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Mazurkiewicz sets Dec 2017

A Mazurkiewicz set is a subset of $\mathbb{R}^2$ which meets every straight line in $\mathbb{R}^2$ in exactly two points. It is easy to construct such a set using a well-order of $\mathbb{R}$. We now produce a model of ZF + DC with a Mazurkiewicz set which does not have a w.o. of $\mathbb{R}$.

Let g be $\mathbb{C}(\omega_1)$-generic over L. Write $\mathbb{R}^* = \mathbb{R} \cap L[g]$. Our model will be a forcing extension of $L(\mathbb{R}^*)$.

Working inside $L(\mathbb{R}^*)$, we define a partial order $\mathbb{P}_M$ as follows. $p \in \mathbb{P}_M$ iff

$$\exists x \in \mathbb{R}^* \; (p \in L[x] ,$$
$$L[x] \models \text{"} p \text{ is a Mazurkiewicz set," and}$$
$$\exists y \in p \;\; x \leq_T y).$$

Notice that if $p \in \mathbb{P}_M$ and $x, x' \in \mathbb{R}^*$ both witness this, then $x \leq_T$ some element of $p \in L[x']$, so $x \in L[x']$, and also $x' \in L[x]$ by symmetry, so $L[x] = L[x']$. Let us write $x(p)$ for the constructibility degree of some/all reals $x \in \mathbb{R}^*$ witnessing $p \in \mathbb{P}_M$.

By $L[x(p)]$ we mean $L[x]$ for some/all $x \in x(p)$.

We say $p \leq_{\mathbb{P}_M} q$ iff $p \supset q$ and $p \setminus q \subset L[x(p)] \setminus L[x(q)]$.

Let m be $\mathbb{P}_M$-generic over $L[g]$. We claim that $L(\mathbb{R}^*)[m]$ is a model of $ZF + DC$ with a Mazurkiewicz set which does not have a w.o. of $\mathbb{R}$.

<u>Lemma</u> 1. Let $p \in \mathbb{P}_M$, and $x \in \mathbb{R}^*$ s.t. $L[x] \supsetneqq L[x(p)]$. There is then some $q \leq p$ s.t. $x \in x(q)$.

Proof: Work in $L[x]$, and let $(\ell_i : i < \omega_1)$ enumerate all the straight lines s.t. $\overline{\overline{\ell_i \cap L[x(p)]}} \leq 1$. Let us construct $(p_i : i \leq \omega_1)$

as follows. $p_0 = p$. $p_\lambda = \bigcup_{i<\lambda} p_i$ for $\lambda \leq \omega_1$ a limit. Suppose p_i is constructed. Pick $a \subset \mathbb{R}^2$, $\bar{\bar{a}} \leq 2$ s.t.

(a) $a \cap \ell(y,z) = \emptyset$ for all $y, z \in p_i$, $y \neq z$, where $\ell(y,z)$ is the line ℓ with $y, z \in \ell$,

and

(b) $\overline{\overline{(p_i \cup a) \cap \ell_i}} = 2$.

Set $p_{i+1} = p_i \cup a$. Finally, set $q = p_{\omega_1}$. q is as desired. $\dashv$

The same proof shows:

<u>Lemma</u> 2. $(\mathbb{P}_M ; \leq_{\mathbb{P}_M})$ is ω-closed in both $L(\mathbb{R}^*)$ as well as $L[g]$.

Proof: Let $\ldots \leq p_{n+1} \leq p_n \leq \ldots$, $p_n \in \mathbb{P}_M$, and let $x \in \mathbb{R}^*$ be s.t. $(x(p_n) : n < \omega)$, $(p_n : n < \omega) \in L[x]$. Then proceed basically as in the proof of Lemma 1. $\dashv$

This shows that $L(\mathbb{R}^*)[m] \models ZF + DC$.

Also, $L(\mathbb{R}^*)[m] \models$ "$\bigcup m$ is a Mazurkiewicz set."

We are left with having to verify that $L(\mathbb{R}^*)[m]$ doesn't have a w.o. of its reals, which by Lemma 2 is $\mathbb{R}^*$.

Let us assume that $p \in m$ and

$$p \Vdash^{\mathbb{P}_M}_{L(\mathbb{R}^*)} \text{"there is a w.o. of } \mathbb{R}\text{, in fact } \varphi(-,-,\check{\vec{z}},\dot{m}) \text{ defines a w.o. of } \mathbb{R}^*\text{,"}$$

where $\dot{m}$ is the canonical name for m.

By Lemma 1, we may assume that $\vec{z} \in L[z(p)] = L[g \upharpoonright \alpha]$ for some $\alpha < \omega_1$.

Let g^* be $\mathbb{C}([\alpha, \omega_1))$-generic over $L[g]$. There must then be $p_0 \leq p$, $p_0 \in g$, $p_1 \leq p$, $p_1 \in g \upharpoonright \alpha ^\frown g^*$, $\gamma \in OR$, $k, \ell_0, \ell_1 < \omega$, $\ell_0 \neq \ell_1$ s.t.

(1) $p_0 \Vdash^{\mathbb{P}_M}_{L(\mathbb{R}^*)}$ "if y is the $\check{\gamma}^{th}$ real acc. to $\varphi(-,-,\check{\vec{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_0$," and

(2) $p_1 \Vdash^{L(\mathbb{R}\cap L[g\restriction\alpha^\frown g^*])}_{\mathbb{P}_M} {}_{L(\mathbb{R}\cap L[g\restriction\alpha^\frown g^*])}$ "if y is the $\check{j}^{th}$ real acc. to $\varphi(-,-,\vec{\check{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_1$."

The $\dot{m}$ of the 2^{nd} statement is formally a different object from the $\dot{m}$ of the 1^{st} statement.

Again by Lemma 1, we may assume that there is some $\beta<\omega_1$, $\beta>\alpha$, s.t. $L[x(p_0)] = L[g\restriction\beta]$ and $L[x(p_1)] = L[g\restriction\alpha^\frown g^*\restriction[\alpha,\beta)]$.

Let $u\in\mathbb{R}\cap L[g,g^*]$ be s.t. $x(p_0), x(p_1)\in L[u]$. The following is the key claim.

<u>Lemma</u> 3. If ℓ is a straight line in $L[u]$, then $\overline{\overline{\ell\cap(p_0\cup p_1)}} \leq 2$.

<u>Proof</u>: As p_0 is a Mazurkiewicz set in $L[x(p_0)]$, $\overline{\overline{\ell\cap p_0}} \leq 2$. Symmetrically, $\overline{\overline{\ell\cap p_1}} \leq 2$, so that $\overline{\overline{\ell\cap(p_0\cup p_1)}} \leq 4$.

Assume that $x_1, x_2 \in \ell\cap p_0$, $x_1\neq x_2$, and

$x_1', x_2' \in \ell \cap p_1$, $x_1' \neq x_2'$.

$g \restriction \beta \frown g^* \restriction [\alpha, \beta)$ is generic over L. L has $\mathbb{C}(\beta)$-names τ_1, τ_2 for x_1, x_2, and $L[g^* \restriction [\alpha, \beta)]$ has $\mathbb{C}(\alpha)$-names τ_1', τ_2' for x_1', x_2'. I.e.,

$L[g^* \restriction [\alpha, \beta)] \models$ "there are $\mathbb{C}(\alpha)$-names τ_1', τ_2' s.t.

$\Vdash$ '$\tau_1' \neq \tau_2'$, $\tau_1 \neq \tau_2$, and $\tau_1', \tau_2' \in \ell(\tau_1, \tau_2)$'."

By absoluteness,

$L \models$ "there are $\mathbb{C}(\alpha)$-names $\overline{\tau_1'}, \overline{\tau_2'}$ s.t.

$\Vdash$ '$\overline{\tau_1'} \neq \overline{\tau_2'}$, $\tau_1 \neq \tau_2$, and $\overline{\tau_1'}, \overline{\tau_2'} \in \ell(\tau_1, \tau_2)$'."

Write $\overline{x_1'} = \overline{\tau_1'}^g$, $\overline{x_2'} = \overline{\tau_2'}^g$. Then

(a) $\overline{x_1'}, \overline{x_2'} \in L[g \restriction \alpha]$, $\overline{x_1'} \neq \overline{x_2'}$,

(b) $\ell(\overline{x_1'}, \overline{x_2'}) = \ell(x_1, x_2)$, and

(c) $x_1, x_2 \in p_0$.

By the fact that $p \subset p_0$ is a Mazurkiewicz set in $L[x(p)]$ we must then have that actually $x_1, x_2 \in p$.

Symmetrically, $x_1', x_2' \in p$. But then

$\overline{\overline{\{x_1, x_2, x_1', x_2'\}}} \leq 2$.

We have shown that $\overline{\overline{\ell \cap (p_0 \vee p_1)}} = 4$ is impossible.

Now let us assume that $\overline{\overline{\ell \cap (p_0 \vee p_1)}} = 3$, say $x_1, x_2, x_3 \in \ell$ are pairwise different with $x_1, x_2 \in p_0$, $x_3 \in p_1$. The previous argument showed that if $\overline{\overline{\ell \cap L[g \restriction \alpha \frown g^* \restriction [\alpha, \beta)]}} \geq 2$, then $\overline{\overline{\ell \cap L[g \restriction \alpha]}} \geq 2$, and then $x_1, x_2 \in p$, so that $\{x_1, x_2, x_3\} \subset p_1$, which contradicts the fact that they are pairwise different.

Hence $\overline{\overline{\ell \cap L[g \restriction \alpha \frown g^* \restriction [\alpha, \beta)]}} \leq 1$, so that x_3 is the <u>only</u> element of $\ell \cap L[g \restriction \alpha \frown g^* \restriction [\alpha, \beta)]$.

x_3 is then definable inside $L[g \restriction \beta \frown g^* \restriction [\alpha, \beta)]$ from the parameters $x_1, x_2 \in L[g \restriction \beta]$, so that $x_3 \in L[g \restriction \beta]$. Therefore,

$x_3 \in L[g\restriction\beta) \cap L[g\restriction\alpha ^\frown g^*\restriction[\alpha,\beta)] =$ $L[g\restriction\alpha]$. But then $x_3 \in L[x(p)] \cap p_1$ implies that $x_3 \in p$ by the definition of $\leq_{\mathbb{P}_M}$. Hence $\{x_1, x_2, x_3\} \subset p_0$, which is a contradiction. Lemma 3 is verified. ⊣

Now let $y \in \mathbb{R} \cap L[g\restriction\beta ^\frown g^*\restriction[\alpha,\beta)]$ be such that $L[y] = L[g\restriction\beta ^\frown g^*\restriction[\alpha,\beta)]$. In the light of Lemma 3, the proof of Lemma 1 may be used to show the following

<u>Lemma 4.</u> There is some $q \in \mathbb{P}_M^{L[g ^\frown g^*\restriction[\alpha,\beta)]}$ s.t. $y \in x(q)$ and $q \supset p_0 \cup p_1$.

Let us write $\mathbb{R}^{**} = \mathbb{R} \cap L[g\restriction\beta ^\frown g^*\restriction[\alpha,\beta)]$. We have $q, p_0, p_1 \in \mathbb{P}_M^{L(\mathbb{R}^{**})}$, and $q \leq_{\mathbb{P}_M^{L(\mathbb{R}^{**})}} p_0, p_1$ by Lemma 4.

But we have that $L(\mathbb{R}^*) \equiv L(\mathbb{R}^{**})$ in the language of set theory with parameters from $\mathbb{R}^* \cup OR$, and we also have that $L(\mathbb{R} \cap L[g \upharpoonright \alpha \frown g^*]) \equiv L(\mathbb{R}^{**})$ in the language of set theory with parameters from $\mathbb{R} \cap L[g \upharpoonright \alpha \frown g^*] \cup OR$.

(1) and (2) on pages 4 and 5 then imply that

$p_0 \Vdash^{\mathbb{P}_M^{L(\mathbb{R}^{**})}}_{L(\mathbb{R}^{**})}$ "if y is the $\check{\gamma}^{\text{th}}$ real acc. to $\varphi(-,-,\check{\vec{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_0$," and

$p_1 \Vdash^{\mathbb{P}_M^{L(\mathbb{R}^{**})}}_{L(\mathbb{R}^{**})}$ "if y is the $\check{\gamma}^{\text{th}}$ real acc. to $\varphi(-,-,\check{\vec{z}},\dot{m})$, then $y(\check{k}) = \check{\ell}_1$."

By $\ell_0 \neq \ell_1$, this contradicts $p_0 \parallel p_1$ in $\mathbb{P}_M^{L(\mathbb{R}^{**})}$.

We have shown that there is no well-order of $\mathbb{R}^*$ inside $L(\mathbb{R}^*)[m]$.

Question: Is there a Mazurkiewicz set in the Cohen-Halpern-Levy model?