Mazurkiewicz Sets

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Abstract

We produce a model of ZF+DC with no wellorderig of the reals in which there are Mazurkiewicz sets.

Definition. A Mazurkiewicz Set is a subset of $\mathbb{R}^2$ which meets every straight line in $\mathbb{R}^2$ in exactly two points. It is easy to construct such a set using a well-order of $\mathbb{R}$. We now produce a model of $ZF + DC$ with a Mazurkiewicz set which does not have a well-order of $\mathbb{R}$.

Let g be $\mathbb{C}(\omega_1)$-generic over L. Write $\mathbb{R}^* = \mathbb{R} \cap L[g]$. Our model will be a forcing extension of $L(\mathbb{R}^*)$ Working inside $L(\mathbb{R}^*)$, we define a partial order $\mathbb{P}_M$ as follows: $p \in \mathbb{P}_M$ iff

$$\exists x \in \mathbb{R}^*(p \in L[x], L[x] \models \text{"}p \text{ is a Mazurkiewicz set"}$$

and

$$(\exists y \in p)(x \leq_T y).$$

Notice that if $p \in \mathbb{P}_M$ and $x, x' \in \mathbb{R}^*$ both witness this, then $x \leq_T$ some element of $p \in L[x']$ so $x \in L[x']$, and also $x' \in L[x]$ by symmetry, so $L[x] = L[x']$. Let us write $x(p)$ for the constructibility degree of some/all reals $x \in \mathbb{R}^*$ witnessing $p \in \mathbb{P}_M$.

By $L[x(p)]$ we mean $L[x]$ for some/all $x \in x(p)$. We say $p \leq_{\mathbb{P}_M} q$ iff $p \supset q$ and $p \setminus q \subset L[x(p)] \setminus L[x(q)]$.

Let m be $\mathbb{P}_M$-generic over $L[g]$. We claim that $L(\mathbb{R}^*)[m]$ is a model of $ZF + DC$ with a Mazurkiewicz set which does not have a well-ordering of $\mathbb{R}$.

Lemma 1. Let $p \in \mathbb{P}_M$, and $x \in \mathbb{R}^*$ such that $L[x] \supsetneqq L[x(p)]$. There is then some $q \leq p$ such that $x \in x(q)$.

Proof. Work in $L[x]$ and let $(l_i : i < \omega_1)$ enumerate all the straight lines such that $\overline{\overline{l_i \cap L[x(p)]}} \leq 1$. Let us construct $(p_i : i \leq \omega_1)$ as follows: $p_0 = p$. $p_\lambda = \bigcup_{i<\lambda} p_i$ for $\lambda \leq \omega_1$ a limit. Suppose p_i is constructed. Pick $a \subset \mathbb{R}^2$, $\overline{\overline{a}} \leq 2$ such that

1. $a \cap l(x,y) = \emptyset$ for all $y, z \in p_i, y \neq z$, where $l(y,z)$ is the line l with $y, z \in l$,

   and

2. $\overline{\overline{(p_i \cup a) \cap l_i}} = 2$.

Set $p_{i+1} = p_i \cup a$. Finally, set $q = p_{\omega_1}$. q is desired. $\dashv$

The same proof shows:

Lemma2 $(\mathbb{P}_M; \leq_{\mathbb{P}_M})$ is ω-closed in both $L(\mathbb{R}^*)$ as well as $L[g]$.

Proof. Let $... \leq p_{n+1} \leq p_n \leq ..., p_n \in \mathbb{P}_M$ and let $x \in \mathbb{R}^*$ be such that

$$(x(p_n) : n < \omega), (p_n : n < \omega) \in L[x]$$

. Then proceed basically as in the proof of Lemma 1. $\dashv$

This shows that $L(\mathbb{R}^*)[m] \models ZF + DC$.

Also, $L(\mathbb{R}^*)[m] \models$ "$\bigcup m$ is a Mazurkiewicz Set". We are left with having to verify that $L(\mathbb{R}^*)[m]$ does not have a well-ordering of its reals, which by Lemma 2 is $\mathbb{R}^*$.

Let us assume that $p \in m$ and

$$p \Vdash^{P_M}_{(L(\mathbb{R}^*)} \text{"there is a well-ordering of } \mathbb{R},$$

in fact

$$p \Vdash^{P_M}_{(L(\mathbb{R}^*)} \phi(-,-,\check{\vec{z}} \in \mathbb{R}^*, \dot{m}) \text{ defines a well-ordering of } \mathbb{R}^*,\text{"}$$

where $\dot{m}$ is the canonical name for m.

By Lemma 1, we may assume that $\vec{z} \in L[x(p)] = L[g \restriction \alpha]$ for some $\alpha < \omega_1$. Let g^* be $\mathbb{C}([\alpha, \omega_1])$-generic over $L[g]$. There must then be $p_0 \leq p$, $p_0 \in g, p_1 \leq p$, $p_1 \in g \restriction \alpha^\frown g^*$, $\gamma \in OR, k, l_0, l_1 < \omega, l_0 \neq l_1$ such that

1. $p_0 \Vdash^{\mathbb{P}_M}_{L(\mathbb{R}^*)}$ "if y is the $\check{\gamma}^{-th}$ real acc. to $\phi(-,-,\check{\vec{z}}, \dot{m})$, then $y(\check{k}) = \check{l}_0$", and

2. $p_1 \Vdash^{\mathbb{P}_M^{\mathbb{R} \cap L[g \restriction \alpha^\frown g^*]}}_{(L(\mathbb{R} \cap L[\restriction \alpha^\frown g^*])}$ "if y is the $\check{\gamma}^{-th}$ real acc. to $\phi(-,-,\check{\vec{z}}, \dot{m})$ then $y(\check{k}) = \check{l}_1$."

The $\dot{m}$ of the second statement is formally a different object from the $\dot{m}$ of the first statement. Again by Lemma 1, we may assume that there is some $\beta < \omega_1, \beta > \alpha$, such that $L[x(p_0)] = L[g \restriction \beta]$ and $L[x(p_1)] = L[g \restriction \alpha^\frown g^* \restriction [\alpha, \beta]]$.

Let $u \in \mathbb{R} \cap L[g, g^*]$ be such that $x(p_0), x(p_1) \in L[u]$.
The following is the key claim.

Lemma3. If l is a straight line in $L[u]$, then $\overline{\overline{l \cap (p_0 \cup p_1)}} \leq 2$.

Proof. As p_0 is a Mazurkiewicz set in $L[x(p_0)], \overline{\overline{l \cap p_0}} \leq 2$.

Symmetrically, $\overline{\overline{l \cap p_1}} \leq 2$, so that $\overline{\overline{l \cap (p_0 \cup p_1)}} \leq 4$.

Assume that $x_1, x_2 \in l \cap p_0, x_1 \neq x_2$, and $x_1', x_2' \in l \cap p_1, x_1' \neq x_2'$.
$g \restriction \beta\}^\frown g^* \restriction [\alpha, \beta]$ is generic over L.

L has $\mathbb{C}(\beta)$-names τ_1, τ_2 for x_1, x_2 and $L[g^* \restriction [\alpha, \beta]]$ has $\mathbb{C}(\alpha)$-names τ_1', τ_2 for x_1', x_2' l.e.,
$L[g^* \restriction [\alpha, \beta]] \models$ "there are $\mathbb{C}(\alpha)-$names τ_1', τ_2' such that $\Vdash' \tau_1' \neq \tau_2'$, $\tau_1 \neq \tau_2$, and $\tau_1', \tau_2' \in l(\tau_1, \tau_2)'$ "
By absoluteness,

$$L \models \text{"there are } \mathbb{C}(\alpha) - \text{names } \overline{\tau_1'}, \overline{\tau_2'}$$

such that $\Vdash' \overline{\tau_1'} \neq \overline{\tau_2'}$, $\tau_1 \neq \tau_2$, and $\overline{\tau_1'}, \overline{\tau_2'} \in l(\tau_1, \tau_2)'$ ".
Write $\overline{x_1'} = \overline{\tau_1'}^g, \overline{x_1'} = \overline{\tau_1'}^g$. Then

a. $\overline{x_1'}, \overline{x_2'} \in L[g \restriction \alpha]$, $\overline{x_1'} \neq \overline{x_2'}$

b. $l(\overline{x_1'}, \overline{x_2'}) = l(x_1, x_2)$

c. $x_1, x_2 \in p_0$.

By the fact that $p \subset p_0$ is a Mazuirkiewicz set in $L[x(p)]$ we must then have that actually $x_1, x_2 \in p$.

Symmetrically, $x_1', x_2' \in p$. But, then $\overline{\overline{\{x_1, x_2, x_1', x_2'\}}} \leq 2$.

We have shown, that $\overline{\overline{l \cap (l_0 \cup l_1)}} = 4$ is impossible.

Now let us assume that $\overline{\overline{l \cap (l_0 \cup l_1)}} = 3$, say $x_1, x_2, x_3 \in l$ are pairwise different wits $x_1, x_2 \in p_0, x_3 \in p_1$. The previous argument showed that if $\overline{\overline{l \cap L[g \restriction \alpha^\frown g^* \restriction [\alpha, \beta]]}} \geq 2$, then $\overline{\overline{l \cap L[g \restriction \alpha]}} \geq 2$, and then $x_1, x_2 \in p$, so

that $\{x_1, x_2, x_3\} \subset p_1$, which contradicts the fact that they are pairwise different.

hence $l \cap L[g \restriction \alpha^\frown g^* \restriction [\alpha, \beta]] \leq 1$, so that x_3 is the only element of $l \cap L[g \restriction \alpha \frown g^* \restriction [\alpha, \beta]]$. x_3 is then definable inside $L[g \restriction \beta \frown g^* \restriction [\alpha, \beta]]$ from the parameters $x_1, x_2 \in L[g \restriction \beta]$, so that $x_3 \in L[g \restriction \beta]$. Therefore, $x_3 \in L[g \restriction \beta] \cap L[g \restriction \alpha \frown g^* \restriction [\alpha, \beta]] = L[g \restriction \alpha]$. But then $x_3 \in L[x(p)] \cap p_1$ implies that $x_3 \in p$ by the definition of $\leq_{\mathbb{P}_M}$. Hence $\{x_1, x_2, x_3\} \subset p_0$, which is a contradiction. Lemma 3 is verified. $\dashv$

Now let $y \in \mathbb{R} \cap L[g \restriction \beta \frown g^* \restriction [\alpha, \beta]]]$ be such that $L[y] = L[g \restriction \beta \frown g^* \restriction [\alpha, \beta]]$. In the light of Lemma 3, the proof of Lemma 1 may be need to show the following.//

Lemma4. There is some $q \in P_M^{L[g \restriction \beta^\frown g^* \restriction [\alpha,\beta]]}$ such that $y \in x(q)$ and $q \supset p_0 \cup p_1$.

Let us write $\mathbb{R}^*{*} = \mathbb{R} \cap L[g \restriction \beta \frown g^* \restriction [\alpha, \beta]]$. We have $q, p_0, p_1 \in \mathbb{P}_M^{L(\mathbb{R}^*{*})}$ and $q \leq_{\mathbb{P}_M^{L(\mathbb{R}^*{*})}} p_0, p_1$ by Lemma 4.

But we have that $L(\mathbb{R}^*) \equiv L(\mathbb{R}^{**})$ in the language of set theory with parameters from $\mathbb{R}^* \cup OR$ and he also have that $L(\mathbb{R} \cap L[g \restriction \alpha \frown g^*) \equiv L(\mathbb{R}^{**})$ in the language of set theory with parameters from $\mathbb{R} \cap L[g \restriction \alpha \frown g^* \cup OR$. (1) and (2) on pages 4 and 5 then imply that $p_0 \Vdash^{\mathbb{P}_M^{L(\mathbb{R}^{**})}}_{(L(\mathbb{R}^{**})}$ "if y is the $\check{\gamma}^{-th}$ real acc. to $\phi(-, -, \check{\vec{z}}, \dot{m})$, then $y(\check{k} = \check{l_0})$", and

$p_1 \Vdash^{\mathbb{P}_M^{L(\mathbb{R}^*{*})}}_{(L(\mathbb{R}^*{*})} M(L(\mathbb{R} \cap L[g \restriction \alpha \frown g^*])$ "if y is the $\check{\gamma}^{-th}$ real acc. to $\phi(-, -, \check{\vec{z}}, \dot{m})$ then $y(\check{k}) = \check{l_1}$." By $l_0 \neq l_1$ this contradicts $p_0 || p_1$ in $\mathbb{P}_M^{L(\mathbb{R}^*{*})}$. We have shown that there is no well-order of $\mathbb{R}^*$ inside $L(\mathbb{R}^*[m])$.

References

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