Mariam Beriashvili, Ralf Schindler

Mazurkiewicz sets

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A Mazurkiewicz set is a subset of $\mathbb{R}^2$ which meets every straight line in $\mathbb{R}^2$ in exactly two points. It is easy to construct such a set using a well-order of $\mathbb{R}$. We now produce a model of ZF + DC with a Mazurkiewicz set which does not have a w.o. of $\mathbb{R}$.

Let g be $\mathbb{C}(\omega_1)$-generic over L. Write $\mathbb{R}^* = \mathbb{R} \cap L[g]$. Our model will be a forcing extension of $L(\mathbb{R}^*)$.

Working inside $L(\mathbb{R}^*)$, we define a partial order $\mathbb{P}_M$ as follows. $p \in \mathbb{P}_M$ iff

$$\exists x \in \mathbb{R}^* \; (p \in L[x],$$
$$L[x] \models \text{"} p \text{ is a Mazurkiewicz set,"} \text{ and}$$
$$\exists y \in p \;\; x \leq_T y).$$

Notice that if $p \in \mathbb{P}_M$ and $x, x' \in \mathbb{R}^*$ both witness this, then $x \leq_T$ some element of $p \in L[x']$, so $x \in L[x']$, and also $x' \in L[x]$ by symmetry, so $L[x] = L[x']$. Let us write $x(p)$ for the constructibility degree of some/all reals $x \in \mathbb{R}^*$ witnessing $p \in \mathbb{P}_M$.

By $L[x(p)]$ we mean $L[x]$ for some/all $x \in x(p)$.

We say $p \leq_{\mathbb{P}_M} q$ iff $p \supset q$ and $p \setminus q \subset L[x(p)] \setminus L[x(q)]$.

Let m be $\mathbb{P}_M$-generic over $L[g]$. We claim that $L(\mathbb{R}^*)[m]$ is a model of ZF + DC with a Mazurkiewicz set which does not have a w.o. of $\mathbb{R}$.

<u>Lemma</u> 1. Let $p \in \mathbb{P}_M$, and $x \in \mathbb{R}^*$ s.t. $L[x] \supsetneq L[x(p)]$. There is then some $q \leq p$ s.t. $x \in x(q)$.

Proof: Work in $L[x]$, and let $(\ell_i : i < \omega_1)$ enumerate all the straight lines s.t. $\overline{\overline{\ell_i \cap L[x(p)]}} \leq 1$. Let us construct $(p_i : i \leq \omega_1)$

as follows. $p_0 = p$. $p_\lambda = \bigcup_{i<\lambda} p_i$ for $\lambda \le \omega_1$ a limit. Suppose p_i is constructed. Pick $a \subset \mathbb{R}^2$, $\bar{\bar{a}} \le 2$ s.t.

(a) $a \cap \ell(y,z) = \emptyset$ for all $y, z \in p_i$, $y \ne z$, where $\ell(y,z)$ is the line ℓ with $y, z \in \ell$,

and

(b) $\overline{\overline{(p_i \cup a) \cap \ell_i}} = 2$.

Set $p_{i+1} = p_i \cup a$. Finally, set $q = p_{\omega_1}$. q is as desired. $\dashv$

The same proof shows:

<u>Lemma</u> 2. $(\mathbb{P}_M ; \le_{\mathbb{P}_M})$ is ω-closed in both $L(\mathbb{R}^*)$ as well as $L[g]$.

Proof: Let $\ldots \le p_{n+1} \le p_n \le \ldots$, $p_n \in \mathbb{P}_M$, and let $x \in \mathbb{R}^*$ be s.t. $(x(p_n) : n<\omega)$, $(p_n : n<\omega) \in L[x]$. Then proceed basically as in the proof of Lemma 1. $\dashv$

This shows that $L(\mathbb{R}^*)[m] \models ZF + DC$.

Also, $L(\mathbb{R}^*)[m] \models$ "$\bigcup m$ is a Mazurkiewicz set."

We are left with having to verify that $L(\mathbb{R}^*)[m]$ doesn't have a w.o. of its reals, which by Lemma 2 is $\mathbb{R}^*$.

Let us assume that $p \in m$ and

$$p \Vdash^{\mathbb{P}_M}_{L(\mathbb{R}^*)} \text{" there is a w.o. of } \mathbb{R}\text{, in fact } \varphi(-,-,\underset{\underset{\mathbb{R}^*}{\cap}}{\check{\vec{z}}},\dot{m}) \text{ defines a w.o. of } \mathbb{R}^*\text{,"}$$

where $\dot{m}$ is the canonical name for m.

By Lemma 1, we may assume that $\vec{z} \in L[x(p)] = L[g \restriction \alpha]$ for some $\alpha < \omega_1$.

Let g^* be $\mathbb{C}([\alpha, \omega_1))$-generic over $L[g]$. There must then be $p_0 \leq p$, $p_0 \in g$, $p_1 \leq p$, $p_1 \in g \restriction \alpha ^\frown g^*$, $\gamma \in OR$, $k, \ell_0, \ell_1 < \omega$, $\ell_0 \neq \ell_1$ s.t.

(1) $p_0 \Vdash^{\mathbb{P}_M}_{L(\mathbb{R}^*)}$ "if y is the $\check{\gamma}^{\text{th}}$ real acc. to $\varphi(-,-,\check{\vec{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_0$," and

(2) $p_1 \Vdash^{\mathbb{P}_M^{L(\mathbb{R}\cap L[g\restriction\alpha^\frown g^*])}}_{L(\mathbb{R}\cap L[g\restriction\alpha^\frown g^*])}$ "if y is the $\check{j}^{th}$ real acc. to $\varphi(-,-,\vec{\check{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_1$."

The $\dot{m}$ of the 2^{nd} statement is formally a different object from the $\dot{m}$ of the 1^{st} statement.

Again by Lemma 1, we may assume that there is some $\beta < \omega_1$, $\beta > \alpha$, s.t. $L[x(p_0)] = L[g\restriction\beta]$ and $L[x(p_1)] = L[g\restriction\alpha^\frown g^*\restriction[\alpha,\beta)]$.

Let $u \in \mathbb{R}\cap L[g, g^*]$ be s.t. $x(p_0), x(p_1) \in L[u]$.

The following is the key claim.

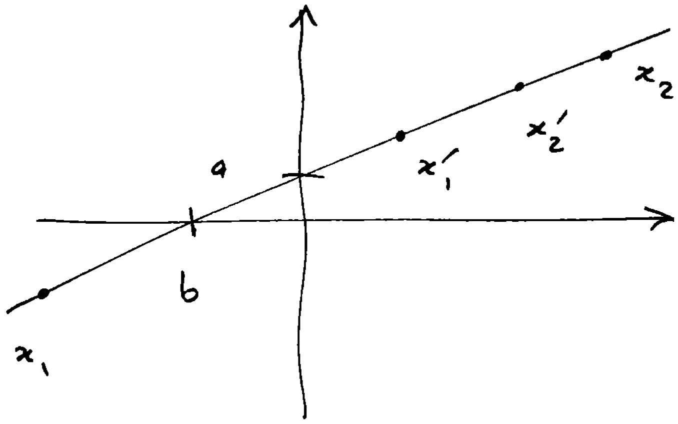
<u>Lemma</u> 3. If ℓ is a straight line in $L[u]$, then $\overline{\overline{\ell\cap(p_0\cup p_1)}} \leq 2$.

Proof: As p_0 is a Mazurkiewicz set in $L[x(p_0)]$, $\overline{\overline{\ell\cap p_0}} \leq 2$. Symmetrically, $\overline{\overline{\ell\cap p_1}} \leq 2$, so that $\overline{\overline{\ell\cap(p_0\cup p_1)}} \leq 4$.

Assume that $x_1, x_2 \in \ell\cap p_0$, $x_1 \neq x_2$, and

$x_1', x_2' \in \ell \cap p_1, \quad x_1' \neq x_2'$.

Let us assume w.l.o.g. that there are unique a and b s.t. $(a, 0), (0, b) \in \ell$ — otherwise the argument is a simple variant of what is to follow.



We will then have that $a, b \in L[x(p_0)] \cap L[x(p_1)] = L[g \restriction \alpha] = L[x(p)]$.

But then $\overline{\overline{\ell \cap p}} = 2$, and because $p_0, p_1 \leq p$, $\{x_1, x_2\} = \{x_1', x_2'\} \in p$.

This shows that if $\overline{\overline{\ell \cap (p_0 \cup p_1)}} > 2$, then $\overline{\overline{\ell \cap (p_0 \cup p_1)}} = 3$ and none of the three points in $\ell \cap (p_0 \cup p_1)$ is in both p_0 and in p_1.

Let us assume that $x_1, x_2 \in \ell \cap p_0$, $x_1 \neq x_2$, and $\overline{\overline{\ell \cap p_1}} = 1$. As p_1 is a Mazurkiewicz set in $L[x(p_1)]$, we must then have that

$$\overline{\overline{\ell \cap L[g \restriction \alpha \wedge g^* \restriction [\alpha,\beta)]}} = 1 .$$

For notational simplicity, let us identify $g^* \restriction [\alpha,\beta)$ with a single Cohen real c.

Let $\tau \in L[g \restriction \beta]$ be such that τ^c is the only element of $\ell \cap L[g \restriction \alpha \wedge c]$, and let $s \in \tau$ be such that

$$s \Vdash^{\mathbb{C}}_{L[g \restriction \beta]} \text{" } \tau \text{ is the only element of } \ell(\check{x}_1, \check{x}_2) \text{ which is in } L[g \restriction \alpha \wedge \dot{c}] \text{,"}$$

where $\ell = \ell(x_1, x_2)$ is the line given by x_1, x_2, and $\dot{c}$ is the name for c.

Let $d_1, d_2 \in L[g \restriction \beta \wedge c]$, $d_1 \neq d_2$, ~~both~~ be mutually Cohen generic over $L[g \restriction \beta)$ with $s \in d_1$, $s \in d_2$.

Then $\tau^{d_1}, \tau^{d_2} \in \ell \cap L[g \restriction \alpha \wedge c]$, so that $\tau^{d_1} = \tau^{d_2} = \tau^s$ and hence $\tau^s \in L[g \restriction \alpha \wedge d_1] \cap L[g \restriction \alpha \wedge d_2] = L[g \restriction \alpha]$. This then contradicts the fact that p_0 is a Mazurkiewicz set in $L[x(p_0)]$. *)

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*) The 2nd author thanks Philipp Lücke for a discussion of this argument.

~~$x_3 \in L[g\restriction\beta] \cap L[g\restriction\alpha{}^\frown g^*\restriction[\alpha,\beta)] = L[g\restriction\alpha]$. But then $x_3 \in L[x(p)] \cap p_1$ implies that $x_3 \in p$ by the definition of $\leq_{\mathbb{P}_M}$. Hence $\{x_1, x_2, x_3\} \subset p_0$, which is a contradiction.~~

Lemma 3 is verified. $\dashv$

Now let $y \in \mathbb{R} \cap L[g\restriction\beta{}^\frown g^*\restriction[\alpha,\beta)]$ be such that $L[y] = L[g\restriction\beta{}^\frown g^*\restriction[\alpha,\beta)]$. In the light of Lemma 3, the proof of Lemma 1 may be used to show the following

<u>Lemma 4</u>. there is some $q \in \mathbb{P}_M^{L[g^\frown g^*\restriction[\alpha,\beta)]}$ s.t. $y \in x(q)$ and $q \supset p_0 \cup p_1$.

Let us write $\mathbb{R}^{**} = \mathbb{R} \cap L[g\restriction\beta{}^\frown g^*\restriction[\alpha,\beta)]$. We have $q, p_0, p_1 \in \mathbb{P}_M^{L(\mathbb{R}^{**})}$, and $q \leq_{\mathbb{P}_M^{L(\mathbb{R}^{**})}} p_0, p_1$ by Lemma 4.

But we have that $L(\mathbb{R}^*) \equiv L(\mathbb{R}^{**})$ in the language of set theory with parameters from $\mathbb{R}^* \cup OR$, and we also have that $L(\mathbb{R} \cap L[g \restriction \alpha ^\frown g^*]) \equiv L(\mathbb{R}^{**})$ in the language of set theory with parameters from $\mathbb{R} \cap L[g \restriction \alpha ^\frown g^*] \cup OR$.

(1) and (2) on pages 4 and 5 then imply that

$p_0 \Vdash^{\mathbb{P}_M^{L(\mathbb{R}^{**})}}_{L(\mathbb{R}^{**})}$ "if y is the $\check{\gamma}^{\text{th}}$ real acc. to $\varphi(-,-,\check{\vec{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_0$," and

$p_1 \Vdash^{\mathbb{P}_M^{L(\mathbb{R}^{**})}}_{L(\mathbb{R}^{**})}$ "if y is the $\check{\gamma}^{\text{th}}$ real acc. to $\varphi(-,-,\check{\vec{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_1$."

By $\ell_0 \neq \ell_1$, this contradicts $p_0 \parallel p_1$ in $\mathbb{P}_M^{L(\mathbb{R}^{**})}$.

We have shown that there is no well-order of $\mathbb{R}^*$ inside $L(\mathbb{R}^*)[m]$.

Question: Is there a Mazurkiewicz set in the Cohen-Halpern-Levy model?