Review of "Forcing over models of determinacy" by Paul Larson

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The forcing $\mathbb{P}_{\max}$ is entirely due to W. Hugh Woodin. It arose out of earlier work of J. Steel and R. Van Wesep, cf. [**StVW82**], who had shown that ZFC plus "$\boldsymbol{\delta}^1_2 = \aleph_2$" plus "$\mathsf{NS}_{\omega_1}$ is saturated" is consistent relative to ZF plus AD plus $\mathbb{R}$–AC.

ZF denotes Zermelo–Fraenkel's axiomatization of set theory without any form of the axiom of choice, and ZFC is ZF with AC, the full axiom of choice. $\mathbb{R}$–AC denotes the axiom of choice for families $(A_x : x \in \mathbb{R})$ indexed by reals, i.e., for any such family such that $A_x \neq \emptyset$ for all $x \in \mathbb{R}$ there is some function f with domain $\mathbb{R}$ such that $f(x) \in A_x$ for all $x \in \mathbb{R}$. AD denotes the full axiom of determinacy, i.e. the statement that if $A \subset {}^\omega\omega$ is any set of reals, then one of the two players has a winning strategy in the game $G(A)$, cf. e.g. [**Sch14**, p. 279f.]. NS_{ω_1} is the ideal of all non–stationary subsets of ω_1.

Notice that $\mathbb{R}$–AC implies full uniformization, i.e., the statement that for every relation $R \subset {}^\omega\omega \times {}^\omega\omega$ there is some function $F\colon {}^\omega\omega \to {}^\omega\omega$ such that for all x,

$$\exists y\,(x,y) \in R \Longrightarrow (x, F(x)) \in R.$$

This full form of uniformization is false in $L(\mathbb{R})$, the least inner model of ZF which contains all the reals, and in fact ZF plus AD plus $\mathbb{R}$–AC is much stronger than just ZF plus AD. Whereas [**StVW82**] showed that they could force over a model of ZF plus AD plus $\mathbb{R}$–AC to get a model in which ZFC plus "$\boldsymbol{\delta}^1_2 = \aleph_2$" plus "$\mathsf{NS}_{\omega_1}$ is saturated" was true, it was W. Hugh Woodin's ambition to force just over a model of ZF plus AD and reach the same conclusion which eventually led to the development of $\mathbb{P}_{\max}$.

Cantor's *continuum problem* is the question as to how many real numbers there are. An effective version of the continuum problem is the question as to whether $L(\mathbb{R})$ may have a counterexample to the continuum problem, or more precisely:

Question 1. Letting

$$\Theta_{L(\mathbb{R})} = \sup\{\alpha\colon \exists f \in L(\mathbb{R})\, f\colon \mathbb{R} \to \alpha \text{ is onto}\}, \tag{1}$$

is it possible or even provable that

$$\Theta_{L(\mathbb{R})} > \omega_2 \tag{2}$$

in the presence of large cardinals and/or reasonable axioms of set theory extending ZFC?

1991 *Mathematics Subject Classification.* Primary 03E57, 03E65; Secondary 03E45, 03E55, 03E60.

Key words and phrases. $\mathbb{P}_{\max}$ forcing, Martin's Maximum.

This question found an in–depth discussion in [**FoMa95**]. ZFC by itself only proves that $\Theta_{L(\mathbb{R})} > \omega_1$. As a matter of fact, ZF plus AD proves even a local form of (2). We write

$$\boldsymbol{\delta}^1_2 = \sup\{\alpha \colon \exists f \in \boldsymbol{\Delta}^1_2\, f \colon \mathbb{R} \to \alpha \text{ is onto}\}, \tag{3}$$

where "$f \in \boldsymbol{\Delta}^1_2$" expresses the fact that $\{(x,y) \in \mathbb{R}^2 \colon f(x) \leq f(y)\} \in \boldsymbol{\Delta}^1_2$. Under AD, $\boldsymbol{\delta}^1_2 = \omega_2$, and hence $\Theta_{L(\mathbb{R})} > \omega_2$; for this and much stronger results cf. [**Ke78**].

W. Hugh Woodin introduced in [**Woo83**] an axiom which he called $*$, cf. [**Woo83**, p. 189]. He shows that ZFC plus $*$ is consistent relative to ZF plus AD and that $*$ implies that $\boldsymbol{\delta}^1_2 = \aleph_2$ and NS_{ω_1} is saturated, cf. [**Woo83**, Theorems 3.2 and 3.3]. The axiom $*$ of [**Woo83**] is a precursor to the axiom $(*)$ which W. Hugh Woodin introduced in [**Woo99**, Definition 5.1].

A $\mathbb{P}_{\max}$–condition p is a countable model of the form

$$p = (M; \in, I, a), \tag{4}$$

where M is a countable transitive model of a reasonable fragment of ZFC plus MA_{ω_1}, Martin's axiom for $\aleph_1$ many dense sets, $a \subset \omega_1^M$, and I is a normal ideal on ω_1 from the point of view of p such that p is *generically iterable* via I and its images. A generic iteration of p as in (4) is a commuting system of the form

$$((M_i; \in, I_i, a_i) \colon i \leq \theta), (\pi_{ij} \colon i \leq j \leq \theta), (g_i \colon i < \theta), \tag{5}$$

where $\theta \leq \omega_1^V$, $(M_0; \in, I_0, a_0) = p$, each M_i is transitive, g_i is generic over M_i for forcing with the positive sets of I_i,

$$\pi_{ii+1} \colon (M_i; \in, I_i, a_i) \to (M_{i+1}; \in, I_{i+1}, a_{i+1})$$

is the generic ultrapower embedding given by g_i, and direct limits are taken at limit stages. p is generically iterable iff no attempt to build a generic iteration of p leads to an ill–founded structure.

The set of all $\mathbb{P}_{\max}$ is thus Π^1_2 in the codes. We say that $p \leq_{\mathbb{P}_{\max}} q$, i.e., $p = (M; \in, I, a)$ is stronger than $q = (N; \in, J, b)$ iff M can see a generic iteration of q with last model $(N^*, \in, J^*, b^*)$ such that $J^* = I \cap N^*$ and $b^* = a$. If the reals are closed under the † operator, then every real is an element of a $\mathbb{P}_{\max}$ condition.

The axiom $(*)$ of [**Woo99**] says that

(1) AD holds in $L(\mathbb{R})$, and
(2) there is some G which is $\mathbb{P}_{\max}$–generic over $L(\mathbb{R})$ such that $\mathcal{P}(\omega_1) \subset L(\mathbb{R})[G]$.

As $*$, the $\mathbb{P}_{\max}$ axiom $(*)$ also implies that $\boldsymbol{\delta}^1_2 = \aleph_2$ and NS_{ω_1} is saturated – the latter under the additional hypothesis that $V = L(\mathbb{R})[G]$, where G witnesses $(*)$, cf. [**Woo99**, Theorems 4.50 and 4.53].

Working in ZFC, if NS_{ω_1} is saturated and there is a measurable cardinal Ω, then every countable substructure of $V_{\Omega+2}$ collapses to a $\mathbb{P}_{\max}$ condition which admits a generic iteration as in (5) such that $\theta = \omega_1$ and all the M_i may be reembedded into $V_{\Omega+2}$ in a way that those embeddings commute with the iteration maps. This, together with a boundedness argument, yields W. Hugh Woodin's celebrated theorem according to which ZFC plus "NS_{ω_1} is saturated" plus "there is a measurable cardinal" proves $\boldsymbol{\delta}^1_2 = \omega_2$, cf. [**Woo99**, Theorem 3.17], and hence gives a positive answer to Question 1 above.

Besides, "$\delta^1_2 = \aleph_2$," $(*)$ yields many other interesting statements whose complexity is Π_2 over H_{ω_2}, e.g. $\phi_{\rm AC}$, [**Woo99**, Corollary 5.7], and $\psi_{\rm AC}$, [**Woo99**, Lemma 5.18], cf. also [**Woo99**, Theorems 5.74 and 5.76].

The axiom $(*)$ may in fact be construed as a maximality principle with respect to truths which are Π_2 over H_{ω_2}. E.g., every sentence of that complexity which holds true in V already holds true in every $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, cf. [**Woo99**, Theorem 4.64], and $(*)$ implies that every sentence which is Π_2 over H_{ω_2} and which is Ω–consistent holds true in V, cf. [**Woo99**, Theorem 10.149].

Another way of spelling out the Π_2 maximality feature of $(*)$ is given by [**AspSch14**, Theorem 2.7] which states that $(*)$ is in fact *equivalent* to a generalized version of Bounded Martin's Maximum^{++}. [**Sch**∞, Theorem 4.2] in turn is an expansion of [**AspSch14**, Theorem 2.7].

There is a discussion in [**Woo99**] and also in [**SchWoo**∞] of the relationship of $(*)$ with forcing axioms, but to this date this relationship still remains a mystery.

Martin's Maximum, MM (cf.[**FoMaSh88**]), expresses the idea that V is maximal in the sense that if certain Σ_1 truths may be forced to hold in stationary set preserving forcing extensions of V, then these truths already hold in V. Cf. e.g. [**ClaSch12**, Theorem 1.3] for a precise formulation. Many consequences of $(*)$ which are Π_2 over H_{ω_2} have been verified to follow also from MM, cf. [**Woo99**, Theorems 3.17, 5.9, and 5.14], [**ClaSch09**], and [**DoeSch09**].

Recall that Martin's Maximum^{++}, MM^{++} for short, is the statement that for every stationary set preserving poset $\mathbb{P}$, for every family $\{D_i \colon i < \omega_1\}$ of dense subsets of $\mathbb{P}$, and for every collection $\{\tau_i \colon i < \omega_1\}$ of names for stationary subsets of ω_1 there is a filter G such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$ and $\tau_i^G = \{\xi < \omega_1 \colon \exists p \in G\, p \Vdash \check{\xi} \in \tau_i\}$ is stationary in ω_1 for every $i < \omega_1$.

It is fair to say that ZFC plus $(*)$ and ZFC plus MM^{++} are the two most prominent axiomatizations of set theory which both negatively decide the continuum problem. However, the following questions are still wide open, cf. [**Woo99**, pp. 769ff. and p.924 Question (18) a)] and [**Lar08**, Question 7.2].

Question 2. Assuming ZFC plus the existence of large cardinals, must there be a (semi–proper) forcing $\mathbb{P}$ such that if G is $\mathbb{P}$–generic over V, then $V[G] \models (*)$?

Question 3. Is Martin's Maximum^{++} consistent with $(*)$? Or even: Does Martin's Maximum^{++} imply $(*)$?

The reader should consult [**Woo99**, Theorem 10.14 and 10.70], [**Lar00**], [**Lar08**], and [**SchWoo**∞] to find out what is known concerning these questions.

Inspired by the reviewer's work on Jensen's $\mathcal{L}$–forcing which led to the papers [**ClaSch09**], [**DoeSch09**], and [**AspSch14**, Definition 2.6], he formulated in [**Sch**∞] an axiom which he calls *Martin's Maximum*$^{*,++}$, MM$^{*,++}$, and which amalgamates $(*)$ and MM^{++}, cf. [**Sch**∞, Definition 2.10].

The paper under review, [**Lar10**], gives an excellent introduction to the theory of $\mathbb{P}_{\max}$ forcing by developping its apparatus and producing applications via forcing over $L(\mathbb{R})$ or larger models of AD.

The heart of the pure $\mathbb{P}_{\max}$ theory is the proof according to which any $\mathbb{P}_{\max}$ extension of a reasonable model of AD satisfies the axiom of choice. For this, one needs to verify that for every $A \subset \mathbb{R}$ in the ground model there are densely many $\mathbb{P}_{\max}$ conditions as in (4) which *capture* A in that

(a) $A \cap M \in M$, and
(b) if M_θ is any iterate of M as in (5), then $\pi_{0\theta}(A \cap M) = A \cap M_\theta$.

Such capturing phenomena were also identified by W. Hugh Woodin in the context of the analysis of HOD of various models of AD. With $\mathbb{P}_{\max}$, besides being needed as a tool for verifying AC, capturing also establishes the key property that every $A \subset \omega_1$ which exists in the generic extension is in the range of some $\pi_{0\omega_1}$ for an iteration as in (5), where all the models from the iteration are in the generic filter. Capturing in fact produces principles like ϕ_{AC} and ψ_{AC} which give elegant proofs of AC in the generic extension.

Apparently the only way one can prove capturing for $\mathbb{P}_{\max}$ conditions is with the help of robust Suslin representations for the given set of reals and its complement, i.e., by making use of a degree of universally Baireness. This is also how [**Lar10**] proceeds to prove capturing. Capturing is where the full strength of AD to hold in the ground model is exploited in order to make $\mathbb{P}_{\max}$ work out.

[**Lar10**, Theorem 8.7] gives a summary of what may be achieved by forcing with $\mathbb{P}_{\max}$ over larger models of AD. This issue is also discussed in [**CSLSSSZ∞**] where it is investigated how much of MM^{++} one can show to hold in $\mathbb{P}_{\max}$ extensions of large models of determinacy.

[**Lar10**] also addresses the connections of $\mathbb{P}_{\max}$ with Ω logic.

Finally, [**Lar10**] discusses variants of $\mathbb{P}_{\max}$ which for instance give the consistency of the fact that NS_{ω_1} is ω_1–dense. It is a seminal theorem of W. Hugh Woodin which says that the following theories are equiconsistent.

(1) ZF plus AD.
(2) ZFC plus "there are infinitely many Woodin cardinals."
(3) ZFC plus "NS_{ω_1} is ω_1–dense."

A variant of $\mathbb{P}_{\max}$ proves that (3) is consistent relative to (1). Another key technique of contemporary set theory, the *core model induction*, cf. e.g. [**SchSt∞**], was first explored by W. Hugh Woodin in order to prove that (1) is consistent relative to (3).

Paul Larson's Handbook article [**Lar10**] gives a thorough and nicely readable introduction into one of the most powerful forcing techniques of current–day set theory.

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