Mazurkiewicz Sets
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January 18, 2018

Abstract
We produce a model of ZF+DC with no wellordering of the reals in which there are Mazurkiewicz sets.

Definition. A Mazurkiewicz Set is a subset of \( \mathbb{R}^2 \) which meets every straight line in \( \mathbb{R}^2 \) in exactly two points. It is easy to construct such a set using a well-order of \( \mathbb{R} \). We now produce a model of ZF + DC with a Mazurkiewicz set which does not have a well-ordering of \( \mathbb{R} \).

Let \( g \) be \( C(\omega_1) \)-generic over \( L \). Write \( \mathbb{R}^* = \mathbb{R} \cap L[g] \). Our model will be a forcing extension of \( L(\mathbb{R}^*) \) Working inside \( L(\mathbb{R}^*) \), we define a partial order \( \mathbb{P}_M \) as follows: \( p \in \mathbb{P}_M \) iff

\[
\exists x \in \mathbb{R}^*(p \in L[x], L[x] \models "p \text{ is a Mazurkiewicz set"}
\]

and

\[
(\exists y \in p)(x \leq_T y).
\]

Notice that if \( p \in \mathbb{P}_M \) and \( x, x' \in \mathbb{R}^* \) both witness this, then \( x \leq_T \) some element of \( p \in L[x'] \) so \( x \in L[x'] \), and also \( x' \in L[x] \) by symmetry, so \( L[x] = L[x'] \). Let us write \( x(p) \) for the constructibility degree of some/all reals \( x \in \mathbb{R}^* \) witnessing \( p \in \mathbb{P}_M \).

By \( L[x(p)] \) we mean \( L[x] \) for some/all \( x \in x(p) \). We say \( p \leq_{\mathbb{P}_M} q \) iff \( p \supset q \) and \( p \setminus q \subseteq L[x(p)] \setminus L[x(q)] \).

Let \( m \) be \( \mathbb{P}_M \)-generic over \( L[g] \). We claim that \( L(\mathbb{R}^*),[m] \) is a model of ZF + DC with a Mazurkiewicz set which does not have a well-ordering of \( \mathbb{R} \).

Lemma 1. Let \( p \in \mathbb{P}_M \), and \( x \in \mathbb{R}^* \) such that \( L[x] \nsubseteq L[x(p)] \). There is then some \( q \leq p \) such that \( x \in x(q) \).

Proof. Work in \( L[x] \) and let \( (l_i : i < \omega_1) \) enumerate all the straight lines such that \( l_i \cap L[x(p)] \leq 1 \). Let us construct \( (p_i : i \leq \omega_1) \) as follows: \( p_0 = p \).

\[
p_\lambda = \bigcup_{i < \lambda} p_i \text{ for } \lambda \leq \omega_1 \text{ a limit. Suppose } p_i \text{ is constructed. Pick } a \subset \mathbb{R}^2, \text{ \overline{\lambda} } \leq 2 \text{ such that}
\]

...
1. \( a \cap l(x, y) = \emptyset \) for all \( y, z \in p_i, y \neq z \), where \( l(y, z) \) is the line \( l \) with \( y, z \in l \), and
2. \( (p_i \cup a) \cap l_i = 2 \).

Set \( p_{i+1} = p_i \cup a \). Finally, set \( q = p_{\omega_1} \). \( q \) is desired.

The same proof shows:

**Lemma 2** \((\mathbb{P}_M; \leq_{\mathbb{P}_M})\) is \( \omega \)-closed in both \( L(\mathbb{R}^*) \) as well as \( L[g] \).

**Proof.** Let \( \cdots \leq p_{n+1} \leq p_n \leq \cdots, p_n \in \mathbb{P}_M \) and let \( x \in \mathbb{R}^* \) be such that
\[
(x(p_n) : n < \omega), (p_n : n < \omega) \in L[x]
\]
. Then proceed basically as in the proof of Lemma 1. \( \dashv \)

This shows that \( L(\mathbb{R}^*)[m] \models ZF + DC \).

Also, \( L(\mathbb{R}^*)[m] \models \text{"\( \bigcup m \) is a Mazurkiewicz Set"} \). We are left with having to verify that \( L(\mathbb{R}^*)[m] \) does not have a well-ordering of its reals, which by Lemma 2 is \( \mathbb{R}^* \).

Let us assume that \( p \in m \) and
\[ p \Vdash_{L(\mathbb{R}^*)} \phi(-, -, \vec{z}, \dot{m}) \] ”there is a well-ordering of \( \mathbb{R} \),

in fact
\[ p \Vdash_{L(\mathbb{R}^*)} \phi(-, -, \vec{z}, \dot{m}) \] defines a well-ordering of \( \mathbb{R}^* \),”

where \( \dot{m} \) is the canonical name for \( m \).

By Lemma 1, we may assume that \( \vec{z} \in L[x(p)] = L[g \upharpoonright \alpha] \) for some \( \alpha < \omega_1 \). Let \( g^* \) be \( C(\alpha, \omega_1) \)-generic over \( L[g] \). There must then be \( p_0 \leq p \), \( p_0 \in g, p_1 \leq p, p_1 \in g \upharpoonright \alpha \cdot g^*, \gamma \in OR, k, l_0, l_1 < \omega, l_0 \neq l_1 \) such that

1. \( p_0 \Vdash_{L(\mathbb{R}^*)} \phi(-, -, \vec{z}, \dot{m}) \) ”if \( y \) is the \( \check{\gamma} \)-th real acc. to \( \phi(-, -, \vec{z}, \dot{m}) \), then \( y(\check{k}) = \check{l_0} \),”

and

2. \( p_1 \Vdash_{L(\mathbb{R}^*)} \phi(-, -, \vec{z}, \dot{m}) \) ”if \( y \) is the \( \check{\gamma} \)-th real acc. to \( \phi(-, -, \vec{z}, \dot{m}) \) then \( y(\check{k}) = \check{l_1} \).”

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The \( \dot{m} \) of the second statement is formally a different object from the \( \dot{m} \) of the first statement. Again by Lemma 1, we may assume that there is some \( \beta < \omega_1, \beta > \alpha \), such that \( L[x(p_0)] = L[g \upharpoonright \beta] \) and \( L[x(p_1)] = L[g \upharpoonright \alpha^* g^* \upharpoonright [\alpha, \beta]] \).

Let \( u \in \mathbb{R} \cap L[g, g^*] \) be such that \( x(p_0), x(p_1) \in L[u] \).

The following is the key claim.

**Lemma 3.** If \( l \) is a straight line in \( L[u] \), then \( \overline{l \cap (p_0 \cup p_1)} \leq 2 \).

**Proof.** As \( p_0 \) is a Mazurkiewicz set in \( L[x(p_0)], \overline{l \cap p_0} \leq 2 \).

Symmetrically, \( \overline{l \cap p_1} \leq 2 \), so that \( \overline{l \cap (p_0 \cup p_1)} \leq 4 \).

Assume that \( x_1, x_2 \in l \cap p_0, x_1 \neq x_2 \), and \( x_1', x_2' \in l \cap p_1, x_1' \neq x_2' \).

By absoluteness, \( g \upharpoonright \beta \upharpoonright g^* \upharpoonright [\alpha, \beta] \) is generic over \( L \).

Let \( \mathcal{C}(\beta) \)-names \( \tau_1, \tau_2 \) for \( x_1, x_2 \) and \( L[g^* \upharpoonright [\alpha, \beta]] \) has \( \mathcal{C}(\alpha) \)-names \( \tau_1', \tau_2' \) for \( x_1', x_2' \) i.e.,

\[
L[g^* \upharpoonright [\alpha, \beta]] \models \text{"there are } \mathcal{C}(\alpha)\text{ - names } \tau_1', \tau_2' \text{ such that } \models \tau_1' \neq \tau_2', \tau_1 \neq \tau_2, \text{ and } \tau_1', \tau_2' \in l(\tau_1, \tau_2)' \text{"}.
\]

By absoluteness,

\[
L \models \text{"there are } \mathcal{C}(\alpha)\text{ - names } \tau_1', \tau_2' \text{ such that } \models \lnot \tau_1' \neq \tau_2', \tau_1 \neq \tau_2, \text{ and } \tau_1', \tau_2' \in l(\tau_1, \tau_2)' \text{"}.
\]

Write \( \overline{x_1} = \tau_1^l, \overline{x_1'} = \tau_1'^l \). Then

a. \( \overline{x_1}, \overline{x_2} \in L[g \upharpoonright \alpha], \overline{x_1} \neq \overline{x_2} \)
b. \( l(\overline{x_1}, \overline{x_2}) = l(x_1, x_2) \)
c. \( x_1, x_2 \in p_0 \).

By the fact that \( p \subset p_0 \) is a Mazuirkiewicz set in \( L[x(p)] \) we must then have that actually \( x_1, x_2 \in p \).

Symmetrically, \( x_1', x_2' \in p \). But, then \( \overline{x_1, x_2, x_1', x_2'} \leq 2 \).

We have shown, that \( \overline{l \cap (l_0 \cup l_1)} = 4 \) is impossible.

Now let us assume that \( \overline{l \cap (l_0 \cup l_1)} = 3 \), say \( x_1, x_2, x_3 \in l \) are pairwise different with \( x_1, x_2 \in p_0, x_3 \in p_1 \). The previous argument showed that if \( \overline{l \cap L[g \upharpoonright \alpha^* g^* \upharpoonright [\alpha, \beta]]} \geq 2 \), then \( \overline{l \cap L[g \upharpoonright \alpha]} \geq 2 \), and then \( x_1, x_2 \in p \), so
that \( \{x_1, x_2, x_3\} \subset p_1 \), which contradicts the fact that they are pairwise different.

hence \( l \cap L[g \upharpoonright \alpha \sim g^* \upharpoonright [\alpha, \beta]] \leq 1 \), so that \( x_3 \) is the only element of \( l \cap L[g \upharpoonright \alpha \sim g^* \upharpoonright [\alpha, \beta]] \). \( x_3 \) is then definable inside \( L[g \upharpoonright \beta \sim g^* \upharpoonright [\alpha, \beta]] \) from the parameters \( x_1, x_2 \in L[g \upharpoonright \beta] \), so that \( x_3 \in L[g \upharpoonright \beta] \). Therefore, \( x_3 \in L[g \upharpoonright \beta] \cap L[g \upharpoonright \alpha \sim g^* \upharpoonright [\alpha, \beta]] = L[g \upharpoonright \alpha] \). But then \( x_3 \in L[x(p)] \cap p_1 \) implies that \( x_3 \in p \) by the definition of \( \leq \). Hence \( \{x_1, x_2, x_3\} \subset p_0 \), which is a contradiction. Lemma 3 is verified. \( \Box \)

Now let \( y \in \mathbb{R} \cap L[g \upharpoonright \beta \sim g^* \upharpoonright [\alpha, \beta]] \) be such that \( L[y] = L[g \upharpoonright \beta \sim g^* \upharpoonright [\alpha, \beta]] \). In the light of Lemma 3, the proof of Lemma 1 may be need to show the following.//

**Lemma 4.** There is some \( q \in P_L^{L[g \upharpoonright \beta \sim g^* \upharpoonright [\alpha, \beta]]} \) such that \( y \in x(q) \) and \( q \supset P_0 \cup p_1 \).

Let us write \( \mathbb{R}^* = \mathbb{R} \cap L[g \upharpoonright \beta \sim g^* \upharpoonright [\alpha, \beta]] \). We have \( q, p_0, p_1 \in P_{\mathbb{M}}^{L(\mathbb{R}^*)} \) and \( q \leq_{P_{\mathbb{M}}^{L(\mathbb{R}^*)}} p_0, p_1 \) by Lemma 4.

But we have that \( L(\mathbb{R}^*) \equiv L(\mathbb{R}^*) \) in the language of set theory with parameters from \( \mathbb{R}^* \cup OR \) and he also have that \( L(\mathbb{R} \cap L[g \upharpoonright \alpha \sim g^*]) \equiv L(\mathbb{R}^*) \) in the language of set theory with parameters from \( \mathbb{R} \cap L[g \upharpoonright \alpha \sim g^*] \cup OR \). (1) and (2) on pages 4 and 5 then imply that \( p_0 \models_{P_{\mathbb{M}}^{L(\mathbb{R}^*)}} \phi(\gamma, \gamma, \bar{z}, \bar{m}) \) if \( y \) is the \( \gamma \)-th real acc. to \( \phi(-, -, \bar{z}, \bar{m}) \), then \( y(\bar{k} = \bar{l}_0) \), and

\[ p_1 \models_{P_{\mathbb{M}}^{L(\mathbb{R}^*)}} M(\mathbb{R} \cap L[g \upharpoonright \alpha \sim g^*]) \) if \( y \) is the \( \gamma \)-th acc. to \( \phi(-, -, \bar{z}, \bar{m}) \) then \( y(\bar{k} = \bar{l}_1) \). By \( l_0 \neq l_1 \) this contradicts \( p_0 \models_{P_{\mathbb{M}}^{L(\mathbb{R}^*)}} \). We have shown that there is no well-order of \( \mathbb{R}^* \) inside \( L(\mathbb{R}^* \{ \bar{m} \}) \).

**References**
