A note on the reals of C^*

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$C^* = L[A]$ where $A = \{\xi \colon \mathrm{cf}(\xi) = \omega\}$, i.e., C^* is the least inner model which knows which ordinals have countable cofinality, see [1].

We aim to get information about $\mathbb{R} \cap C^*$. Theorem 0.7 will generalize Theorem 0.1.

Question 1. Can we have $(2^{\aleph_0})^{C^*} = \aleph_2^V$ in the presence of substantial large cardnals, e.g. supercompact cardinals?

Or just:

Question 2. Can we have $\mathbb{R} \cap C^*$ is not contained in a mouse in the presence of substantial large cardnals, e.g. supercompact cardinals?

Theorem 0.1 (Magidor, Sch, Woodin (?)) *Assume* MM. *Then*

$$\mathbb{R} \cap C^* = \mathbb{R} \cap (C^*_{<\kappa})^{M_1},$$

where M_1 is the least inner model with a Woodin cardinal and κ is any indiscernible for it (e.g. κ is an uncountable V–cardinal). In particular, $\mathbb{R} \cap C^* \subset M_1$.

Proof. "$\Longrightarrow$": Let $x \in \mathbb{R} \cap C^*$. Then $x \in L'_\alpha$, some $\alpha < \omega_2^V$. Pick $z \in \mathbb{R}$ s.t. $(\omega_1^V)^{+L[z]} > \alpha$. Let $j \colon M_1^\# \to N$ be a countable iteration of $M_1^\#$ such that z is generic over N for N's extender algebra. Let $i \colon N \to P$ result from iterating N via its top measure and its images ω_1^V times. Then i lifts to $\hat{i} \colon N[z] \to P[z]$, so that $(\omega_1^V)^{+P} \geq (\omega_1^V)^{L[z]} > \alpha$. This implies that $x \in (C^*_{<\omega_1^V})^P$, hence by pulling back via $i \circ j$, $x \in (C^*_{<\kappa})^{M_1}$ for any M_1–indiscernible κ.

"$\Longrightarrow$": Let $x \in (C^*_{<\kappa})^{M_1}$ for an M_1–indiscernible κ. Let $i \colon M_1^\# \to P$ result from iterating $M_1^\#$ via its top measure and its images ω_1^V times. We get that $x \in \mathbb{R} \cap (C^*_{<\omega_1^V})^P \subset C^*$. $\square$

Corollary 0.2 (Magidor) *Assume* MM. *C^* doesn't have an inner model with a Woodin cardinal. (So it has core model.)*

Proof. Deny. As C^* is closed under #-s, C^* then has its version M of $M_1^\#$. By absoluteness, M is Π^1_2 iterable in V, and hence

$$\mathbb{R} \cap C^* \subset \mathbb{R} \cap M_1 \subset \mathbb{R} \cap M.$$

But if $x \in \mathbb{R} \cap C^*$ codes M, then $x \notin M$. Contradiction! $\square$

We don't need $u_2 = \aleph_2$ to verify the conclusion of Corollary 0.2:

Theorem 0.3 *(Magidor) Assume that there is a measurable cardinal, κ, above a Woodin cardinal, δ. Then $\mathbb{R} \cap C^* \subset C^* | u_2$. Consequently,*

$$\mathbb{R} \cap C^* = \mathbb{R} \cap (C^*_{<\kappa})^{M_1},$$

and C^* does not have an inner model with a Woodin cardinal.

Proof. Fix $x \in \mathbb{R} \cap C^*$, so that $x \in C^*|\omega_2$. Let $\sigma\colon M \to V_{\kappa+2}$, where M is countable and transitive and $\{x, \delta\} \subset \operatorname{ran}(\sigma)$. Let $j\colon M \to M^*$ be a generic iteration of length ω_1 of M via the countable stationary tower $\mathbb{Q}_{<\sigma^{-1}(\delta)}$ in the sense of M and its images. We have that $j(\omega_1^M) = \omega_1^V$ and $x \in (C^*)^{M^*}|\omega_2^{M^*} = (C^*)^V|\omega_2^{M^*}$. By boundedness, $\omega_2^{M^*} < u_2$. □

Question 3. Under MM or the existence of a measurable cardinal above a Woodin cardinal, how does the core model of C^* look like? Is $K^{C^*} = C^*$?

The same proof as the one for Theorem 0.1 shows the following slightly more general result.

Theorem 0.4 (Magidor, Sch, Woodin (?)) *Assume* MM. *Let M^* be the iterate of M_1 obtained by iterating the least (total) measure of M_1 and its images ω_1^V times. Then*

$$\mathrm{HC} \cap C^* = \mathrm{HC} \cap (C^*_{<\kappa})^{M^*}$$

for all indiscernibles for M^* (e.g. κ is a V–cardinal $\geq \aleph_2$).

The question concerning CH in C^* is therefore a question about M_1, as by Thm. 0.4 CH is true in C^* iff it is true in $(C^*_{<\kappa})^{M_1}$ for any (all) M_1–indiscernible(s) κ. Another way to think of it is given by the following.

Theorem 0.5 *Assume* MM. *For a cone of reals x,*

$$\mathbb{R} \cap C^* = \mathbb{R} \cap (C^*_{<\kappa})^{L[x]},$$

where κ is any x–indiscernible (e.g. κ is an uncountable V–cardinal).

Proof. Let z be any real. Let $j\colon M_1 \to N$ be a countable iteration of M_1 such that z is generic over N for N's extender algebra. Let g be $\mathrm{Col}(\omega, \delta^N)$–generic over N such that $z \in N[g]$, and let $x \in N[g]$ be a real such that $N[g] = L[x]$ and $z \leq_T x$. Then $\mathbb{R} \cap C^* = \mathbb{R} \cap (C^*_{<\kappa})^{M_1} = \mathbb{R} \cap (C^*_{<\kappa})^N = (C^*_{<\kappa})^{N[g]} = (C^*_{<\kappa})^{L[x]}$ say for $\kappa = \omega_1^V$. There is hence a $\leq_T$–cofinal set of reals x satisfying the statement of the Thm. which implies that there is a cone of such x. □

Theorem 0.6 *Assume* MM. *For a cone of reals x,*

$$\mathrm{HC} \cap C^* = \mathrm{HC} \cap (C^*_{<\kappa})^{L[x]},$$

where κ is any uncountable x–indiscernible (e.g. κ is an V–cardinal $\geq \aleph_2$).

The following is a crude generalization of Theorem 0.1, Theorem 0.8 gives some more information.

Theorem 0.7 *Let M be an inner model of* AD^+ *such that $\mathbb{R} \subset M$ and*

$$\Theta^M \geq \aleph_2^V.$$

Let P be a countable mouse with ω Woodins such that in $V^{\mathrm{Col}(\omega, 2^{\aleph_0})}$, M can be realized as a derived model of an iterate of P. Then $\mathbb{R} \cap C^* \subset P$. In particular, $\mathrm{Card}(\mathbb{R} \cap C^*) = \aleph_0$.

The hypothesis of Theorem 0.4 holds true e.g. if $M = L(\mathbb{R})$, $\Theta^{L(\mathbb{R})} \geq \aleph_2^V$, and $P = M_\omega^\#$, or M is the least inner model of $\mathsf{AD}_\mathbb{R}$ with $\mathbb{R} \subset M$, $\Theta^M \geq \aleph_2^V$, and $P = M_{\text{adr}}^\#$, but it is much more general.

Proof of Theorem 0.7. Let η be the supremum of the relevant ω Woodin cardinals of P. Inside $V^{\text{Col}(\omega, 2^{\aleph_0})}$, let $i \colon P \to P^*$ be an iteration of P such that $i(\eta) = \omega_1^V$ and $P^*(\mathbb{R}^V) = M$.

Notice that if $\xi < \omega_2^V \leq \Theta^M$, then $\text{cf}^V(\xi) = \omega$ iff $\text{cf}^M(\xi) = \omega$, so that, writing $\mathcal{D}(Q, \rho)$ for the derived model of $\mathcal{Q}$ at ρ,

$$\mathbb{R} \cap C^* = \mathbb{R} \cap (C^*)^{\mathcal{D}(P^*, i(\eta))} = \mathbb{R} \cap (C^*)^{\mathcal{D}(P, \eta)} \subset P.$$

This finishes the proof. □

Theorem 0.8 *Suppose that for every $\xi < \omega_2^V$ there is some countable mouse P with a Woodin cardinal δ and some $\tau \in P^{\text{Col}(\omega,\delta)}$ capturing some prewellordering R on $\mathbb{R}$ with $||R|| \geq \xi$. Then every real in C^* is in a mouse.*

In the absence of $0^\#$, say, Question 1 has an easy answer, cf. Theorem 0.10. Theorems 0.9 and 0.10 compute the consistency strength of "$(2^{\aleph_0})^{C^*} = \aleph_2^V$" over ZFC.

Theorem 0.9 *Suppose that $(2^{\aleph_0})^{C^*} = \aleph_2^V$. Then $\aleph_2^V$ is inaccessible in L.*

Proof. Assume that $\eta^{+L} = \omega_2^V$. Let $A \subset \omega_1^V$ be such that

1. $\omega_1^{L[A]} = \omega_1^V$, and
2. $L[A] \models \text{Card}(\eta) = \aleph_1$.

Let ξ be any ordinal with $\text{cf}(\xi) = \omega$ in V, say $X \subset \xi$ is cofinal and has order type ω. By Jensen Covering, there is some $Y \subset \xi$ such that $Y \supset X$, $Y \in L$ and $\text{otp}(Y) < \omega_2^V$. There is then some bijection $f \colon \omega_1 \to Y$ inside $L[A]$, so that $f"\rho \supset X$ for some $\rho < \omega_1$. In other words,$\text{cf}(\xi) = \omega$ in $L[A]$.

We have shown that $C^* = (C^*)^{L[A]}$. But $\mathbb{R} \cap L[A] \subset L_{\omega_1}[A]$ by condensation, so that $(2^{\aleph_0})^{C^*} < \aleph_2^V$. □

Theorem 0.10 (Kennedy, Magidor, Väänänen) *Assume $V = L$ and κ is inaccessible. There is then a generic extension $V[G]$ of V such that $\omega_1^{V[G]} = \omega_1^V$ and*

$$(2^{\aleph_0})^{(C^*)^{V[G]}} = \kappa = \aleph_2^{V[G]}.$$

Proof. Working over L, we may define define a subproper iteration which adds κ Cohen subsets $\{x_i \colon i < \kappa\}$ of ω and arranges that in the extension,

$$\text{cf}(\aleph_{\omega \cdot i + n + 5}^L) = \omega \iff n \in x_i.$$

Cf. Theorem 7.3 of [1]. □

References

[1] Kennedy, Magidor, Väänänen, Inner Models from Extended Logics. `https://www.newton.ac.uk/files/preprints/ni16006.pdf`