Dilemmas and truths in set theory

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Set Theory and Cantor’s Continuum Hypothesis

- Set theory started with the following theorem of Georg Cantor.
- Cantor (Nov 11, 1873, in a letter to R. Dedekind): \( \mathbb{R} \) is uncountable. I.e., there are uncountably many real numbers.
- Cantor’s first proof of this used nested intervals.
- But how many real numbers are there?
- Continuum Hypothesis (CH): For every uncountable \( A \subseteq \mathbb{R} \) there is a bijection \( f : \mathbb{R} \rightarrow A \).
- Cantor’s Program: Show CH by “induction on the complexity” of \( A \subseteq \mathbb{R} \).
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Cantor’s Program: Show CH by “induction on the complexity” of $A \subset \mathbb{R}$. 
Cantor–Bendixson (1883): Every uncountable *closed* $A \subseteq \mathbb{R}$ contains a perfect subset.

Young (1906): Every uncountable $G_\delta$– oder $F_\sigma$–set $A \subseteq \mathbb{R}$ contains a perfect subset.

Aleksandrov/Hausdorff (1916): Every uncountable *Borel* set $A \subseteq \mathbb{R}$ contains a perfect subset.

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Cantor’s Generalized Continuum Hypothesis

- In addition to sets of natural numbers, of reals, of sets of reals, etc., Cantor started considering sets *in general*.
- “By a ‘set’ we understand any gathering-together $M$ of determined well-distinguished objects $m$ of our intuition or of our thought, into a whole.” (Cantor, 1995)
- This idea leads to the cumulative hierarchy of sets.
- For every set $x$ whatsoever, the *power set* $\mathcal{P}(x)$ exists.
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Cantor’s Theorem (1892): Let $x$ be any set. There is no surjection $f : x \to \mathcal{P}(x)$.

This time, Cantor’s proof uses a diagonal argument.

How big is $\mathcal{P}(x)$ in comparison to $x$?

Generalized Continuum Hypothesis (GCH): For every infinite set $x$ and every $A \subset \mathcal{P}(x)$, there is either a surjection $f : x \to A$ or else a bijection $f : \mathcal{P}(x) \to A$.

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We need to talk about axiomatizations of set theory in order to discuss CH and GCH.
The axiom system ZFC (Zermelo–Fraenkel with choice)

- Any two sets with the same elements are equal.
- For all \( x \) and \( y \), \( \{x, y\} \), \( \bigcup x \), and \( \mathcal{P}(x) \) exist.
- There is an infinite set.
- Separation. For all \( x \) and for all formulae \( \varphi(y) \), \( \{y \in x : \varphi(y)\} \) exists.
- Replacement. For all \( x \) and for all formulae \( \varphi(y, z) \) such that for all \( y \in x \) there is a unique \( z \) with \( \varphi(y, z) \), \( \{z : \exists y \in x \varphi(y, z)\} \) exists.
- Every \( x \) with \( \emptyset \notin x \) admits a choice function.
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The usual formulation of ZFC allows the formulae $\varphi$ in Separation and Replacement to contain *parameters*.

It may be shown, though, that these parameters are not needed:

Levy (1971): If in the formulation of Separation and Replacement, the formulae $\varphi$ are required to be lightface (parameter free), then we get a system which is as strong as ZFC.
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ZFC formalizes the idea (albeit somewhat indirectly) that the universe of set theory arises from nothing (\(\emptyset\)) through the operations \(x \mapsto \mathcal{P}(x)\) and \(x \mapsto \bigcup x\) in a cumulative fashion:

- If we define \(V_\alpha = \bigcup \{\mathcal{P}(V_\beta) : \beta < \alpha\}\) for ordinals \(\alpha\), then ZFC proves that every \(x\) is an element of some \(V_\alpha\). The \(V_\alpha\)'s are called ranks.
- Provably, there is no set of all sets. (By Cantor’s Theorem: if \(v\) were such a set, then there would be a surjection from \(v\) onto \(\mathcal{P}(v)\).)
- However, we may introduce a new category of objects, classes ("inconsistent multiplicities" in the language of Cantor), and there will be a class of all sets.
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However, we may introduce a new category of objects, classes (“inconsistent multiplicities” in the language of Cantor), and there will be a class of all sets.
The introduction of classes is tantamount to adding a *truth predicate* to the language of set theory.

**BGC** (Bernays–Gödel with choice) results from ZFC by adding a new sort of variables, class variables $X$, $Y$, ..., and demanding that the universe of all classes is closed under the logical operations; instead of talking about formulae in Separation and Replacement we now talk about classes.

**A philosophical credo.** In contrast to sets, classes do not exist *de re*, they just exist *de dicto*. Otherwise the collection of all classes would just be another rank of the set theoretical universe, and what appeared to be classes are in fact sets.
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Tarski (1936)/Mostowski (1950): Whereas the truth predicate for set theory cannot be defined in the language of ZFC, it may be defined in the language of BGC in a $\Delta^1_1$ fashion. All instances of the Tarski schema

$$\varphi \iff \Box \varphi \text{ is true}$$

for set theoretical $\varphi$ may be proven in BGC.

Sch (2002): The Tarski sentence of negation,

$$\forall \Box \varphi \left( \Box \neg \varphi \text{ is true } \iff \neg \Box \varphi \text{ is true } \right)$$

is not provable in BGC, though (unless BGC is inconsistent). The Tarski schema of negation is provable in BGC plus $\Sigma^1_1$ induction.
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Sch (1993): We may in fact define a Tarskian hierarchy of meta–languages for the language of set theory.

Each language $\mathcal{L}_\alpha$ comes with a new sort of variables for classes of type $\alpha$. We demand that if $\varphi(x)$ is from $\mathcal{L}_\beta$, some $\beta < \alpha$, then

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Replacement may be construed as a “large cardinal axiom.” It says that for every formula \( \varphi \) there is a rank \( V_\alpha \) which reflects \( \varphi \), i.e.,

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The exploitation of this idea leads to stronger and stronger reflection principles: “If \( V \) has a certain property, then there is a rank \( V_\alpha \) which also has this property.”
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Here is a list of some of the large cardinal concepts which are on the market nowadays.

- Inaccessible < Mahlo < weakly compact < measurable < strong < Woodin < subcompact < supercompact < $\lambda_0$ < ...

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- Bernays has formulated a system of class theory which proves the existence of inaccessible and Mahlo cardinals via reflection principles.

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- Our only apparently good arguments for the existence of large cardinals are based on reflection principles.
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On the other hand, there are many statements which imply the consistency of the existence of large cardinals with ZFC, in fact the existence of canonical inner models with such large cardinals.

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- Where should the journey go?
- Non–option: Forget about the question.
- 1st option: Woodin’s “Ultimate L.” (Yields CH.)
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