\rho \geq \lambda$ and $(\alpha_i \colon i \in \lambda \setminus b)$ witnesses that $\sigma\mathcal{T} \upharpoonright \lambda$ is continuously ill–founded off b.

As it is still true in $V^{\mathrm{Col}(\omega,\kappa)}$ that UBH is true for $\mathcal{E}$, cf. [**SchWoo**∞], in the light of Lemma 2.2 and the choice of κ it is easy to see that T witnesses that Σ is universally Baire in the codes. □

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Institut für Mathematische Logik und Grundlagenforschung, Universität Münster, Einsteinstr. 62, Münster, FRG

E-mail address: `rds@wwu.de`