A dilemma in the philosophy of set theory

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Set Theory and Large Cardinals

- Set theory was discovered by Georg Cantor.
- Cantor (Nov 11, 1873, in a letter to R. Dedekind): $\mathbb{R}$ is uncountable. I.e., there are uncountably many real numbers.
- This led Cantor to a systematic study of cardinalities of sets and to the abstract conception of a set.
- "By a 'set' we understand any gathering-together M of determined well-distinguished objects m of our intuition or of our thought, into a whole." (Cantor, 1995)
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- This idea leads to the cumulative hierarchy of sets and to the theory ZFC.

The axiom system ZFC (Zermelo–Fraenkel with choice)

- Any two sets with the same elements are equal.
- For all x and y, $\{x, y\}$, $\bigcup x$, and $\mathcal{P}(x)$ exist.
- There is an infinite set.
- Separation. For all x and for all formulae $\varphi(y)$, $\{y \in x \colon \varphi(y)\}$ exists.
- Replacement. For all x and for all formulae $\varphi(y, z)$ such that for all $y \in x$ there is a unique z with $\varphi(y, z)$, $\{z \colon \exists y \in x \varphi(y, z)\}$ exists.
- Every x with $\emptyset \notin x$ admits a choice function.
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- ZFC formalizes the idea (albeit somewhat indirectly) that the universe of set theory arises from nothing ($\emptyset$) through the operations $x \mapsto \mathcal{P}(x)$ and $x \mapsto \bigcup x$ in a cumulative fashion:
- If we define $V_\alpha = \bigcup\{\mathcal{P}(V_\beta) \colon \beta < \alpha\}$ for ordinals α, then ZFC proves that every x is an element of some V_α. The V_α's are called *ranks*.
- Provably, there is no set of all sets. (By Cantor's Theorem: if v were such a set, then there would be a surjection from v onto $\mathcal{P}(v)$.)
- However, we may introduce a new category of objects, *classes* ("inconsistent multiplicities" in the language of Cantor), and there will be a class of all sets.

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Classes and Truth

- The introduction of classes is tantamount to adding a *truth predicate* to the language of set theory.
- BGC (Bernays–Gödel with choice) results from ZFC by adding a new sort of variables, class variables X, Y, ..., and demanding that the universe of all classes is closed under the logical oprations; instead of talking about formulae in Separation and Replacement we now talk about classes.
- A philosophical credo. In contrast to sets, classes do not exist *de re*, they just exist *de dicto*. Otherwise the collection of all classes would just be another rank of the set theoretical universe, and what appeared to be classes are in fact sets.

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- Tarski (1936)/Mostowski (1950): Whereas the truth predicate for set theory cannot be defined in the language of ZFC, it may be defined in the language of BGC in a Δ^1_1 fashion. All instances of the Tarski schema

$$\varphi \longleftrightarrow \ulcorner\varphi\urcorner \text{ is true}$$

for set theoretical φ may be proven in BGC.
- Sch (2002): The Tarski *sentence* of negation,

$$\forall\ulcorner\varphi\urcorner\ (\ulcorner\neg\,\varphi\urcorner \text{ is true } \longleftrightarrow \neg\,\ulcorner\varphi\urcorner \text{ is true })$$

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- Each language $\mathcal{L}_\alpha$ comes with a new sort of variables for classes of type α. We demand that if $\varphi(x)$ is from $\mathcal{L}_\beta$, some $\beta < \alpha$, then

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Large cardinals

- Replacement may be construed as a "large cardinal axiom." It says that for every formula φ there is a rank V_α which refects φ, i.e.,

$$\varphi(x_1, ...x_k) \longleftrightarrow V_\alpha \models \varphi(x_1, ..., x_k)$$

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- The exploitation of this idea leads to stronger and stronger *reflection principles*: "If V has a certain property, then there is a rank V_α which also has this property."

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- Here is a list of some of the large cardinal concepts which are on the market nowadays.
- Inaccessible < Mahlo < weakly compact < measurable < strong < Woodin < subcompact < supercompact < I_0 < ...
- Large cardinals are ubiquitous in set theory.
- Many questions which are indepenedent from ZFC may be decided by assuming large cardinals. E.g., the determinacy of "definable" sets of reals.
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- Do large cardinals exist?
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Bernays' System of Class Theory

- Bernays has formulated a system of class theory which proves the existence of inaccessible and Mahlo cardinals via reflection principles.
- Bernays' System $B_{\texttt{refl}}$ is BGC together with the following schema of reflection. For every formula φ in the language of BGC with no class quantifiers,

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- Bernays (1961): $\mathrm{B}_{\texttt{refl}}$ proves the existence of inaccessible and in fact Mahlo cardinals.
- Sch (1995): $\mathrm{B}_{\texttt{refl}}$ proves $\Delta_1^{1,\mathrm{BGC}}$ class comprehension.
- Sch (1995): $\Delta_1^{1,\mathrm{BGC}}$ class comprehension implies the existence of non–predicative classes.
- Sch (1995): If κ is weakly compact in L, then $(L_\kappa; \Delta_1^{1,L}(L_\kappa))$ is a model of $\mathrm{B}_{\texttt{refl}}$.

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- Our only apparently good arguments for the existence of large cardinals are based on reflection principles.
- The weakest successful system which expoits this idea, namely Bernays' $B_{\texttt{refl}}$, presupposes the existence of non–predicative classes.
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The Consistency of Large Cardinals

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