Let $A = (A; \ldots)$, $B = (B; \ldots)$ be two models in the same language, $L$.

Let us consider the following game, $G(A, B)$.

$$
\begin{array}{c|ccccc}
\hline
\text{I} & a_0 & a_1 & \cdots \\
\hline
\text{II} & b_0 & b_1 & \cdots \\
\end{array}
$$

The game lasts $\omega$ steps. Rules:

- $\{a_0, \ldots\} \subseteq A$, $\{b_0, \ldots\} \subseteq B$, and for all $L$-formulas $\varphi$,

$$
A \models \varphi(a_0, \ldots, a_{k-1}) \iff B \models \varphi(b_0, \ldots, b_{k-1}).
$$

$G(A, B)$ is closed: If $\text{II}$ loses, she loses after finitely many steps.

Hence $G(A, B)$ is determined by Gale–Stewart.
Lemma 1. Let $A$ be c.c.c. Then II wins $G(A, B)$ iff there is an elementary embedding $j: A \rightarrow B$.

Proof: "$\Rightarrow"$ Let I play $a_0, a_1, \ldots$, while $A = \{a_0, a_1, \ldots\}$. The winning strategy for II will produce an elementary embedding $A \rightarrow B$.

"$\Leftarrow"$ Let $j: A \rightarrow B$ be elementary. In response to $a \in A$, played by I, let II play $b = j(a)$.

Generalization:

Lemma 2. II wins $G(A, B)$ iff in $V_{Con(w, A)}$ there is an elementary embedding $j: A \rightarrow B$.

Proof: "$\Rightarrow"$ Let $\tau$ be a w.s. for II. Then $\tau$ is still a w.s. for II in $V_{Con(w, A)}$, then use Lemma 1.
Let \( \varphi : A \rightarrow B \). Look at \( \bar{\varphi}(A, B) \), played in \( V \):

\[
\begin{array}{c|cccc}
I & a_0 & a_1 & \ldots \\
\hline
II & p_0, b_0 & p_1, b_1 & \ldots \\
\end{array}
\]

Rules: \( \{a_0, \ldots\} \in A, \{b_0, \ldots\} \in B, \ p_0 \leq p, \ p_{n+1} \leq p_n, \ p_n \vdash \bar{\varphi}(\hat{a}_n) = \hat{b}_n \).

Clearly, \( II \) has a w.s. in \( \bar{\varphi}(A, B) \). But this results in a w.s. for \( II \) in \( \bar{\varphi}(A, B) \) by hiding the side moves \( p_0, p_1, \ldots \).

**Corollary 1.** If there is an elementary embedding \( \bar{\varphi} : A \rightarrow B \) (in \( V \)), then \( II \) wins \( \bar{\varphi}(A, B) \).

We shall now be interested in such initial segments of \( V \) as our models.
Recall: \( \kappa \) is supercompact iff for every \( \lambda > \kappa \) there is an elementary embedding \( j: V \rightarrow M \) with critical point \( \kappa \) s.t. \( M \) is transitive and \( j(\kappa) > \lambda \), \( V^M \subseteq M \).

**Lemma 3 (Magidor?)** Let \( \kappa \) be supercompact.

For all \( \alpha > \kappa \) there is some \( \beta < \kappa \) and some \( j^*: V^\beta \rightarrow V^\alpha \) s.t. \( j^*(\text{crit}(j^*)) = \kappa \).

**Proof:** Fix \( \alpha \). Pick \( j: V \rightarrow M \) with \( \text{crit}(j) = \kappa \) s.t. \( \nu^{j(M)} \subseteq M \), \( \alpha \notin \nu^{M} \). In particular, \{\( V_\alpha, j^V_\alpha \}\} \subseteq M \) and also \( V^M = V_\alpha \). Hence

\[
M \models \exists \beta < j(\kappa) \exists j^*: V^\beta \rightarrow j(\nu^{\alpha}), j^*(\text{crit}(j^*)) = j(\kappa)
\]

(True as being witnessed by \( \alpha, j^V_\alpha \)). So

\[
V \models \exists \beta < \kappa \exists j^*: V^\beta \rightarrow V^\alpha, j^*(\text{crit}(j^*)) = \kappa.
\]

The converse of Lemma 3 is true also.
Corollary 2. Let $\kappa$ be supercompact.

For all $\alpha > \kappa$ there is $\beta < \kappa$ s.t. $\neg \exists \eta \in G(\beta, \eta)$.

We may obviously reformulate the conclusion of Cor. 2 as follows. Consider the following game, $G^\kappa_0$:

\[
\begin{array}{c|cccc}
I & \alpha & x_0 & x_1 & \ldots \\
\hline
\neg I & \beta & y_0 & y_1 & \ldots \\
\end{array}
\]

Rules: $\alpha > \kappa$, $\beta < \kappa$, $\{x_0, x_1, \ldots\} \subseteq V_\beta$, $\{y_0, y_1, \ldots\} \subseteq V_\alpha$, and for all formulae $\gamma$,

$V_\beta \models \gamma(x_0, \ldots, x_{k-1}) \iff V_\alpha \models \gamma(y_0, \ldots, y_{k-1})$.

Cor. 3 If $\kappa$ is supercompact, then $\neg \exists \eta \in G^\kappa_0$.

\[\neg \exists \eta \in G^\kappa_0\]
Let us also consider a variant of $C^\alpha$, call it $C^\alpha\,^*$:

\begin{align*}
\alpha & \quad x_0 \quad x_1 \quad \ldots \\
\beta & \quad \bar{\alpha} \quad y_0 \quad y_1 \quad \ldots
\end{align*}

Rules: As in $C^\alpha$, plus: $\bar{\alpha} < \beta$,
if $x_n \in \mathcal{V}_{\bar{\alpha}}$, then $y_n = x_n$, and if $x_n = \bar{\alpha}$,
then $y_n = \alpha$.

**Definition 1.** $\alpha$ is called remarkable iff $\beta$ wins $C^\alpha\,^*$.

By Lemma 3 and the proof of Lemma 2:

**Corollary 4.** If $\alpha$ is supercompact, then $\alpha$ is remarkable.
In contrast to supercompact cardinals, remarkable cardinals exist in $L$.

Lemma 4. Let $M$ be a transitive model of $\text{ZFC}$, and let $A, B \in M$. Then

$V \models \text{I wins } G(A, B) \iff M \models \text{I wins } G(A, B)$.

Proof: "$\leq"$

"$\Rightarrow$" If $M \models \text{I wins } G(A, B)$ via $T$.

Let $T$ be the tree of all finite initial segments of plays of $G(A, B)$ in which $\text{I}$ follows $T$ and $\text{II}$ didn't yet lose.

If $V \models t$ is not a w.s. for $\text{I}$ in $G(A, B)$, then $T$ is ill-founded in $V$, hence in $M$, hence $M$ has an infinite play of $G(A, B)$ in which $\text{I}$ follows $T$ but does not win. $\exists$
The proof of Lemma 4 plus Cor. 4 give:

Lemma 5. If $\kappa$ is supercompact, then $\kappa$ is remarkable in $L$.

Let's prove a stronger result, building upon Lemma 4.

Lemma 6. Suppose that $M$ is a transitive model of $\text{ZFC}$, and let $A, B \in M$ be such that in $V$, there is an elementary embedding $j: A \rightarrow B$. Then $\text{II}$ wins $G(A, B)$ in $M$.

Proof: Lemma 1 and Lemma 4. 

Theorem 1. Assume $\#$ exists. Every Silver indiscernible is remarkable in $L$.

Proof: Let $\kappa$ be a Silver indiscernible, and let $\alpha > \kappa$. 

Let $j : L \rightarrow L$ be such that $\text{crit}(j) = \alpha$ and $j(\alpha) > \alpha$. As $j^\alpha L \in L$ is elementary, \text{II} wins $G^\alpha_\alpha$ in $L$ by (the proof of) Lemma 6, where \text{I} starts out by playing $j(\alpha)$.

By the elementarity of $j$, \text{II} wins $G^\alpha_\alpha$ in $L$, where \text{I} starts out by playing $\alpha$.

As $\alpha$ was arbitrary, \text{II} wins $G^\alpha_\alpha$ in $L$. \hfill \Box

**Theorem 2.** Let $(\kappa, F)$ be lexicographically least s.t. $\kappa < F \leq \infty$ and $L_F \models \text{"ZFC + II wins } G^\kappa_0\text{"}$. Then $\kappa$ is remarkable in $L_F$.

**Proof.** Suppose that $\sigma$ is a w.r. for \text{II} in $G^\kappa_0$:

<table>
<thead>
<tr>
<th>I</th>
<th>$\alpha$</th>
<th>$x_0$</th>
<th>$x_1$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>$\beta$</td>
<td>$y_0$</td>
<td>$y_1$</td>
<td>...</td>
</tr>
</tbody>
</table>
By the proof of Lemma 2, for all $\alpha < \beta$, $\alpha > \kappa$, there is then some $\beta < \alpha$ s.t. in $L \cap (\omega, \beta^+)$ there is an $\aleph$-embedding $j_{\beta \alpha} : V_\beta \rightarrow V_\alpha$.

It suffices to see that $j_{\beta \alpha} (\text{crit}(j_{\beta \alpha})) = \kappa$.

Suppose this were wrong for some $\alpha, \beta$. Write $j = j_{\beta \alpha}$. Write $\lambda = \text{crit}(j) = j^{-1}(\kappa)$, which we may assume to exist.

Let $\alpha < \min \{\kappa, j(\lambda)\}$. As $j \upharpoonright V_\beta \rightarrow j(\omega)$ is elementary, $I$ wins exactly $G_0(\alpha)$ in $L_\kappa$, where $I$ starts out by playing $j(\alpha)$.

Using $j$, $\Pi$ wins $G_0^2$ in $L_\kappa$, where $I$ starts out by playing $\alpha$.

In other words, $L_\kappa \models \text{"} \Pi \text{ win } G_0^2 \text{"}$.

But $L_\kappa \models \text{ZFC}$, so we have a contradiction to the minimality of $(\kappa, j)$.

On the other hand, if $\kappa$ is reachable in $L_j$, then clearly $\Pi$ wins $G_0^2$. 